

NOTE ON A LITTLEWOOD-PALEY INEQUALITY

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ABSTRACT. We show that a recent result of Littlewood-Paley type, due to the author, is essentially best-possible.

In a recent paper [W], the author proved Littlewood-Paley inequalities for certain finite linear sums $f = \sum_I \lambda_I \phi_{(I)}$, defined on \mathbf{R}^d . The summation here is indexed over \mathcal{D} , the family of dyadic cubes in \mathbf{R}^d ; the λ_I are complex numbers. The functions $\phi_{(I)}$, assumed to be smooth, belong to a family \mathcal{F} that is “almost-orthogonal” and satisfies a mild decay condition. Precisely, we assume that there is an $M > d/2$ such that for all $x \in \mathbf{R}^d$ and all $I \in \mathcal{D}$,

$$(1) \quad |\phi_{(I)}(x)| + \ell(I)|\nabla\phi_{(I)}(x)| \leq |I|^{-1/2}(1 + |x - x_I|/\ell(I))^{-M}.$$

The notation is more or less standard: x_I denotes I 's center, $\ell(I)$ is its sidelength, and $|I|$ is its Lebesgue measure (per tradition, we shall use $|E|$ to mean the Lebesgue measure of any measurable set E). We assume in addition that, for any finite linear sum $\sum_I \gamma_I \phi_{(I)}$,

$$(2) \quad \int_{\mathbf{R}^d} \left| \sum_I \gamma_I \phi_{(I)} \right|^2 dx \leq \sum_I |\gamma_I|^2.$$

Recall that a non-negative $\sigma \in L^1_{loc}(\mathbf{R}^d)$ is said to be an A_∞ weight if there are positive constants a and b such that, for all cubes $Q \subset \mathbf{R}^d$ and measurable subsets $E \subset Q$,

$$\frac{\int_E \sigma}{\int_Q \sigma} \leq a \left(\frac{|E|}{|Q|} \right)^b.$$

The main result from [W] is:

Theorem 1. *Let \mathcal{F} be a family satisfying (1) and (2). Let $\rho > d$. For every $\sigma \in A_\infty$ as above and $0 < p < \infty$, there is a constant $C = C(M, d, \rho, a, b, p)$ such that, for all finite linear sums $f = \sum_I \lambda_I \phi_{(I)}$ from \mathcal{F} ,*

$$\int_{\mathbf{R}^d} |f|^p \sigma dx \leq C \int_{\mathbf{R}^d} \left(\sum_I \frac{|\lambda_I|^2}{|I|} (1 + |x - x_I|/\ell(I))^{-2M+\rho} \right)^{p/2} \sigma dx.$$

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The thing to watch is the exponent $-2M + \rho$. If there were only one term in the sum, we could do no better than $-2M$. The author finds it remarkable that, with such weak hypotheses on the family \mathcal{F} , one can get a result that seems so close to best-possible.

In this note we show that Theorem 1 is, in fact, within ϵ of best-possible.

Theorem 2. *Theorem 1 fails for $p = 2$ and all $\rho < d$.*

Proof of Theorem 2. First we will give our original proof, which uses well-known facts about Bochner-Riesz kernels. The argument has the disadvantage of only working for $d > 1$. At the end we will indicate how to extend the proof to $d = 1$. The extension is easy, but it uses somewhat lesser-known facts about more general oscillatory kernels, due to Jurkat and Sampson [JS].

For $\delta > 0$, let K^δ be the function whose Fourier transform is $(1 - |\xi|^2)_+^\delta$. From [St] (pages 338 and 390), we have:

$$|K^\delta(x)| + |\nabla K^\delta(x)| \leq C(1 + |x|)^{-(\delta+(d+1)/2)}.$$

Asymptotic estimates for K^δ imply that, as long as $\delta < (d - 1)/2$, the convolution operator $K^\delta * f$ cannot be bounded on $L^p(\mathbf{R}^d)$ unless

$$\frac{2d}{d + 1 + 2\delta} < p < \frac{2d}{d - 1 - 2\delta};$$

in particular, for such δ , convolution with K^δ must be unbounded on $L^p(\mathbf{R}^d)$ for large values of p .

We now prove a simple lemma.

Lemma. *Let $\{I_j\}_j$ be the collection of dyadic cubes with sidelength equal to 1. Suppose that, for each j , we have a function h_j satisfying:*

$$\begin{aligned} \text{supp } h_j &\subset I_j, \\ \int |h_j|^2 dx &\leq 1. \end{aligned}$$

For each j , set $\phi_j = K^\delta * h_j$. Then, with $M = \delta + (d + 1)/2$, we have

$$(3) \quad |\phi_j(x)| + |\nabla \phi_j(x)| \leq C(1 + |x - x_{I_j}|)^{-M};$$

and, for all finite linear sums $\sum_j \lambda_j \phi_j$,

$$(4) \quad \int \left| \sum_j \lambda_j \phi_j \right|^2 dx \leq \sum_j |\lambda_j|^2.$$

Proof of the Lemma. Inequality (3) follows from the bounds on K^δ given above. Inequality (4) follows because

$$\int \left| \sum_j \lambda_j h_j \right|^2 dx \leq \sum_j |\lambda_j|^2$$

and $\|\widehat{K^\delta}\|_\infty \leq 1$.

We will now suppose that Theorem 1 holds for some $\rho < d$, and see what happens.

Let $f \in L^2(\mathbf{R}^d)$ have compact support. For each j , set $\gamma_j = (\int_{I_j} |f|^2)^{1/2}$. Define:

$$h_j = \begin{cases} f\chi_{I_j}/\gamma_j & \text{if } \gamma_j > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and set $\phi_j = K^\delta * h_j$, as in the Lemma, where we choose $\delta > 0$ to satisfy:

$$(5) \quad \frac{\rho - 1}{2} < \delta < \frac{d - 1}{2}.$$

Inequality (5) implies that $2M - \rho > d$ and that convolution with K^δ is unbounded on L^p for large p . Let us fix a p for which this is the case.

Note that $\{\phi_j\}_j$ is a finite collection, and that

$$\sum_j \gamma_j \phi_j = K^\delta * f.$$

Let $s > 1$ be the dual exponent to $p/2$, and let $1 < r < s$. Since we have already used M to mean an exponent, we shall denote the Hardy-Littlewood maximal operator by T . Set $T_r g = (T(|g|^r))^{1/r}$. It is a standard fact that, if $g \not\equiv 0$, then $T_r g \in A_\infty$, with A_∞ parameters only depending on r and d . In fact, $T_r g$ belongs to the family A_1 : There is a constant $C = C(r, d)$ such that $T(T_r g) \leq C T_r g$ almost everywhere. One more fact we shall need: There is a constant $C = C(r, s, d)$ such that

$$\|T_r g\|_s \leq C \|g\|_s$$

for all $g \in L^s$.

If Theorem 1 held for the given ρ , then, for all f as described and all $g \in L^s$, we would have:

$$\begin{aligned} \int |K^\delta * f|^2 g \, dx &\leq \int |K^\delta * f|^2 T_r g(x) \, dx \\ &\leq C \sum_j |\gamma_j|^2 \int T_r g(x) (1 + |x - x_{I_j}|)^{-2M+\rho} \, dx \\ &\leq C \sum_j |\gamma_j|^2 T(T_r g)(x_{I_j}) \\ &\leq C \sum_j |\gamma_j|^2 \int_{I_j} T_r g \, dx \\ &\leq C \int (\sum_j \gamma_j^2 \chi_{I_j}) T_r g \, dx \\ &\leq C \left(\int (\sum_j \gamma_j^2 \chi_{I_j})^{p/2} \, dx \right)^{2/p} \|T_r g\|_s \\ &\leq C \left(\int \sum_j \gamma_j^p \chi_{I_j} \, dx \right)^{2/p} \|g\|_s \\ &\leq C \|f\|_p^2 \|g\|_s, \end{aligned}$$

which would imply that the convolution with K^δ was bounded on $L^p(\mathbf{R}^d)$. This proves Theorem 2 when $d > 1$.

To handle $d = 1$, we replace K^δ with a kernel $K_{\alpha,\beta}$, defined by:

$$K_{\alpha,\beta}(t) = (1 + |t|)^{-\beta} e^{i|t|^\alpha}.$$

Here α and β are positive numbers which will be chosen presently (depending on ρ).

Notice that, if α and β are both less than 1, then convolution with $K_{\alpha,\beta}$ must be unbounded on $L^p(\mathbf{R})$ when $p < \beta^{-1}$ (look at $K_{\alpha,\beta} * \chi_B$, where B is a small interval centered at the origin); and, therefore, it must be unbounded on $L^p(\mathbf{R})$ for large p 's. Notice also that, for such α and β , we have:

$$(6) \quad |K_{\alpha,\beta}(t)| + |K'_{\alpha,\beta}(t)| \leq C(\alpha, \beta)(1 + |t|)^{-\beta}$$

for all t . (Strictly speaking, $K_{\alpha,\beta}$ does not satisfy (6) at $t = 0$, but this can be fixed by convolving $K_{\alpha,\beta}$ with a suitable mollifier: what is crucial here is the Lipschitz smoothness.)

We use the following result from [JS] (page 410):

Theorem 3. *Let α, β be positive, $\alpha \neq 1$, $\beta < 1$, and $(\alpha/2) + \beta \geq 1$. Then convolution with $K_{\alpha,\beta}$ is bounded on $L^p(\mathbf{R})$ when*

$$\frac{\alpha}{\alpha + \beta - 1} \leq p \leq \frac{\alpha}{1 - \beta}.$$

In particular, the convolution will be bounded on L^2 .

It is easy to see that the one-dimensional analogue of the “simple lemma” goes through verbatim with $K_{\alpha,\beta}$ in place of K^δ , and with $M \equiv \beta$. Now let $0 < \rho < 1$. Pick $0 < \beta < 1$, so close to 1 that $2\beta - \rho > 1$, and so there exists an α , $0 < \alpha < 1$, such that $(\alpha/2) + \beta \geq 1$. Fix these α and β .

If Theorem 1 held for this ρ , then, by a virtual repetition of our earlier argument, convolution with $K_{\alpha,\beta}$ would be bounded on $L^p(\mathbf{R})$ for large p 's.

Theorem 2 is proved.

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REFERENCES

- [JS] W. B. Jurkat, G. Sampson, “The complete solution to the (L^p, L^q) mapping problem for a class of oscillating kernels,” *Indiana University Mathematics Journal* **30** (1981), 403–413. MR **84i**:42033
- [St] E. M. Stein, *Harmonic Analysis*, Princeton University Press, Princeton (1993).
- [W] J. M. Wilson, “Global orthogonality implies local almost-orthogonality,” to appear in *Revista Matemática Iberoamericana*.

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