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DUALS AND TOPOLOGICAL CENTER OF A CLASS OF MATRIX ALGEBRAS WITH APPLICATIONS

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ABSTRACT. We characterize the topological center of a class of matrix algebras, which are called ℓ^1-Munn algebras. This involves a characterization of Arens regular semigroup algebras for a large class of semigroups, which is an extension of Young's Theorem for semigroups. We show by some counter examples that only up to a certain extent Young's Theorem can be generalized.

1. INTRODUCTION

The second dual of a Banach algebra can be made into a Banach algebra in two different ways, as was first shown by Arens [1]. Different authors studied the second dual of group algebras and Fourier algebras [3], [8], [9], [15]. One of the related problems is specifying the topological center, which is the largest subalgebra where these two multiplications coincide. For a recent and abstract approach see [15]. Isik, Pym and Ulger [12] showed that for any compact group G, the topological center of $L^1(G)^{**}$ is $L^1(G)$. This was extended to any locally compact group by Lau and Losert [13]. Also Young [19] showed that for a locally compact group G, $L^1(G)$ is Arens regular if and only if G is finite. ℓ^1-Munn algebras were introduced in [6] and some of their properties were considered in the same article. Here we study duals of ℓ^1-Munn algebras and we show that their second dual is indeed another ℓ^1-Munn algebra, built on the second dual of the original algebra. Then we provide a characterization of their topological center and in particular a characterization of Arens regular ℓ^1-Munn algebras. This is an analog of Young's theorem for ℓ^1-Munn algebras. Application of the latter result to semigroup algebras provides some generalizations of Young's Theorem for semigroups. We give some examples to show the restrictions on generalizing Young's Theorem.

This paper is organized as follows: In section 2 we introduce our notations. In section 3 we show some basic results on the duals of ℓ^1-Munn algebras. Topological center of ℓ^1-Munn algebras are characterized in section 4. In section 5 we apply the results of previous sections to semigroup algebras. Further study of the structure of the ℓ^1-Munn algebras, in particular their representation theory, is done in [7].

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2. Notations

Throughout we use notations of [6]. Let I and J be arbitrary index sets and P be a $J \times I$ matrix over $\mathcal{A}$ which has no zero row or column, $\|P\|_\infty = sup\{\|P_{ji}\| : j \in J, i \in I\} \leq 1$ and nonzero entries of P are invertible. The set $\mathcal{LM}(\mathcal{A}, P)$ of all $I \times J$ matrices A over $\mathcal{A}$ such that $\|A\|_1 = \sum_{i \in I, j \in J} \|A_{ij}\| < \infty$ with ℓ^1-norm and product $A \circ B = APB,\ A, B \in \mathcal{LM}(\mathcal{A}, P)$ is a Banach algebra that we call the *ℓ^1-Munn $I \times J$ matrix algebra over $\mathcal{A}$ with sandwich matrix P* or briefly the *ℓ^1-Munn algebra*. When $I = J$ and P is the identity $J \times J$ matrix over $\mathcal{A}$, we denote $\mathcal{LM}(\mathcal{A}, P)$ by $\mathcal{LM}_J(\mathcal{A})$. Also we denote $\mathcal{LM}_J(\mathbb{C})$ simply by $\mathcal{LM}_J$. In particular when $|J| = m < \infty$, $\mathcal{LM}_J$ is the algebra $\mathcal{M}_m$ of $m \times m$ complex matrices.

Throughout $\{\varepsilon_{ij} : i \in I,\ j \in J\}$ is the standard matrix unit system of the matrix algebra under discussion. Let $\mathcal{A}$ be an arbitrary Banach algebra. The *first* and *second Arens multiplications on* $\mathcal{A}^{**}$ that we denote by "Δ" and "." respectively, are defined in three steps. For $a, b \in \mathcal{A},\ f \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$, the elements $f\Delta a,\ a.f,\ m\Delta f,\ f.m$ of $\mathcal{A}^*$ and $m\Delta n,\ m.n$ of $\mathcal{A}^{**}$ are defined in the following way:

$$\Big\langle f\Delta a\ ,\ b\Big\rangle = \Big\langle f\ ,\ ab\Big\rangle, \qquad \Big\langle a.f\ ,\ b\Big\rangle = \Big\langle f\ ,\ ba\Big\rangle,$$
$$\Big\langle m\Delta f\ ,\ a\Big\rangle = \Big\langle m\ ,\ f\Delta a\Big\rangle, \qquad \Big\langle f.m\ ,\ b\Big\rangle = \Big\langle m\ ,\ b.f\Big\rangle,$$
$$\Big\langle m\Delta n\ ,\ f\Big\rangle = \Big\langle m\ ,\ n\Delta f\Big\rangle, \qquad \Big\langle m.n\ ,\ f\Big\rangle = \Big\langle n\ ,\ f.m\Big\rangle.$$

Here we assume $\mathcal{A}^{**}$ has the first Arens multiplication, unless stated otherwise. For fixed $n \in \mathcal{A}^{**}$ the map $m \mapsto m\Delta n$ $[m \mapsto n.m]$ is weak*-weak* continuous, but the map $m \mapsto n\Delta m$ $[m \mapsto m.n]$ is not necessarily weak*-weak* continuous, unless n is in $\mathcal{A}$. The *topological center* $\mathcal{Z}(\mathcal{A}^{**})$ *of* $\mathcal{A}^{**}$ is defined by

$$\mathcal{Z}(\mathcal{A}^{**}) = \{n \in \mathcal{A}^{**}\ :\ \text{The map } m \mapsto n\Delta m \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.$$

It can be shown that

$$\mathcal{Z}(\mathcal{A}^{**}) = \{n \in \mathcal{A}^{**}\ :\ n\Delta m = n.m \text{ for all } m \in \mathcal{A}^{**}\}.$$

If $\mathcal{Z}(\mathcal{A}^{**}) = \mathcal{A}^{**}$, then $\mathcal{A}$ is called *Arens Regular*. If $\mathcal{A}$ is commutative, then $\mathcal{Z}(\mathcal{A}^{**})$ is precisely the algebraic center of $\mathcal{A}^{**}$.

Let X be an arbitrary Banach space. It is well known that $\ell^1(I, X)^*$ is isometrically isomorphic to $\ell^\infty(I, X^*)$. In particular $\mathcal{LM}(\mathcal{A}, P)^* = \ell^\infty(I \times J, \mathcal{A}^*)$. Note that if I is a finite index set, then $\ell^1(I, X) = X^I = \ell^\infty(I, X)$ as vector spaces and equivalence of the two norms in this case implies that these two are topologically isomorphic. So in this case all of the Banach spaces $\ell^\infty(I, X)^*$, $\ell^\infty(I, X^*)$, $\ell^1(I, X)^*$ and $\ell^1(I, X^*)$ are topologically isomorphic.

In the algebraic notations for semigroups we mainly follow [4]. Throughout S $[G]$ is a semigroup [group] and E_S is the set of idempotent elements of S. If T is an ideal of S, then the *Rees factor semigroup* S/T is the result of collapsing T into a single element 0 and retaining the identity of elements of $S \setminus T$. We make the convention that $S/\varnothing = S$. If S has an identity, then $S^1 = S$; otherwise $S^1 = S \cup \{1\}$ where 1 is the identity joined to S. For $a \in S,\ J(a)$ is the principal ideal $S^1 a S^1$ and J_a is the set of elements $b \in J(a)$ such that $J(b) = J(a)$. By $I(a)$ we mean the ideal $\{b \in J(a)\ :\ J(b) \subsetneqq J(a)\}$, i.e. $I(a) = J(a) \setminus J_a$. On E_S we have a usual order: $e, f \in E_S,\ e \leq f$ if $ef = fe = e$. An idempotent $e \in E_S$ is called *primitive* if it is nonzero and is minimal in the set of nonzero idempotents. A semigroup S with zero is 0-*simple* if $\{0\}$ and S are the only ideals of S. S is called *completely* [0-]*simple* if it is [0-]simple and contains a nonzero primitive idempotent. The factors $J(a)/I(a),\ a \in S$ are called the *principal factors* of S. A [relative] ideal series $S = S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} = \varnothing$ that has no proper refinement is called a *principal* [*composition*] *series*. If S has a principal series, then the factors of this series are isomorphic in some order to the principal factors of S [4, Theorem 2.40].

A semigroup S is called *regular* if for every $a \in S$ there is $b \in S$ such that $a = aba$. S is an *inverse semigroup* if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$. Let G be a group, I and J be arbitrary nonempty sets and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. An $I \times J$ matrix A over G^0 that has at most one nonzero entry $a = A(i,j)$ is called a *Rees* $I \times J$ *matrix over* G^0 and is denoted by $(a)_{ij}$. Let P be a $J \times I$ matrix over G. The set $S = G \times I \times J$ with the composition $(a,i,j) \circ (b,l,k) = (aP_{jl}b, i, k)\ ,\ (a,i,j),(b,l,k) \in S$ is a semigroup that we denote by $\mathcal{M}(G,P)$ [18, page 68]. Similarly if P is a $J \times I$ matrix over G^0, then $S = G \times I \times J \cup \{0\}$ is a semigroup under the following composition operation which is denoted by $\mathcal{M}^0(G,P)$:

$$(a,i,j) \circ (b,l,k) = \begin{cases} (aP_{jl}b, i, k) & \text{if } P_{jl} \neq 0, \\ 0 & \text{if } P_{jl} = 0, \end{cases}$$
$$(a,i,j) \circ 0 = 0 \circ (a,i,j) = 0 \circ 0 = 0.$$

$\mathcal{M}^0(G,P)$ is isomorphic to the semigroup of all Rees $I \times J$ matrices over G^0 with binary operation $A \circ B = APB$, which is called the *Rees* $I \times J$ *matrix semigroup over* G^0 *with the sandwich matrix* P [11, pages 61-63]. An $I \times J$ matrix P over G^0 is called *regular* [*invertible*] if every row and every column of P contains at least [exactly] one nonzero entry.

If S has a zero, then we call the algebra $\ell^1(S)/\ell^1(0)$ the *contracted semigroup algebra of* S, where $\ell^1(0) = \ell^1(\{0\})$.

3. First and second duals of ℓ^1-Munn algebras

Convention. If $F \in \ell^\infty(I \times J, \mathcal{A}^*)$, then we denote $F(i,j)$ by F_{ij}. Also by $f\varepsilon_{ij} \in \ell^\infty(I \times J, \mathcal{A}^*)$ we mean an $I \times J$ matrix over $\mathcal{A}^*$ that has f as its (i,j)th entry and 0 elsewhere. From now on by ψ we mean the map $\psi : \mathcal{LM}(\mathcal{A},P)^* \longrightarrow \ell^\infty(I \times J, \mathcal{A}^*)\ ,\ \left\langle \psi(F)(i,j)\ ,\ a \right\rangle = \left\langle F\ ,\ a\varepsilon_{ij} \right\rangle$. We will denote $\psi(F)$ by $\widetilde{F}$. We will use similar notations for $\ell^\infty(I \times J, \mathcal{A}^{**})$ and its elements. Also when I and J are finite, we will use the same notations to identify the two Banach spaces $\mathcal{LM}(\mathcal{A},P)^{**}$ and $\mathcal{LM}(\mathcal{A}^{**},P)$.

Remark 3.1. Although in the above definition of ℓ^1-Munn algebras we assumed nonzero entries of P are invertible, this assumption is not used in some of the results of [6]; in particular [6, Lemma 3.5] and also in Lemma 3.2, Lemma 3.5 and Proposition 4.1 of this paper. We will drop this condition in those results as we need this case in the proof of Theorem 4.2(i).

Lemma 3.2. *If the index sets are finite, then* $\mathcal{LM}(\mathcal{A},P)^{**}$ *is topologically algebra isomorphic to* $\mathcal{LM}(\mathcal{A}^{**},P)$ *when both of* $\mathcal{A}^{**}$ *and* $\mathcal{LM}(\mathcal{A},P)^{**}$ *are equipped with the first [second] Arens product.*

Proof. Using the facts that were mentioned at the beginning of this section, we need only to show that the linear isomorphism $\psi\ : \mathcal{LM}(\mathcal{A},P)^{**} \longrightarrow \mathcal{LM}\,(\mathcal{A}^{**},P)$ is multiplicative. Throughout we will use the fact that restriction of Arens product of $\mathcal{A}^{**}$ to $\mathcal{A}$ agrees with the multiplication of $\mathcal{A}$. Let $A, X \in \mathcal{LM}(\mathcal{A},P)$, $F \in \mathcal{LM}(\mathcal{A},P)^*$ and $M, N \in \mathcal{LM}(\mathcal{A},P)^{**}$. Then,

$$\begin{aligned}\Big\langle (\widetilde{F\Delta A})_{ij}\ ,\ X_{ij}\Big\rangle &= \Big\langle F\ ,\ A\circ X_{ij}\varepsilon_{ij}\Big\rangle = \sum_{k,l}\Big\langle F\ ,\ A_{kl}P_{li}X_{ij}\varepsilon_{kj}\Big\rangle\\ &= \sum_{k,l}\Big\langle \widetilde{F}_{kj}\ ,\ A_{kl}P_{li}X_{ij}\Big\rangle = \Big\langle \sum_{k,l}\widetilde{F}_{kj}\Delta(A_{kl}P_{li})\ ,\ X_{ij}\Big\rangle,\end{aligned}$$

so

$$(\widetilde{F\Delta A})_{ij} = \sum_{k,l}\widetilde{F}_{kj}\Delta(A_{kl}P_{li}).$$

Applying this relation to $\widetilde{M\Delta F}$, we get

$$\begin{aligned}\Big\langle (\widetilde{M\Delta F})_{ij}\ ,\ A_{ij}\Big\rangle &= \Big\langle M\ ,\ F\Delta(A_{ij}\varepsilon_{ij})\Big\rangle = \sum_{r,s}\Big\langle \widetilde{M}_{rs}\ ,\ \widetilde{F}_{is}\Delta(A_{ij}P_{jr})\Big\rangle\\ &= \sum_{r,s}\Big\langle A_{ij}\Delta P_{jr}\ ,\ \widetilde{M}_{rs}\Delta\widetilde{F}_{is}\Big\rangle = \Big\langle \sum_{r,s}P_{jr}\Delta(\widetilde{M}_{rs}\Delta\widetilde{F}_{is})\ ,\ A_{ij}\Big\rangle,\end{aligned}$$

so

$$(\widetilde{M\Delta F})_{ij} = \sum_{r,s}P_{jr}\Delta(\widetilde{M}_{rs}\Delta\widetilde{F}_{is}).$$

Now

$$\begin{aligned}\Big\langle (\widetilde{N\Delta M})_{ij}\ ,\ \widetilde{F}_{ij}\Big\rangle &= \Big\langle N\ ,\ M\Delta(\widetilde{F}_{ij}\varepsilon_{ij})\Big\rangle = \sum_{r,l}\Big\langle \widetilde{N}_{il}\ ,\ P_{lr}\Delta(\widetilde{M}_{rj}\Delta\widetilde{F}_{ij})\Big\rangle\\ &= \Big\langle \sum_{r,l}\widetilde{N}_{il}\Delta(P_{lr}\Delta\widetilde{M}_{rj})\ ,\ \widetilde{F}_{ij}\Big\rangle.\end{aligned}$$

Thus

$$(\widetilde{N\Delta M})_{ij} = \sum_{r,l}\widetilde{N}_{il}\Delta(P_{lr}\Delta\widetilde{M}_{rj}). \tag{1}$$

Similarly for the second Arens product we can prove the following identities:

$$\begin{aligned}(\widetilde{A.F})_{ij} &= \sum_{k,l}(P_{jl}A_{lk}).\widetilde{F}_{ik},\\ (\widetilde{F.M})_{ij} &= \sum_{r,s}(\widetilde{F}_{rj}.\widetilde{M}_{rs}).P_{si},\\ (\widetilde{N.M})_{ij} &= \sum_{r,l}(\widetilde{N}_{il}.P_{lr}).\widetilde{M}_{rj}. \qquad (2)\end{aligned}$$

Therefore $\psi(N\Delta M) = \psi(N)\Delta\psi(M)$ and $\psi(N.M) = \psi(N).\psi(M)$. □

Corollary 3.3. *Suppose the index sets are finite. Then $\mathcal{LM}(\mathcal{A},P)$ is an ideal in $\mathcal{LM}(\mathcal{A},P)^{**}$ if and only if $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$.*

Proof. It is easy to check that $\psi(\mathcal{LM}(\mathcal{A},P))=\mathcal{LM}(\mathcal{A},P)$.

Suppose $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$. Let $A\in\mathcal{LM}(\mathcal{A},P)$ and $M\in\mathcal{LM}(\mathcal{A},P)^{**}$. By Lemma 3.2 and relation (1) in its proof we get

$$(\widetilde{A\Delta M})_{ij}=\sum_{r,l}(A_{il}P_{lr})\Delta\widetilde{M}_{rj}\in\mathcal{A}\text{ for all } i\in I,\ j\in J.$$

Thus $\widetilde{A\Delta M}\in\mathcal{LM}(\mathcal{A},P)$. Similarly $\widetilde{M\Delta A}\in\mathcal{LM}(\mathcal{A},P)$; therefore $\mathcal{LM}(\mathcal{A},P)$ is an ideal in $\mathcal{LM}(\mathcal{A},P)^{**}$.

Conversely suppose $\mathcal{LM}(\mathcal{A},P)$ is an ideal in $\mathcal{LM}(\mathcal{A},P)^{**}$ and fix $i\in I,\ j\in J$ such that $P_{ji}\neq 0$. Let $m\in\mathcal{A}^{**}$ and $a\in\mathcal{A}$. Let $M=\psi^{-1}(m\varepsilon_{ij})$ and $A=P_{ji}^{-1}a\varepsilon_{ij}$. By relation (1) in the proof of Lemma 3.2 $(\widetilde{M\Delta A})_{ij}=m\Delta a$. Now by assumption $M\Delta A\in\mathcal{LM}(\mathcal{A},P)$ and hence $\widetilde{M\Delta A}\in\mathcal{LM}(\mathcal{A},P)$. Therefore $m\Delta a\in\mathcal{A}$. □

Definition 3.4. We will call the following relations *mixed associativity relations.* They can be proved easily just by using the definition of Arens product.

For every $a,x\in\mathcal{A},\ f\in\mathcal{A}^*$ and $m\in\mathcal{A}^{**}$,

$$\begin{aligned}&(f\Delta x).a=f\Delta(xa), && a\Delta(x.f)=(ax).f, && (x.f)\Delta a=x.(f\Delta a),\\ &(a.m)\Delta f=a.(m\Delta f), && f.(m\Delta a)=(f.m)\Delta a,\\ &(m\Delta a).x=m\Delta(ax), && a\Delta(x.m)=(ax).m.\end{aligned}$$

Suppose I and J are finite. If V and W are invertible $J\times J$ and $I\times I$ matrices over $\mathcal{A}$ respectively, then the map $\theta\ :\ \mathcal{LM}(\mathcal{A},P)\longrightarrow\mathcal{LM}(\mathcal{A},VPW)$ defined by $\theta(A)=W^{-1}AV^{-1}$ is a topological algebra isomorphism [6, Lemma 3.5].

Lemma 3.5. *Suppose the index sets are finite and let $\theta\ :\ \mathcal{LM}(\mathcal{A}^{**},Q)\longrightarrow\mathcal{LM}(\mathcal{A}^{**},P)$ be the above topological algebra isomorphism. Then the restriction of θ to $\mathcal{LM}(\mathcal{A},P)$, which we will denote by θ again, maps $\mathcal{LM}(\mathcal{A},Q)$ onto $\mathcal{LM}(\mathcal{A},P)$ and makes the following diagram commutative:*

$$\begin{array}{ccc}\mathcal{LM}(\mathcal{A},Q)^{**} & \xrightarrow{\theta^{**}} & \mathcal{LM}(\mathcal{A},P)^{**}\\ \Big\downarrow\psi & & \Big\downarrow\psi\\ \mathcal{LM}(\mathcal{A}^{**},Q) & \xrightarrow{\theta} & \mathcal{LM}(\mathcal{A}^{**},P).\end{array}$$

Proof. The first statement of the Lemma follows from the argument of [6, Lemma 3.5]. Let $M\in\mathcal{LM}(\mathcal{A},Q)^{**}$ and $f\in\mathcal{A}^*$. Whenever necessary, we will assume $f\varepsilon_{ij}\in\ell^\infty(I\times J,\mathcal{A}^{***})=\mathcal{LM}(\mathcal{A}^{**},P)^*$.

Let $\lambda\ :\mathcal{LM}(\mathcal{A},Q)\hookrightarrow\mathcal{LM}(\mathcal{A}^{**},Q)$ and $\mu\ :\mathcal{LM}(\mathcal{A}^{**},Q)\hookrightarrow\mathcal{LM}(\mathcal{A}^{**},Q)^{**}$ be the natural embeddings. It is easy to check that $\lambda^{**}=\mu\psi$. So,

$$\begin{aligned}\Big\langle(\psi\theta^{**}(M))_{ij}\ ,\ f\Big\rangle&=\Big\langle M\ ,\ \theta^*(f\varepsilon_{ij})\Big\rangle=\Big\langle\lambda^{**}(M)\ ,\ \theta^*(f\varepsilon_{ij})\Big\rangle\\&=\Big\langle\theta^*(f\varepsilon_{ij})\ ,\ \psi(M)\Big\rangle=\Big\langle(\theta\psi(M))_{ij}\ ,\ f\Big\rangle.\end{aligned}$$

Therefore $\psi\theta^{**}=\theta\psi$ as required. □

4. Topological center of the second dual of ℓ^1-Munn algebras

Proposition 4.1. *If the index sets I and J are finite, then*

$$\mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right) \subseteq \psi(\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right)).$$

In particular if $\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right) = \mathcal{LM}(\mathcal{A}, P)$, then $\mathcal{Z}(\mathcal{A}^{**}) = \mathcal{A}$.

Proof. Let $\widetilde{M} \in \mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right)$ and $M = \psi^{-1}(\widetilde{M}) \in \mathcal{LM}(\mathcal{A}, P)^{**}$. As $\mathcal{A} \subseteq \mathcal{Z}(\mathcal{A}^{**})$, by relation (1) in the proof of Lemma 3.2 for every $N \in \mathcal{LM}(\mathcal{A}, P)^{**}$ we have

$$(\widetilde{M\Delta N})_{ij} = \sum_{r,l} \widetilde{M}_{il} \Delta(P_{lr} \Delta \widetilde{N}_{rj}) = \sum_{r,l} \widetilde{M}_{il}.(P_{lr}.\widetilde{N}_{rj}) = (\widetilde{M.N})_{ij}.$$

So $M\Delta N = M.N$ and hence $\widetilde{M} \in \psi(\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right))$. The second statement follows from the first part and the fact that $\psi(\mathcal{LM}(\mathcal{A}, P)) = \mathcal{LM}(\mathcal{A}, P)$. □

Now we prove the main result of this section which explains the relation between the topological center of the second dual of the ℓ^1-Munn algebras and some other finiteness conditions, in particular existence of bounded approximate identities in the ℓ^1-Munn algebras. Throughout the proof we will use the following fact without any specific reference: If $\theta : \mathcal{B} \longrightarrow \mathcal{C}$ is a topological algebra isomorphism between two arbitrary Banach algebras $\mathcal{B}$ and $\mathcal{C}$, then $\theta^{**}\left(\mathcal{Z}(\mathcal{B}^{**})\right) = \mathcal{Z}(\mathcal{C}^{**})$.

Theorem 4.2. (i) *$\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity if and only if the index sets are finite and $\psi(\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right)) = \mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right)$.*

(ii) *Suppose $\mathcal{A}$ admits a nonzero multiplicative linear functional. Then $\mathcal{LM}(\mathcal{A}, P)$ is Arens regular if and only if one of the following conditions holds:*

(a) *$\mathcal{A}$ is Arens regular and both of the index sets are finite.*

(b) *$\mathcal{A}$ is finite dimensional and one of the index sets is finite.*

Proof. (i)($\Longrightarrow$) Suppose $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity. By [6, Lemma 3.7] I and J are finite and P is invertible. Assume $\theta : \mathcal{LM}\left(\mathcal{A}^{**}, P\right) \longrightarrow \mathcal{LM}_J(\mathcal{A}^{**})$ is the topological algebra isomorphism that was mentioned prior to Lemma 3.5. By Proposition 4.1,

$$\mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right) \subseteq \psi(\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right)).$$

Let $M \in \mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**}),\ n \in \mathcal{A}^{**}$ and $i, j \in J$. Let $\widetilde{N} = n\varepsilon_{jj}$ and $N = \psi^{-1}(\widetilde{N})$. Then by relations (1) and (2) in the proof of Lemma 3.2 we have

$$\widetilde{M}_{ij} \Delta n = (\widetilde{M\Delta N})_{ij} = (\widetilde{M.N})_{ij} = \widetilde{M}_{ij}.n.$$

So $\psi(M) \in \mathcal{LM}_J(\mathcal{Z}(\mathcal{A}^{**}))$ and for this special case the equality $\mathcal{LM}_J(\mathcal{Z}(\mathcal{A}^{**})) = \psi(\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**}))$ holds.

For the rest of this part we follow the terminology of Lemma 3.5. The equality $\theta^{**}(\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**})) = \mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})$ implies that if we restrict the maps ψ and θ^{**} to the $\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**})$ in the commutative diagram of Lemma 3.5, we will get the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**}) & \xrightarrow{\theta^{**}} & \mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**}) \\
\Big\downarrow \psi & & \Big\downarrow \psi \\
\mathcal{LM}_J(\mathcal{A}^{**}) & \xrightarrow{\theta} & \mathcal{LM}\left(\mathcal{A}^{**}, P\right).
\end{array}$$

So we have

$$\begin{aligned}\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A},P)^{**})) &= \theta\psi(\theta^{**})^{-1}(\mathcal{Z}(\mathcal{LM}(\mathcal{A},P)^{**})) = \theta\psi(\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**})) \\ &= \theta(\mathcal{LM}_J(\mathcal{Z}(\mathcal{A}^{**}))) = \mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}),P\right).\end{aligned}$$

($\Longleftarrow$) Suppose I and J are finite and $\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A},P)^{**})) = \mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}),P\right)$. By [6, Lemma 3.7] it is enough to show that P is invertible. We will do this by induction on $k=\max\{|I|,|J|\}$. If $k=1$, trivially P is invertible. So assume $k>1$. If P is not invertible, then by [6, Lemma 3.6] there is an $r<k$ such that P is equivalent to $Q=\begin{bmatrix} I_r & 0\\ 0 & E\end{bmatrix}$ and E is noninvertible. By induction assumption there is an $M\in\mathcal{Z}(\mathcal{LM}(\mathcal{A},E)^{**})$ such that $\psi(M)\notin\mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}),E)$. So $\widetilde{M}_{ij}\notin\mathcal{Z}(\mathcal{A}^{**})$ for some $i\in I,\ j\in J$. Thus

$$\begin{bmatrix} 0 & 0\\ 0 & \widetilde{M}\end{bmatrix}\notin\mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}),Q). \tag{1}$$

On the other hand $M\in\mathcal{Z}(\mathcal{LM}(\mathcal{A},E)^{**})$ together with relations (1) and (2) in the proof of Lemma 3.2 imply that for every $N\in\mathcal{LM}(\mathcal{A},E)^{**}$ we have $\widetilde{M}\Delta E\Delta\widetilde{N} = \widetilde{M}.E.\widetilde{N}$, where these products are ordinary matrix products when $\mathcal{A}^{**}$ is equipped with the first and the second Arens products respectively. By applying this identity for the special case that $\widetilde{N}$ has only one nonzero column, we conclude that it is true for every finite matrix $\widetilde{N}$ of appropriate size on $\mathcal{A}^{**}$ that the above matrix product is defined. So for every $\begin{bmatrix}\widetilde{N}_1 & \widetilde{N}_2\\ \widetilde{N}_3 & \widetilde{N}_4\end{bmatrix}\in\mathcal{LM}(\mathcal{A}^{**},Q)$,

$$\begin{aligned}\psi^{-1}\left(\begin{bmatrix} 0 & 0\\ 0 & \widetilde{M}\end{bmatrix}\right)\circ\psi^{-1}\left(\begin{bmatrix}\widetilde{N}_1 & \widetilde{N}_2\\ \widetilde{N}_3 & \widetilde{N}_4\end{bmatrix}\right) &= \psi^{-1}\left(\begin{bmatrix} 0 & 0\\ \widetilde{M}\Delta E\Delta\widetilde{N}_3 & \widetilde{M}\Delta E\Delta\widetilde{N}_4\end{bmatrix}\right)\\ &= \psi^{-1}\left(\begin{bmatrix} 0 & 0\\ \widetilde{M}.E.\widetilde{N}_3 & \widetilde{M}.E.\widetilde{N}_4\end{bmatrix}\right)\\ &= \psi^{-1}\left(\begin{bmatrix} 0 & 0\\ 0 & \widetilde{M}\end{bmatrix}\right).\psi^{-1}\left(\begin{bmatrix}\widetilde{N}_1 & \widetilde{N}_2\\ \widetilde{N}_3 & \widetilde{N}_4\end{bmatrix}\right).\end{aligned}$$

Thus

$$\begin{bmatrix} 0 & 0\\ 0 & \widetilde{M}\end{bmatrix}\in\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A},Q)^{**})).$$

Comparing this with relation (1), we conclude that

$$\mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}),Q)\neq\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A},Q)^{**})). \tag{2}$$

On the other hand by using the commutative diagram of Lemma 3.5 and our assumption we get

$$\begin{aligned}\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A},Q)^{**})) &= \theta^{-1}\psi\theta^{**}(\mathcal{Z}(\mathcal{LM}(\mathcal{A},Q)^{**})) = \theta^{-1}\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A},P)^{**}))\\ &= \theta^{-1}(\mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}),P\right)) = \mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}),Q)\end{aligned}$$

which contradicts (2). Therefore P is invertible.

(ii)($\Longleftarrow$) Suppose $\mathcal{LM}(\mathcal{A},P)$ is Arens regular. First assume both I and J are infinite and choose countable infinite subsets $\{i_n\}$ and $\{j_m\}$ from them respectively.

Choose $k \in J,\ l \in I$ such that $P_{kl} \neq 0$ and let V be a closed subspace of $\mathcal{A}$ such that $\mathcal{A} = V \oplus \left\langle P_{kl} \right\rangle$. Suppose $h \in \mathcal{A}^*$ is such that $h(P_{kl}) = 1$ and $h = 0$ on V.

Define $\widetilde{F} :\ I \times J \longrightarrow \mathcal{A}^*$ by $\widetilde{F}_{i_n j_m} = h$ if $n > m$ and $\widetilde{F}_{ij} = 0$ otherwise. Let $X_n = \varepsilon_{i_n k},\ Y_m = \varepsilon_{l j_m} \in \mathcal{LM}(\mathcal{A}, P),\ m, n \in \mathbb{N}$. Then

$$\left\langle F\ ,\ X_n \circ Y_m \right\rangle = \sum_{i,j} \left\langle \widetilde{F}_{ij}\ ,\ (X_n \circ Y_m)_{ij} \right\rangle = \left\langle \widetilde{F}_{i_n j_m}\ ,\ P_{kl} \right\rangle.$$

So $\lim_m \lim_n \left\langle F\ ,\ X_n \circ Y_m \right\rangle = h(P_{kl}) = 1$ and $\lim_n \lim_m \left\langle F\ ,\ X_n \circ Y_m \right\rangle = 0$ which contradicts Arens regularity of $\mathcal{LM}(\mathcal{A}, P)$ by [17, Theorem 1.4.11]. Therefore at least one of the I and J must be finite.

If $\dim \mathcal{A} < \infty$, we have condition (b). If $\mathcal{A}$ is infinite dimensional, then we will show that the other index set is finite too.

We may assume that J is finite but I is infinite. By regularity of P there is at least one row of P that has infinitely many nonzero entries, say row k. Let $\{i_n :\ n \in \mathbb{N}\} \subseteq I$ be such that $P_{k i_n} \neq 0,\ n \in \mathbb{N}$. Choose a sequence $\{a_m :\ m \in \mathbb{N}\}$ in $\mathcal{A}$ such that $\{a_m :\ m \in \mathbb{N}\} \cup \{1\}$ is linearly independent and $\|a_m\| = \frac{1}{2}$. Then by [18, Theorem 10.7] for every $m \in \mathbb{N}$, $b_m = 1 - a_m$ is invertible and $\|b_m^{-1}\| \leq \frac{13}{8}$. Moreover $\{b_m :\ m \in \mathbb{N}\}$ is linearly independent. For every $n > 1$, let $V_n = \left\langle b_1, ..., b_{n-1} \right\rangle$ and W_n be a closed subspace of $\mathcal{A}$ such that $\mathcal{A} = V_n \oplus W_n$. Then $V_1 \subsetneqq V_2 \subsetneqq ...$ and $W_1 \supsetneqq W_2 \supsetneqq ...$.

Now let h be a nonzero multiplicative linear functional on $\mathcal{A}$, $X_n = \varepsilon_{i_n k}$ and $Y_m = (P_{k i_m})^{-1} b_m \varepsilon_{i_m k} \in \mathcal{LM}(\mathcal{A}, P), \quad n, m \in \mathbb{N}$.

Define the map $\widetilde{F} :\ I \times J \longrightarrow \mathcal{A}^*$ in the following way: For every $n > 1$, let $\widetilde{F}_{i_n k} = h$ on V_n, $\widetilde{F}_{i_n k} = 0$ on W_n and $\widetilde{F}_{ij} = 0$ otherwise. Then $F(X_n \circ Y_m) = F(b_m \varepsilon_{i_n k}) = \widetilde{F}_{i_n k}(b_m)$. So $\lim_n \lim_m F(X_n \circ Y_m) = 0$ but since h is multiplicative, we have

$$|\lim_m \lim_n F(X_n \circ Y_m)| = \lim_m |h(b_m)| = \lim_m \frac{1}{|h(b_m^{-1})|} \geq \frac{8}{13}.$$

Again by [17, Theorem 1.4.11] this contradicts Arens regularity of $\mathcal{LM}(\mathcal{A}, P)$. So I is also finite. Now we need only to show that $\mathcal{A}$ is Arens regular. Let $i \in I,\ j \in J$ be such that $P_{ji} \neq 0$. Suppose $m, n \in \mathcal{A}^{**}$, $\widetilde{M} = m\varepsilon_{ij}$ and $\widetilde{N} = (n\Delta(P_{ji})^{-1})\varepsilon_{ij}$. Arens regularity of $\mathcal{LM}(\mathcal{A}, P)$ implies that $\widetilde{N\Delta M} = \widetilde{N.M}$. So by using relations (1) and (2) in the proof of Lemma 3.2 and mixed associativity identities we get

$$n\Delta m = (\widetilde{N\Delta M})_{ij} = (\widetilde{N.M})_{ij} = n.m.$$

($\Longrightarrow$) Suppose condition (a) holds. By Proposition 4.1,

$$\mathcal{LM}(\mathcal{A}, P)^{**} = \psi^{-1}(\mathcal{LM}\left(\mathcal{A}^{**}, P\right)) = \psi^{-1}(\mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right)) \subseteq \mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**}).$$

So in this case $\mathcal{LM}(\mathcal{A}, P)$ is Arens regular.

Now suppose condition (b) holds. We may assume that J is finite. Let $\{X^n : n \in \mathbb{N}\}$ and $\{Y^m : m \in \mathbb{N}\}$ be sequences in $\mathcal{LM}(\mathcal{A}, P)$, bounded by $M \in \mathbb{R}^+$ and $F \in \mathcal{LM}(\mathcal{A}, P)^*$ such that both of the limits $\lim_n \lim_m F(X^n \circ Y^m)$ and $\lim_m \lim_n F(X^n \circ Y^m)$ exist.

Define the sequence $\{g_m : m \in \mathbb{N}\}$ of functions on $\mathbb{N} \times J \times J$ by $g_m(n, j, k) = \sum_{i \in I} \widetilde{F}_{ij} \left(X^n_{ik} \sum_{l \in I} P_{kl} Y^m_{lj} \right)$. Since $\{g_m : m \in \mathbb{N}\}$ is a uniformly bounded sequence of complex functions (with the uniform bound $\|F\|M^2$) on a countable set, then it has a pointwise convergent subsequence, say $\{g_{m_r} : r \in \mathbb{N}\}$.

With a similar discussion we can find a sequence $\{n_s\}$ in $\mathbb{N}$ such that $\{\lim_r \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n_s} \sum_{l\in I} P_{kl} Y_{lj}^{m_r}\right) : s \in \mathbb{N}\}$ is convergent for every $j, k \in J$. Therefore

$$\begin{aligned}\lim_n \lim_m F(X^n \circ Y^m) &= \lim_s \lim_r \sum_{j,k\in J} \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n_s} \sum_{l\in I} P_{kl} Y_{lj}^{m_r}\right)\\ &= \sum_{j,k\in J} \lim_s \lim_r \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n_s} \sum_{l\in I} P_{kl} Y_{lj}^{m_r}\right).\end{aligned}$$

Now by doing the same process on the $\lim_r \lim_s F(X_{n_s} \circ Y_{m_r})$ and denoting the new subsequences by $\{X^n : n \in \mathbb{N}\}$ and $\{Y^m : m \in \mathbb{N}\}$ again, we will get

$$\begin{aligned}\lim_n \lim_m F(X^n \circ Y^m) &= \sum_{j,k\in J} \lim_n \lim_m \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n} \sum_{l\in I} P_{kl} Y_{lj}^{m}\right),\\ \lim_m \lim_n F(X^n \circ Y^m) &= \sum_{j,k\in J} \lim_m \lim_n \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n} \sum_{l\in I} P_{kl} Y_{lj}^{m}\right).\end{aligned} \tag{3}$$

Suppose $\{f_t\}$ is a (finite) basis for $\mathcal{A}^*$ and $\widetilde{F}_{ij} = \sum_{t=1}^q \alpha_{ij}^t f_t, \quad i \in I,\ j \in J$. Since $\mathcal{A}$ is finite dimensional, like the previous step we can pass to subsequences iteratively and rename the subsequences by $\{X^n : n \in \mathbb{N}\}$ and $\{Y^m : m \in \mathbb{N}\}$ again, to get

$$\begin{aligned}\lim_n \lim_m F(X^n \circ Y^m) &= \sum_{j,k\in J} \sum_t \lim_n \lim_m f_t\left(\sum_{i\in I} \alpha_{ij}^t X_{ik}^n \sum_{l\in I} P_{kl} Y_{lj}^m\right),\\ \lim_m \lim_n F(X^n \circ Y^m) &= \sum_{j,k\in J} \sum_t \lim_m \lim_n f_t\left(\sum_{i\in I} \alpha_{ij}^t X_{ik}^n \sum_{l\in I} P_{kl} Y_{lj}^m\right).\end{aligned} \tag{4}$$

Now fix j, k, t and define

$$a_n = \sum_{i\in I} \alpha_{ij}^t X_{ik}^n, \quad b_m = \sum_{l\in I} P_{kl} Y_{lj}^m, \quad m, n \in \mathbb{N}.$$

Then by Arens regularity of $\mathcal{A}$ (as it is finite dimensional), we have

$$\begin{aligned}\lim_n \lim_m f_t\left(\sum_{i\in I} \alpha_{ij}^t X_{ik}^n \sum_{l\in I} P_{kl} Y_{lj}^m\right) &= \lim_n \lim_m f_t(a_n b_m) = \lim_m \lim_n f_t(a_n b_m)\\ &= \lim_m \lim_n f_t\left(\sum_{i\in I} \alpha_{ij}^t X_{ik}^n \sum_{l\in I} P_{kl} Y_{lj}^m\right).\end{aligned}$$

So the right hand sides of the relations (4) are equal and hence by [16, Theorem 1.4.11] $\mathcal{LM}(\mathcal{A}, P)$ is Arens regular. □

5. Young's Theorem for semigroup algebras

Young [19] showed that for a locally compact group G, $L^1(G)$ is Arens regular if and only if G is finite. The following theorem is an extension Young's Theorem to semigroups. Suppose S is a regular semigroup with E_S finite. Then S has a principal series $S = S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} = \varnothing$. Moreover for every

$k = 1, ..., m-1$ there are natural numbers $n_k,\ l_k$, a group G_k and a regular $l_k \times n_k$ matrix P_k on G_k^0 such that $S_k/S_{k+1} = \mathcal{M}^0(G_k, P_k)$. Also $S_m = \mathcal{M}(G_m, P_m)$ for some $l_m \times n_m$ matrix P_m over a group G_m [6, Lemma 5.2]. We use these notations in the following theorem.

Theorem 5.1. *In any of the following two cases if $\ell^1(S)$ is Arens regular, then S is finite:*

(i) *S is a regular semigroup with a finite number of idempotents.*

(ii) *S is an inverse semigroup which admits a principal series.*

Proof. (i) By [6, Proposition 5.6] and [17, Corollary 1.4.12], $\mathcal{LM}\left(\ell^1(G_k), P_k\right)$ is Arens regular. Now Theorem 4.2(ii) implies $\ell^1(G_k)$ is Arens regular, $k = 1, ..., m$. So by Young's Theorem [19], G_k is finite, $k = 1, ..., m$. Therefore each principal factor of S and consequently S itself is finite.

(ii) Using the same argument of part (i) except the fact that the index sets of principal factors can be infinite initially, we conclude that $\mathcal{LM}\left(\ell^1(G_k), P_k\right)$ is Arens regular, $k = 1, ..., m$. Let $1 \leq k \leq m$. By Theorem 4.2(ii) we have two possibilities:

Case 1: $\ell^1(G_k)$ is Arens regular and both index sets are finite. Then by Young's Theorem G_k and consequently S_k/S_{k+1} is finite.

Case 2: $\ell^1(G_k)$ is finite dimensional and one of the index sets is finite. In this case by [5, Theorem 3.9, page 102] both of the index sets are of the same cardinality and hence are finite. So S_k/S_{k+1} is finite.

Therefore in any case we conclude that principal factors of S are finite which implies that S is finite. □

Remark 5.2. In general Young's Theorem [19] is not true for regular semigroups even if they admit a principal series, as we will see in the following example. On the other hand Theorem 5.1 says that it is true for regular semigroups with a finite number of idempotents. These two facts together say up to what extent Young's Theorem can be generalized.

Example 5.3. Let G be a finite group, $|I| = n$ for some $n \in \mathbb{N}$, and let J be an arbitrary infinite index set. Let P be a regular matrix over G^0. Then $S = \mathcal{M}^0(G, P)$ is a regular semigroup with a principal series. By Theorem 4.2(ii) $\mathcal{LM}\left(\ell^1(G), P\right)$ is Arens regular and hence so is $\ell^1(S)/\ell^1(0)$ as it is topologically algebra isomorphic to $\mathcal{LM}\left(\ell^1(G), P\right)$ [6, Proposition 5.6]. Now it is easy to check that $\ell^1(S)/\ell^1(0) \oplus \ell^1(0)$ is Arens regular. But by [6, Lemma 5.1(ii)], this algebra is topologically algebra isomorphic to $\ell^1(S)$.

Therefore $\ell^1(S)$ is Arens regular, but S is infinite. Note that E_S is infinite in this example which shows the conditions of Theorem 5.1 cannot be weakened.

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