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IRREDUCIBLE CONSTITUENTS OF FAITHFUL INDUCED CHARACTERS

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Abstract. Let G be a finite group, and suppose χ is a character of G obtained by inducing an irreducible character of some subgroup of G. If χ is faithful, we show that some irreducible constituent of χ has a solvable kernel. This yields an improved version of a theorem of Evdokimov and Ponomarenko.

1. Introduction

Suppose G is a finite transitive permutation group with permutation character χ, and let d be the maximum of the degrees of the irreducible constituents of χ. Recently, S. A. Evdokimov and I. N. Ponomarenko have shown that the (unique) largest solvable normal subgroup $\mathbf{S}(G)$ of G must have index bounded above by some function of d. (See [2].) Specifically, they showed that $|G:\mathbf{S}(G)| \leq J(d)^{\log_2(d)}$, where $J(d)$ is the function associated with Jordan's theorem on finite complex linear groups. (See Theorem 14.12 of [4].) In other words, $J(d)$ is the smallest integer with the property that every finite subgroup F of $GL(d,\mathbb{C})$ has an abelian normal subgroup A such that $|F:A| \leq J(d)$.

One purpose of this paper is to give an easier proof of a more general result that yields a better bound.

Theorem A. *Let $\psi \in \mathrm{Irr}(H)$, where $H \subseteq G$, and suppose that the induced character $\chi = \psi^G$ is faithful. If d is the maximum of the degrees of the irreducible constituents of χ, then G has a solvable normal subgroup S such that $|G:S| \leq J(d)$.*

Note that if ψ is the principal character of H, then χ is a faithful transitive permutation character, and we are in the situation considered by Evdokimov and Ponomarenko. We see, therefore, that Theorem A generalizes and strengthens the Evdokimov and Ponomarenko theorem, as promised

Theorem A is an easy corollary of the following result, which we consider to be the main theorem of this paper.

Theorem B. *Let $\psi \in \mathrm{Irr}(H)$, where $H \subseteq G$, and suppose that the induced character $\chi = \psi^G$ is faithful. Then $\ker(\theta)$ is solvable for some irreducible constituent θ of χ.*

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Proof of Theorem A. Let θ be an irreducible constituent of χ such that $K = \ker(\theta)$ is solvable. Then G/K is isomorphic to a subgroup of $GL(n, \mathbb{C})$, where $n = \theta(1) \le d$, and hence G/K is isomorphic to a subgroup of $GL(d, \mathbb{C})$. By Jordan's theorem, there exists an abelian subgroup $A/K \lhd G/K$ such that $|G : A| = |G/K : A/K| \le J(d)$. Of course, $A \lhd G$ and A is solvable, and the proof is complete. □

It is possible to improve Theorem B slightly, and to obtain an upper bound on the derived length of $\ker(\theta)$ in terms of the embedding of H in G. This somewhat technical extension of Theorem B is the following.

Theorem C. *In the situation of Theorem B, let $H = H_0 < H_1 < \cdots < H_r = G$ be a saturated chain of subgroups. (In other words, for each subscript i with $0 \le i < r$, assume that H_i is maximal in H_{i+1}.) Then there exists an irreducible constituent θ of χ such that* $\ker(\theta)$ *is solvable of derived length at most equal to the number of indices $|H_{i+1} : H_i|$ that are prime powers.*

As Evdokimov and Ponomarenko pointed out, an important special case of Theorem A is where $H = 1$ and χ is the regular character of G. In that case, of course, every member of $\mathrm{Irr}(G)$ is a constituent of χ, and thus $d = b(G)$, the maximum of the degrees of the irreducible characters of G. While Theorem A only guarantees the existence of a solvable normal subgroup of bounded index in G, an old theorem of D. S. Passman and the author [3] shows that G actually has an *abelian* normal subgroup of bounded index. It is unclear, however, whether or not Theorem A can be improved to yield an abelian normal subgroup of bounded index in the general case. (As an alternative to [3], see Theorem 12.23 of [4] for a proof of the existence of a not necessarily normal abelian subgroup of index at most $(b(G)!.)^2$ This, of course, yields an normal abelian subgroup of index at most $((b(G)!)^2)!.)$

What happens in Theorem B if χ is the regular character? The conclusion is that for any finite group G, there exists $\theta \in \mathrm{Irr}(G)$ such that $\ker(\theta)$ is solvable. But a better result is known. In fact, there exists $\theta \in \mathrm{Irr}(G)$ such that $\ker(\theta)$ is *nilpotent.* (See [1] or Theorem 12.20 of [4] for this reeult of D. Broline.) We have been unable to decide, however, whether or not in the general situation of Theorem B, there must exist an irreducible constituent of χ having a nilpotent kernel.

2. Proofs

We begin with a lemma.

(2.1) Lemma. *Let M be a maximal subgroup of G, and suppose that $\alpha \in \mathrm{Irr}(M)$ is faithful, but that no irreducible constituent of α^G is faithful. Then $|G : M|$ is a prime power and the kernels of all irreducible constituents of α^G are abelian.*

Proof. Let $\beta \in \mathrm{Irr}(G)$ be a constituent of α^G. Write $K = \ker(\beta)$, and note that since α is a constituent of β_M, we have $K \cap M \subseteq \ker(\alpha) = 1$. But $K > 1$ by assumption, and thus $K \not\subseteq M$, and we conclude from the maximality of M that $KM = G$. Also, since K is a normal complement for the maximal subgroup M, it follows that K is minimal normal in G, and thus K is abelian if and only if it is a p-group for some prime p. Furthermore, $|K| = |G : M|$, and thus K is abelian if and only if $|G : M|$ is a prime power.

Assuming, now, that $|G : M|$ is not a prime power, we work to derive a contradiction. Since $G = MK = M\ker(\beta)$, we see that β_M is irreducible. Thus $\beta_M = \alpha$, and we have $[\alpha^G, \beta] = [\alpha, \beta_M] = 1$. Since β was an arbitrary irreducible constituent of α^G, we see that α^G is a sum of distinct extensions of α, and the number of these is $\alpha^G(1)/\alpha(1) = |G : M|$. Also, we know that the kernels of these extensions are normal complements for M in G.

We claim that the kernels of the $|G : M|$ extensions of α to G are distinct. To see why this is so, observe that, since $KM = G$, we have

$$[(\alpha^G)_K, 1_K] = [(\alpha_{K\cap M})^K, 1_K] = [\alpha_{K\cap M}, 1_{K\cap M}] \le \alpha(1) = \beta(1) = [\beta_K, 1_K]\,.$$

It follows that K is not the kernel of any irreducible constituent of α^G other than β.

We now know that M has $|G : M|$ distinct normal complements in G, and each of these is a nonabelian minimal normal subgroup. The product U of these complements is therefore direct, and we see that $|U| = |G : M|^{|G:M|}$ and that $\mathbf{Z}(U) = 1$. Also, we can write $U = K \times C$, where K is a normal complement for M and $C \triangleleft G$ with $|C| = |G : M|^{|G:M|-1}$. Because $|G : M|$ is not a prime power, we have $|G : M| > 2$, and thus $|C| > |G : M|$. It follows that $C \cap M > 1$.

Now K normalizes $C \cap M$ since $K \subseteq \mathbf{C}_G(C)$. Also, $C \cap M \triangleleft M$ because $C \triangleleft G$, and it follows that $C \cap M \triangleleft KM = G$. But each of the normal complements to M intersects $C \cap M$ trivially, and thus the normal subgroup $C \cap M$ centralizes each of them. It follows that $1 < C \cap M \subseteq \mathbf{Z}(U) = 1$, and this contradiction completes the proof. □

Proofs of Theorems B and C. If $H = G$, we see that $\psi = \chi$ is faithful, and thus $1 = \ker(\psi)$ has derived length 0, as desired. We can thus assume that $H < G$, and we work by induction on $|G : H|$. Let $H \subseteq M$, where M is maximal in G, and let r be the number of indices that are prime powers in a saturated chain of subgroups running from H to M. Our task is to find an irreducible constituent θ of ψ^G such that $\ker(\theta)$ is solvable of derived length at most r if $|G : M|$ is not a prime power, and of derived length at most $r + 1$ if $|G : M|$ is a prime power.

Let $\eta = \psi^M$, and write $N = \ker(\eta) \subseteq \ker(\psi) \subseteq H$. By the inductive hypothesis applied in the group M/N, we can choose an irreducible constituent α of η, with $\ker(\alpha) = L$, and such that L/N is solvable with derived length at most r. Thus $L^{(r)} \subseteq N$, and if K is a normal subgroup of G contained in L, then $K^{(r)} \subseteq N \subseteq \ker(\psi)$. Also $K^{(r)} \triangleleft G$, and it follows that $K^{(r)} \subseteq \ker(\psi^G) = \ker(\chi) = 1$. In other words, every normal subgroup of G contained in L is solvable with derived length at most r.

Now let θ be any irreducible constituent of α^G, and write $U = \ker(\theta) \triangleleft G$. Then $U \cap M \subseteq \ker(\theta_M) \subseteq \ker(\alpha) = L$. If $U \subseteq M$, therefore, we have $U \subseteq L$, and since $U \triangleleft G$, we see by the result of the previous paragraph that U is solvable with derived length at most r. There is nothing further to prove in this case, and so we can assume $U \not\subseteq M$, and that this holds for every choice of the irreducible constituent θ of α^G. In particular, since $U \triangleleft G$ and M is maximal in G, we have $UM = G$. But $U = \ker(\theta)$, and it follows that θ_M is irreducible, and thus each of the irreducible constituents of α^G is an extension of α. We can now apply Garrison's lemma, which is Lemma 12.17 of [4], and we deduce that $\mathbf{V}(\alpha) \triangleleft G$. (Recall that, by definition, $\mathbf{V}(\alpha)$ is the subgroup of M generated by all elements $m \in M$ such that $\alpha(m) \neq 0$.) Now

$$U \cap \mathbf{V}(\alpha) \subseteq U \cap M = \ker(\theta_M) = \ker(\alpha) \subseteq U \cap \mathbf{V}(\alpha)\,,$$

and so $L = \ker(\alpha) = U \cap \mathbf{V}(\alpha) \triangleleft G$.

It follows that L is solvable of derived length at most r, and that α is a faithful character of the maximal subgroup M/L of G/L. We know that none of the irreducible constituents θ of α^G has kernel contained in M, and thus when viewed as a character of G/L, none of them is faithful. It follows by Lemma 2.1 that $|G : M|$ is a prime power and that U/L is abelian. This shows that $\ker(\theta) = U$ is solvable of derived length at most $r + 1$, and the proof is complete. $\square$

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