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ISOMETRICAL EMBEDDINGS OF SEPARABLE BANACH SPACES INTO THE SET OF NOWHERE APPROXIMATIVELY DIFFERENTIABLE AND NOWHERE HÖLDER FUNCTIONS

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Abstract. The well-known Banach-Mazur theorem says that every separable Banach space can be isometrically embedded into $C([0,1])$. We prove that this embedding can have the property that the image of each nonzero element is a nowhere approximatively differentiable and nowhere Hölder function. It improves a recent result of L. Rodriguez-Piazza where the images are nowhere differentiable functions.

Introduction

The well-known Banach-Mazur theorem says that every separable Banach space can be isometrically embedded into $C([0,1])$. We prove that this embedding can have the property that the image of each nonzero element is a nowhere approximatively differentiable and nowhere Hölder function. It improves a recent result of L.Rodriguez-Piazza [2] where the images are nowhere differentiable functions (for references about results concerning embeddings into subsets of $C([0,1])$ see [2]). The basic idea of our proof is the same as in [2] but we use a more complicated construction which uses an idea of Malý and Zajíček [1].

Let Δ be the Cantor set. It is well known that every separable Banach space is isometric to a subspace of $C(\Delta)$ so the following theorem will be enough to obtain the announced result.

Theorem 1. *There exist a closed subset K of $[0,1]$ homeomorphic to the Cantor set Δ and a linear operator $F: C(K) \to C([0,1])$ such that for every $f \in C(K)\setminus\{0\}$ we have:*

(i) *$Ff(t) = f(t)$ for every $t \in K$, so Ff is a continuous extension of f to the whole interval.*

(ii) *$\| f \|_\infty = \| Ff \|_\infty$, so Ff is an isometry.*

(iii) *Ff is nowhere approximatively differentiable and nowhere Hölder function.*

In fact we will prove a stronger result:

Proposition 2. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $\varphi(0) = 0$. Then there exist a closed subset K of $[0,1]$ homeomorphic to the Cantor set Δ and a linear operator $F : C(K) \to C([0,1])$ such that for every $f \in C(K) \setminus \{0\}$ we have:*

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(i) *$Ff(t) = f(t)$ for every $t \in K$.*

(ii) $\| f \|_\infty = \| Ff \|_\infty$.

(iii) *For all $n \in \mathbf{N}$ it holds that the set $D_z = \left\{ y \in [0,1] : \left| \frac{Ff(z)-Ff(y)}{\varphi(|z-y|)} \right| > n \right\}$ has the symmetric upper density 1 at z for all $z \in (0,1)$, the set D_0 has the right upper density 1 at point 0 and the set D_1 has the left upper density 1 at point 1.*

Remark. Proposition 2 not only gives us that Ff does not have finite approximate derivative but also that Ff does not have a finite one-sided preponderant derivative as well (for definition of the preponderant derivative see [3, pages 112–113]).

As a by-product of our construction we will obtain:

Proposition 3. *There exists a nonempty perfect set $K \subset [0,1]$ such that every continuous function on K can be extended to $[0,1]$ such that every point $z \in [0,1]$ is a Jarník point of the extension.*

Recall definitions of some notions used above. Suppose that f is measurable on $[0,1]$.

Let $x \in [0,1]$ and $r \in \overline{\mathbf{R}}$. We say that $\text{ap-lim}_{y\to x} f = r$, if for each neighborhood U of r

$$\lim_{h\to 0+} \frac{|\{y \in [x-h, x+h] \cap [0,1] : f(y) \in U\}|}{|[x-h, x+h] \cap [0,1]|} = 1 \ ,$$

where $|M|$ denotes the Lebesgue measure on $\mathbf{R}$. The function f is said to be approximatively differentiable at a point $x \in [0,1]$ if there exists $L \in \mathbf{R}$ such that

$$\text{ap-lim}_{y\to x} \frac{f(y)-f(x)}{y-x} = L \ .$$

A point $x \in [0,1]$ is a Jarník point of f if

$$\text{ap-lim}_{y\to x} \left| \frac{f(y)-f(x)}{y-x} \right| = \infty \ .$$

We say that f on $[0,1]$ is a nowhere Hölder function if for all $x \in [0,1]$ and $\alpha > 0$

$$\sup_{y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = \infty$$

holds.

If we apply Proposition 2 for $\varphi(t) = -\dfrac{1}{\ln t}$ we get that property (iii) of Proposition 2 clearly implies property (iii) of Theorem 1. Thus we devote the rest of the paper to the proofs of Proposition 2 and Proposition 3.

Construction of useful sequences

Let φ be as in Proposition 2. Put $\psi = \sqrt{\varphi}$. We can clearly find a sequence $\{a_n\}_{n=1}^\infty$ such that $1 > a_n > 0$,

$$\sum_{j=n+1}^{\infty} a_j \le \frac{a_n}{2n}, \tag{1}$$

$$\sum_{j=n}^{\infty} a_j \le \frac{1}{2^n}. \tag{2}$$

Further define inductively a sequence $\{p_n\}_{n=1}^{\infty}$ such that $p_n > 0$,

$$p_1 < \frac{1}{10}, \tag{3}$$

$$2\psi((n-1)p_{n-1}) \le \frac{a_n}{n} \text{ for } n = 2, 3, \dots, \tag{4}$$

$$2\pi p_n \sum_{j=1}^{n-1} \frac{a_j}{p_j} \le \frac{a_n}{n} \qquad \text{for } n = 2, 3, \dots, \tag{5}$$

$$a_n\sqrt{n-1}p_{n-1} > 10np_n \text{ for } n = 2, 3, \dots, \tag{6}$$

$$np_n \searrow 0. \tag{7}$$

Put

$$\lambda_n = \sqrt{n}p_n \qquad \text{for } n \in \mathbf{N}. \tag{8}$$

Construction of K

Define $I_{0,1} = [0,1]$ and $\lambda_0 = 1$. For every $n \in \mathbf{N}$ we will choose 2^n pairwise disjoint closed intervals $\{I_{n,j}\}_{j=1}^{2^n}$ in $[0,1]$, $I_{n,j} = [c_{n,j}, d_{n,j}]$, such that $|I_{n,j}| = \lambda_n$. We will also require that

$$I_{n,2j-1} \cup I_{n,2j} \subset \operatorname{Int}(I_{n-1,j}), \tag{9}$$

$$\frac{c_{n,1}}{p_n} \in \mathbf{N},\ \frac{c_{n,j+1} - d_{n,j}}{p_n} \in \mathbf{N} \text{ for all } j \in \{1, ..., 2^n - 1\}, \tag{10}$$

(11)

$$c_{n,2j-1} - c_{n-1,j} \ge \frac{\lambda_{n-1}}{5},\ c_{n,2j} - d_{n,2j-1} \ge \frac{\lambda_{n-1}}{5} \text{ and } d_{n-1,j} - d_{n,2j} \ge \frac{\lambda_{n-1}}{5}.$$

Let us define intervals $I_{n,j}$. Suppose that for a certain $n \in \mathbf{N}$ we have defined all $I_{n-1,j}$. Divide each $I_{n-1,j}$ into five intervals of equal length $\frac{\lambda_{n-1}}{5}$. If we choose the interval $I_{n,2j-1}$ inside the second one, and $I_{n,2j}$ inside the fourth one, then (9) and (11) clearly hold. Since by (6) and (8) $p_n + \lambda_n < \frac{\lambda_{n-1}}{5}$, it is easy to see that we can choose subsequently $I_{n,1}, I_{n,2}, \dots, I_{n,2^n}$ such that moreover $|I_{n,j}| = \lambda_n$ and (10) holds.

Put $K_n = I_{n,1} \cup I_{n,2} \cup \dots \cup I_{n,2^n}$ and $K = \bigcap_{n\ge 1} K_n$. Clearly K is homeomorphic to the Cantor set.

Lemma 1. *Let $n > 2$ and $(z-h, z+h) \subset [0,1]$ be an interval such that $h \le (n-1)p_{n-1}$. Then $(z+h, z-h)$ intersects at most two components of K_n.*

Proof. Thanks to (6) and (8) we have

$$2h \le 2(n-1)p_{n-1} < \frac{1}{5}\sqrt{n-2}p_{n-2} = \frac{\lambda_{n-2}}{5}\ .$$

So by (11) we obtain that $(z-h, z+h)$ intersects at most one component of K_{n-1} and thus at most two components of K_n. □

Construction of T

Construction of T is analogous to the construction in Lemma 2 from Rodriguez-Piazza [2].

Lemma 2. *There exists a linear $T : C(K) \to C([0,1])$ such that for all $f \in C(K)$:*

(a) $Tf(t) = f(t)$ *for all* $t \in K$.

(b) $|Tf(t)| \le \Big(1 - \frac{1}{2^n}\Big)\| f \|_\infty$ *for all* $t \notin K_n$.

(c) *Let* $n > 1$. *If* $I \subset [0,1]$ *is an interval such that* $I \cap K_n = \emptyset$, *then* Tf *is Lipschitz on* I *with the constant* $\Big(\frac{2\| f \|_\infty}{\frac{1}{5}\sqrt{n-1}p_{n-1}}\Big)$.

Proof. For every $n \ge 0$ and every $j \in \{1, \dots, 2^n\}$ pick a point $x_{n,j} \in I_{n,j} \cap K$. We define $Tf(t) = f(t)$ for every $t \in K$. For every $n \ge 0$ and every $j \in \{1, \dots, 2^n\}$, we define

$$Tf(c_{n,j}) = Tf(d_{n,j}) = f(x_{n,j})\Big(1 - \frac{1}{2^n}\Big) \ .$$

Extend Tf affinely on every interval $[a, b]$ where f has been defined in points a, b above and f has not been defined in points of interval (a, b) above. Conditions (a) and (b) are clearly fulfilled. It is easy to see that F is a linear operator and it is obvious that Ff is a continuous function on $[0, 1]$. Now verify (c). Clearly $|f(a) - f(b)| \le 2\| f \|_\infty$ for endpoints a, b of any interval (a, b) on which f has been defined affinely and (11) implies that every such interval which does not intersect K_n has the length at least $\frac{1}{5}\sqrt{n-1}p_{n-1} = \frac{\lambda_{n-1}}{5}$. □

Construction of F

Choose a sequence $\{y_n\}_{n=1}^\infty$ dense in K and define functions f_n:

$$\begin{aligned} f_n(t) &= 0 \text{ for } t \in K_n, \\ f_n(t) &= a_n \sin\Big(\frac{2\pi t}{p_n}\Big) \text{ for } t \in [0, c_{n,1}], \\ f_n(t) &= a_n \sin\Big(\frac{2\pi(t - d_{n,j})}{p_n}\Big) \text{ for } t \in [d_{n,j}, c_{n,j+1}], \\ f_n(t) &= a_n \sin\Big(\frac{2\pi(t - d_{n,2^n})}{p_n}\Big) \text{ for } t \in [d_{n,2^n}, 1] \ . \end{aligned}$$

Notice that

$$f_n(c_{n,1}) = f_n(c_{n,j+1}) = 0$$

by (10). Put

$$Ff(t) = Tf(t) + \sum_{n=1}^{\infty} f(y_n) f_n(t).$$

Thanks to (2) this series converges uniformly. Thanks to (2) and Lemma 2, $F : C(K) \to C([0,1])$ is a linear isometry so conditions (i) and (ii) from Proposition 2 are fulfilled.

PROPERTIES OF F

We will need the following simple fact.

Lemma 3. *Let I be an interval of length $p > 0$, $M \subset I$, $0 < \alpha < 1$ and $\beta \in \mathbf{R}$. For all $x, y \in M$ let*

$$\Big|\sin\Big(\frac{2\pi x}{p} + \beta\Big) - \sin\Big(\frac{2\pi y}{p} + \beta\Big)\Big| \le \alpha.$$

Then $|M| \le \dfrac{3p}{\pi}\arccos(1-\alpha)$.

Proof. Lemma 3 is proved in [1, Lemma 1] in the special case $\beta = 0$. Applying this [1, Lemma 1] to the interval $I^\star = I - \dfrac{p\beta}{2\pi}$ and the set $M^\star = M - \dfrac{p\beta}{2\pi}$ we obtain our lemma. $\square$

Choose $z \in [0,1]$ and denote $\tilde f = F(f)$. We shall prove that, if $f \ne 0$, then for $\tilde f$ condition (iii) from Proposition 2 holds. It is enough to prove that

(12)
the set $S = \{x : |\tilde f(x) - \tilde f(z)| \le \psi(|x - z|)\}$ has symmetric lower density 0 at z for $z \in (0,1)$ and one-sided lower density 0 at points 0 and 1.

Choose arbitrary $1 > \delta > 0$ and put $M = \{n : |f(y_n)| > \delta \| f \|_\infty\}$. If $f \ne 0$, then $\mathrm{card}(M) = \infty$. Choose an arbitrary $0 < h < p_1$. Thanks to (7) there is a unique $n = n(h)$ such that $np_n < h \le (n-1)p_{n-1}$. First prove that

(13)
if $n \in M$ is big enough and $I \subset (z-h, z+h) \setminus K_n$ is an interval of length p_n,
then $\dfrac{|I \cap S|}{|I|} < \arccos\Big(1 - \dfrac{C}{n}\Big)$,

where $C = 4\max\Big(\dfrac{1}{\delta}, \dfrac{1}{\delta\| f \|_\infty}\Big)$. Choose $x, y \in S \cap I$. The definition of S and (4) give

$$|\tilde f(x) - \tilde f(y)| \le |\tilde f(x) - \tilde f(z)| + |\tilde f(y) - \tilde f(z)| \le \psi(|x-z|) + \psi(|y-z|)$$

$$\le 2\psi((n-1)p_{n-1}) \le \frac{a_n}{n}\ .$$

Put $s_n(x) = \sum_{j=1}^{n} f(y_j) f_j(x)$ and $r_n(x) = \sum_{j=n+1}^{\infty} f(y_j) f_j(x)$. From (1) and (4) we have

$$|s_{n-1}(x) - s_{n-1}(y)| \le |x-y| \sup_{t \in (x,y)} |(s'_{n-1})(t)|$$

$$\le 2\pi|x-y|\ \| f \|_\infty \sum_{i=1}^{n-1} \frac{a_i}{p_i} \le 2\pi p_n \| f \|_\infty \sum_{i=1}^{n-1} \frac{a_i}{p_i} \le \frac{a_n}{n} \| f \|_\infty \text{ and}$$

$$|r_n(x) - r_n(y)| \le 2\| f \|_\infty \sum_{j=n+1}^{\infty} a_j \le \frac{a_n}{n} \| f \|_\infty\ .$$

Now from Lemma 2 (c) and (6) we obtain

$$|Tf(x)-Tf(y)|\leq |x-y|\frac{2\|\, f\,\|_\infty}{\frac{1}{5}\sqrt{n-1}p_{n-1}}\leq p_n\frac{2\|\, f\,\|_\infty}{\frac{1}{5}\sqrt{n-1}p_{n-1}}\leq a_n\frac{\|\, f\,\|_\infty}{n}\ .$$

Since $f(y_n)f_n=\tilde{f}-r_n-s_{n-1}-Tf$, we obtain

$$\begin{aligned}\Big|\sin\Big(\frac{2\pi x}{p_n}+\beta\Big)-\sin\Big(\frac{2\pi y}{p_n}+\beta\Big)\Big| &= \frac{1}{a_n|f(y_n)|}|f(y_n)f_n(x)-f(y_n)f_n(y)|\\ &\leq \frac{1}{a_n|f(y_n)|}(|\tilde{f}(x)-\tilde{f}(y)|+|s_{n-1}(x)-s_{n-1}(y)|\\ &\qquad +|r_n(x)-r_n(y)|+|Tf(x)-Tf(y)|)\\ &\leq \frac{1}{\delta\|\, f\,\|_\infty n}+\frac{3\|\, f\,\|_\infty}{\delta\|\, f\,\|_\infty n}\leq\frac{4}{n}\max\Big(\frac{1}{\delta},\frac{1}{\delta\|\, f\,\|_\infty}\Big)=\frac{C}{n}\ .\end{aligned}$$

Thus Lemma 3 gives that

$$|I\cap S|\leq\frac{3p_n}{\pi}\arccos\Big(1-\frac{C}{n}\Big)\quad\text{whenever}\quad\frac{C}{n}<1\ .$$

So (13) really holds for n big enough.

Choose arbitrary $z\in[0,1)$ and $h<1-z$. From Lemma 1 we get that $(z,z+h)$ intersects at most two components of $K_{n(h)}$ and $h>n(h)p_n$ so $\frac{|K_{n(h)}\cap(z,z+h)|}{h}\to 0$. Clearly there exist $Z_h\subset(z,z+h)$, $|Z_h|<3p_n$ and $J_1,J_2\dots J_k\subset(z,z+h)$ pairwise disjoint intervals of length p_n such that

$$(z,z+h)\setminus(K_n\cup Z_h)=\bigcup_{j=1}^{k}J_j\ .$$

Clearly $\frac{|Z_h|}{h}\to 0$ and for J_j we can use (13). Since $\lim_{h\to 0+}n(h)=\infty$, we obtain $\arccos\Big(1-\frac{C}{n(h)}\Big)\to 0$. This all together gives us

$$\lim_{\substack{h\to 0+\\ n(h)\in M}}\frac{|S\cap(z,z+h)|}{h}=0\ .$$

Analogously for $z\in(0,1]$ it holds that

$$\lim_{\substack{h\to 0+\\ n(h)\in M}}\frac{|S\cap(z-h,z)|}{h}=0\ .$$

Since $\operatorname{card}(M)=\infty$, we obtain that (12) holds and we are done.

Proof of Proposition 3. If $f(t)\neq 0$ for all $t\in K$, then there exists $1>\delta>0$ such that $M=\mathbf{N}$. In this case our construction gives us that $\lim_{h\to 0+}\frac{|S\cap(z,z+h)|}{h}=0$ and $\lim_{h\to 0+}\frac{|S\cap(z-h,z)|}{h}=0$. So every point $z\in[0,1]$ is a Jarník point of Ff.

For a given continuous function f on K find $c\in\mathbf{R}$ such that $f(t)+c\neq 0$ for all $t\in K$. Then $F(f+c)-c$ is the desired extension of f. □

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References

1. J. Malý and L. Zajíček, *Approximate differentiation: Jarník points*, Fund. Math. **140** (1991), 87–97. MR **92m:**26006
2. L. Rodriguez-Piazza, *Every separable Banach space is isometric to a space of continuous nowhere differentiable functions*, Proc. Amer. Math. Soc. **123** (1995), 3649–3654. MR **96d:**46007
3. A. Bruckner, *Differentation of Real Functions*, CRM Monograph Series, Volume 5 [2nd edition], Providence, Rhode Island, 1994. MR **94m:**26001

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