PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 12, Pages 3563–3570
S 0002-9939(00)05605-7

UNIQUENESS OF THE TRACE AND SIMPLICITY

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(Communicated by David R. Larson)

Abstract. It is proved that for certain large classes of unital C*-algebras, the existence of a unique, faithful tracial state implies simplicity. An example is given to show that this implication does not hold for all unital C*-algebras.

Many of the unital C*-algebras that provide interesting examples for the theory and for its applications are simple with a unique tracial state—for example, UHF algebras, finite factors, irrational rotation algebras and many more. The existence of a unique tracial state and simplicity are not unrelated conditions, at least in certain circumstances. Of course, simple C*-algebras need not have tracial states, but it is our contention that a unital C*-algebra with a unique, faithful tracial state is "likely" to be simple. More precisely, although a unital C*-algebra with a unique, faithful tracial state is not necessarily simple (we give an example to show this), for large classes of unital C*-algebras that arise in practice the existence of a unique, faithful tracial state *is* sufficient to ensure that the C*-algebra is simple. The unital C*-algebras that we consider include those that are nuclear and have stable rank one, and those that arise as direct limits of C*-algebras of Type I.

The question of when the covariance algebra of a C*-dynamical system is simple and has a unique tracial state is one that is often considered. We apply some of our results in this context.

Our methods require the C*-algebras under consideration to be unital. Moreover, a number of the results we obtain are false in the non-unital case.

We thank the anonymous referee for helpful suggestions that improved the presentation of the paper and for suggesting Corollary 2 to us.

We say that a C*-algebra A has the *quotient tracial state* property (QTS property) if, for each proper closed ideal I of A, the quotient A/I admits a tracial state.

Recall, for the following theorem, that the class of exact C*-algebras includes the nuclear C*-algebras [15, p. 21] and that a unital C*-algebra A is of *stable rank one*, in symbols, $\mathrm{sr}(A) = 1$, if the set $\mathrm{Inv}(A)$, of invertible elements of A, is dense in A.

1. Proposition. *If A is an exact, unital C*-algebra of stable rank one, then A has the QTS property.*

Received by the editors June 29, 1998 and, in revised form, February 2, 1999.
2000 *Mathematics Subject Classification.* Primary 46L05; Secondary 46L55.
Key words and phrases. Faithful tracial state, simple C*-algebra.

Proof. Let I be a proper, closed ideal of A and set $B = A/I$. Since $\mathrm{Inv}(A)$ is dense in A, $\mathrm{Inv}(B)$ is dense in B; that is, $\mathrm{sr}(B) = 1$. Also, B is exact, since A is exact [15, Corollary 9.3]. U. Haagerup [9] has shown, building on a well-known result of D. Handelman [10] on quasitraces, that every stably-finite, exact, unital C*-algebra admits a tracial state. In particular, B admits a tracial state and therefore A has the QTS property. ☐

Recently, E. Bèdos [2] has introduced the concept of hypertraciality for C*-algebras: If A is a C*-algebra, it is *hypertracial* if, for each non-zero, non-degenerate representation (H, π) of A, there is a state τ of $B(H)$ centralised by $\pi(A)$; that is, we have $\tau(\pi(a)T) = \tau(T\pi(a))$ for all $a \in A$ and $T \in B(H)$.

The class of hypertracial C*-algebras is very large, since it includes all Type I C*-algebras and their direct limits [2, pp. 300-301]. An obvious, but important, property that unital, hypertracial C*-algebras enjoy is that they necessarily admit tracial states, a fact that we shall use in the sequel.

2. Corollary. *If A is a nuclear, unital C*-algebra of stable rank one, then A is hypertracial.*

Proof. By Proposition 1, A has the QTS property. It follows from [2, Proposition 3.5] that A is hypertracial. ☐

3. Proposition. *If A is a unital, hypertracial C*-algebra, it has the QTS property.*

Proof. Let I be a proper, closed ideal of A. Then A/I is hypertracial, since A is hyertracial [2, Proposition 3.1]. Hence, since A/I is also unital, it admits a tracial state. Therefore, A has the QTS property. ☐

4. Lemma. *Let A be a unital C*-algebra having the QTS property. Then A is simple if, and only if, all its tracial states are faithful.*

Proof. The forward implication is obvious. Suppose that all the tracial states of A are faithful and let I be a proper, closed ideal of A. By the QST property, A/I admits a tracial state and this gives rise to a tracial state τ on A that vanishes on I. Hence, τ is faithful, by hypothesis. It follows immediately that $I = 0$ and therefore A is simple. ☐

5. Theorem. *Let A be an exact, unital C*-algebra of stable rank one. Then A is simple if, and only if, every tracial state of A is faithful.*

Proof. This follows immediately by combining Proposition 1 and Lemma 4. ☐

6. Corollary. *Let A be an exact, unital C*-algebra of stable rank one that admits a unique, faithful tracial state. Then A is simple.*

Proof. The hypothesis implies that A has only one tracial state (faithful or otherwise). For, if τ is the unique faithful tracial state and σ is any tracial state, then their average $(\sigma + \tau)/2$ is a faithful tracial state and therefore, must be equal to τ. Hence, $\sigma = \tau$. The result now follows from the theorem. ☐

Thus, by the theorem, a unital AF-algebra A is simple if and only if all its tracial states are faithful. By the corollary, a unital AF-algebra with a unique, faithful tracial state is necessarily simple.

The faithfulness condition in the corollary is necessary: If K is the C*-algebra of all compact operators on a separable, infinite-dimensional Hilbert space and A

is the unitisation of K, then A is an AF-algebra and has a unique (non-faithful) tracial state, but A is obviously not simple.

Let $\mathbf{T}$ denote the unit circle in $\mathbf{C}$. Recall that a *circle algebra* is a C*-algebra that is a unital direct limit $A = \lim A_n$, where each C*-algebra A_n is a finite direct sum of matrix algebras $M_m(C(\mathbf{T}))$. Circle algebras have become important in recent years and some classical C*-algebras, such as the irrational rotation algebras and the Bunce–Deddens algebras, have been shown to be circle algebras [7, 8].

7. Corollary. *A circle algebra is simple if it has a unique, faithful tracial state.*

Proof. The algebra $C(\mathbf{T})$ has stable rank one, since the covering dimension of $\mathbf{T}$ is one [13, Proposition 1.7]. Hence, $\mathrm{sr}(M_n(C(\mathbf{T}))) = 1$, by [13, Theorem 3.3]. Clearly, then, any finite direct sum of the algebras $M_n(C(\mathbf{T}))$ has stable rank one, and therefore, by [13, Theorem 5.1], a unital inductive limit of such direct sums has stable rank one. Thus, circle algebras have stable rank one. Clearly, such algebras are also nuclear and, therefore, exact. The result now follows from Corollary 6. ☐

8. Theorem. *Let A be a unital, hypertracial C*-algebra. Then A is simple if and only if all of its tracial states are faithful.*

Proof. This is immediate from Proposition 3 combined with Lemma 4. ☐

9. Corollary. *Let (A, G) be a C*-dynamical system, where A is a hypertracial, unital C*-algebra and G is an amenable, discrete group. Then the covariance algebra $C^*(A, G)$ is simple if and only if all its tracial states are faithful.*

Proof. By [2, Proposition 3.7], $C^*(A, G)$ is hypertracial. ☐

10. Corollary. *Let (A, G) be a C*-dynamical system, where A is a hypertracial, unital C*-algebra and G is an amenable, discrete group. Suppose that invariant tracial states of A are necessarily faithful. Then the covariance algebra $C^*(A, G)$ is simple if it has a unique tracial state (equivalently, if it has at most one tracial state).*

Proof. By [2, Proposition 3.7], $C = C^*(A, G)$ is hypertracial. It therefore admits a tracial state; thus, it has a unique tracial state if and only if it has at most one tracial state. Suppose then C has a unique tracial state τ. Let E be the canonical positive linear map from C onto A. Since the restriction τ_A of τ to A is necessarily an invariant tracial state of A, it is faithful, by hypothesis. Since E is also faithful (a non-trivial consequence of the fact that G is amenable [14, pp. 4–5]), the composition $\sigma = \tau_A \circ E$ is a faithful state of C. From the invariance property that τ_A enjoys, it follows that σ is a tracial state and therefore, by the unique-tracial-state assumption, $\sigma = \tau$. Hence, the unique tracial state of C is faithful. Therefore, by the theorem, C is simple. ☐

11. Corollary. *Let (A, G) be a C*-dynamical system, where A is an abelian, unital C*-algebra and G a discrete, abelian group. Suppose that invariant states of A are necessarily faithful. Then $C^*(A, G)$ is simple if it admits a unique tracial state (equivalently, if it admits at most one tracial state).*

Proof. Since A is abelian, it is hypertracial. Also, an abelian group is necessarily amenable. Hence, the result follows from Corollary 10. ☐

There are many cases in practice where we want to show that a C*-algebra is simple and has a unique tracial state. Corollary 11 can sometimes be used in such cases to give an efficient proof. For instance, consider the irrational rotation algebra A_θ. Thus, A_θ is a unital C*-algebra generated by a pair of unitaries u and v for which $vu = \lambda uv$, where $\lambda = \exp(i2\pi\theta)$ and θ is irrational. As is well known, A_θ is simple and has a unique tracial state and we shall show this now using Corollary 11. To do it, let us take a particular realisation of A_θ, namely $A_\theta = C^*(A, \mathbf{Z})$, where $A = C(\mathbf{T})$ and the action of $\mathbf{Z}$ on A is the one associated to the unique automorphism α on A that maps the canonical unitary generator z of A onto λz. If $u = z$ and v is the canonical unitary in $C^*(A, \mathbf{Z})$ for which $\alpha(z) = vzv^*$, then $vu = \lambda uv$ and u and v generate A_θ. In fact, A_θ is the closed linear span of the elements $u^m v^n$, where $m, n \in \mathbf{Z}$. Moreover, by the universal properties of the covariance algebra, if B is any other unital C*-algebra generated by a pair of unitaries U and V such that $VU = \lambda UV$, then there is a unital $*$-homomorphism ϕ from A_θ onto B such that $\phi(u) = U$ and $\phi(v) = V$. Therefore, if we show A_θ is simple, it will follow that A_θ and B are canonically isomorphic.

Thus, we want to show that A_θ has a unique tracial state and is simple. By Corollary 11, we need only show that A_θ has at most one tracial state and any invariant state of A is faithful. Consider the latter condition first. A state on A corresponds to integration against a probability measure on the unit circle, and if the state is invariant, the measure must be equal to normalised Haar measure on $\mathbf{T}$ (using the irrationality of θ). Since the support of Haar measure is equal to $\mathbf{T}$, the corresponding state is clearly faithful. Now suppose that τ is a tracial state of A_θ. To see there can be at most one such tracial state, we need only show that necessarily $\tau(v^m u^n) = 0$, unless $m = n = 0$. However, this is a consequence of the easily-verified fact that $v^m u^n = \lambda^{mn} u^n v^m$ and the tracial condition.

Recall that if (A, G) is a C*-dynamical system, the action of G on A is said to be *effective* if the automorphism of A corresponding to an element g of G is not the identity unless $g = e$, where e is the unit of G.

Many of the ideas in the proof of the following theorem are standard. Nevertheless, the hypotheses of the theorem do not follow along standard lines, since we make no explicit "minimality" assumption on the character space of the abelian C*-algebra. The result is known (see [3, Corollary 6] and [11, Corollary 2.10]).

12. Theorem. *Let (A, G) be a C*-dynamical system, where A is an abelian, unital C*-algebra and G is a discrete, abelian group that acts effectively on A. Suppose that A admits a unique invariant state τ and suppose, moreover, that τ is faithful. Then $C^*(A, G)$ is simple and admits a unique tracial state.*

Proof. In light of Corollary 11, we need only show that $C = C^*(A, G)$ has at most one tracial state. To see this, we need only show that if σ is a tracial state of C, then $\sigma = \sigma \circ E$, where E is the canonical positive linear map from C onto A (for then $\sigma(c) = \tau(E(c))$ for all $c \in C$). (We have $\sigma_A = \tau$, since σ_A is an invariant state of A.) Since $E(aU_g) = 0$ for all $g \in G \setminus \{e\}$ and for all $a \in A$, where $U \colon g \mapsto U_g$ is the canonical unitary representation of G in C, to see that $\sigma = \sigma \circ E$ we need only show that $\sigma(aU_g) = 0$ for all $g \neq e$.

Define $\sigma_g \in A^*$ by setting $\sigma_g(a) = \sigma(aU_g)$. Then, if $h \in G$, $\sigma_g(U_h a U_h^*) = \sigma(U_h a U_h^* U_g) = \sigma(a U_h^* U_g U_h) = \sigma(aU_g) = \sigma_g(a)$, since σ is a tracial state and G is abelian. Thus, σ_g is an invariant bounded linear functional. However, by the Jordan decomposition, σ_g is a linear combination of states, and it is easily verified that each of these is invariant. Hence, by the uniqueness assumption for τ, we have $\sigma_g = \lambda_g \tau$ for some complex number λ_g. We shall show that $\lambda_g = 0$ if $g \neq e$.

Before proceeding, let us denote the character space of A by Ω. As is well known, the action $\alpha\colon g \mapsto \alpha_g$ of G on A induces an action of G on Ω given by $\alpha_g(a)(\omega) = a(g^{-1}\omega)$ for all $a \in A$ and $\omega \in \Omega$. Here we are regarding A as the algebra of continuous functions on Ω. Since the action α is effective, if $g \neq e$, then $g\omega \neq \omega$ for some point ω in Ω. Hence, there exists an open set V in Ω containing ω such that $V \cap gV = \emptyset$. Now choose a non-zero, positive element a in A having support in V. Since $V \cap gV = \emptyset$, we have $a\alpha_g(a) = 0$. Hence,

$$\sigma_g(a^2) = \sigma(a^2U_g) = \sigma(aU_g a) = \sigma(a\alpha_g(a)U_g) = 0;$$

that is, $\lambda_g \tau(a^2) = 0$. Since $\tau(a^2) \neq 0$, because τ is faithful, we must have $\lambda_g = 0$, as required. $\square$

The preceding theorem can be applied to a C*-dynamical system (A, G) in a context where we have not explicitly calculated the character space Ω of the abelian algebra A, and do not check for minimality of Ω under the induced action of the group G. This situation is illustrated in the following examples.

If $s \in \mathbf{R}$, let e_s be the function on $\mathbf{R}$ defined by $e_s(t) = e^{ist}$. Recall that an *almost-periodic function* on $\mathbf{R}$ is an element of the closed linear span in $C_{\mathrm{b}}(\mathbf{R})$ of the functions e_s $(s \in \mathbf{R})$. Clearly, the set $AP(\mathbf{R})$ of almost periodic functions is a C*-subalgebra of $C_{\mathrm{b}}(\mathbf{R})$. If $f \in AP(\mathbf{R})$, then the limit $\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T} f(t)\,dt$ exists and is called the *mean value* of f. If the mean value of $\bar{f}f$ is equal to zero, then $f = 0$ (see [4]).

Suppose now G is a dense subroup of $\mathbf{R}$. The closed linear span A in $AP(\mathbf{R})$ of the functions e_s $(s \in G)$ is clearly a unital C*-algebra. We define a faithful state τ on A by letting $\tau(f)$ be the mean value of f. Clearly, τ is the unique state of A for which $\tau(e_s) = 0$ whenever $s \in G$ and $s \neq 0$.

Consider the effective action α of G on A for which $\alpha_s(e_t) = e^{ist}e_t$ for all $s, t \in G$. Since $\tau\alpha_s(e_t) = e^{ist}\tau(e_t) = 0$ if $t \neq 0$, the state $\tau\alpha_s$ must be the same as τ. Hence, τ is invariant. Moreover, τ is the only invariant state of A. For, if σ is an invariant state, and $s, t \in G$, then $\sigma(e_t) = \sigma\alpha_s(e_t) = e^{ist}\sigma(e_t)$. Hence, if $\sigma(e_t) \neq 0$, then $e^{ist} = 1$ for all $s \in G$. By density of G in $\mathbf{R}$, this implies that $e^{ist} = 1$ for all $s \in \mathbf{R}$. Therefore, $t = 0$. This shows that $\sigma = \tau$, as required.

Summarising, we have shown that if G is endowed with the discrete topology, then (A, G) is a C*-dynamical system that satisfies the hypotheses of Theorem 12. Consequently, $C^*(A, G)$ is a simple C*-algebra with a unique tracial state.

We present another example based on recent work of W. Arveson. This involves almost-periodic sequences, for which a convenient reference is [12]. Let α be the action of $\mathbf{Z}$ on the C*-algebra $\ell^\infty(\mathbf{Z})$ defined by translation. The set $AP(\mathbf{Z})$ of all almost-periodic sequences is the closed linear span in $\ell^\infty(\mathbf{Z})$ of the group characters of $\mathbf{Z}$. It is a C*-subalgebra of $\ell^\infty(\mathbf{Z})$ that is invariant under the action α. This algebra admits an invariant state defined by $\tau(a) = \lim_{n\to\infty}(\sum_{k=-n}^{n} a_k)/(2n+1)$. Moreover, $\tau(a)$ is the unique scalar in the closed convex hull of the translates of a. Because every non-zero, closed, translation-invariant linear subspace of $AP(\mathbf{Z})$ is the closed linear span of characters of the group $\mathbf{Z}$, the left kernel N_τ of τ must be equal to the zero space (since it cannot contain a character); that is, τ is faithful.

With these preliminaries stated, we can now prove the following result of Arveson [1].

13. Theorem. *Let $a = (a_n)_{n\in\mathbf{Z}}$ be a bounded real sequence that is almost periodic, but not periodic, and let M_a be the corresponding multiplication operator on $\ell^2(\mathbf{Z})$. Let U be the forward bilateral shift on the standard orthonormal basis of $\ell^2(\mathbf{Z})$. Then the C*-algebra $C^*(M_a, U)$ generated by M_a and U is simple and has a unique tracial state.*

Proof. Clearly $U^n M_a U^{n*} = M_{\alpha_n(a)}$. Hence, if A is the translation-invariant unital C*-subalgebra of $AP(\mathbf{Z})$ generated by a, then $C^*(M_a, U)$ is the image, under an obvious $*$-homomorphism, of the covariance algebra $C^*(A, \mathbf{Z})$, where the action of $\mathbf{Z}$ on A is (the restriction of) α. Thus, we need only show that $C^*(A, \mathbf{Z})$ is simple and admits a unique tracial state. Hence, by Theorem 12, and the fact that τ_A is faithful, we need only show that α is effective and that τ_A is the unique invariant state of A. The first condition follows immediately from the assumption that a is not periodic. The second is a consequence of the fact that $\tau(b)$ is the only scalar in the closed convex hull of the set of translates of an element b of A. □

All the elements of the preceding proof are to be found in Arveson's proof in [1]. However, the proof given here is much shorter and more elementary and illustrates again the convenience of Theorem 12; in contrast, the proof given in [1] proceeds along standard lines to first invoke the theory of the Connes spectrum to prove simplicity, and then goes on to show separately that a unique tracial state exists.

Arveson has given an application of Theorem 13 to the problem of calculating the spectra of discretised Hamiltonians (*loc. cit.*).

It is not true in general that the existence of a unique, faithful tracial state on a unital C*-algebra implies simplicity, as we shall now show.

14. Theorem. *Let I be a non-unital C*-algebra admitting a unique, faithful tracial state. Let A be the multiplier algebra of I and suppose that the quotient C*-algebra A/I admits no tracial states. Then A is a unital C*-algebra with a unique, faithful tracial state and A is (obviously) not simple.*

Proof. Let τ_I be the unique, faithful tracial state on I and let τ_A be its unique extension to a state on A. It is well known and easily checked that τ_A must also be a tracial state. Moreover, τ_A is faithful. For, if $a \in A$ and $\tau_A(a^*a) = 0$, then for all $b \in I$, we have $\tau_I((ba)^*ba) = 0$ and therefore, $ba = 0$, since τ_I is faithful. Hence, $Ia = 0$ and therefore $a = 0$, since I is an essential ideal of A.

Since the quotient algebra A/I admits no tracial states, the closed linear span of the additive commutators in A/I is equal to A/I, by a theorem of J. Cuntz and G.K. Pedersen [5]. Hence, if $[A, A]$ denotes the linear span of the additive commutators in A, we have $(I + [A, A])^- = A$. Now suppose that σ is any tracial state on A. Then $\sigma = c\tau_A$ on I, for some non-negative number c, since I has unique, faithful tracial state τ_I. Moreover, $\sigma = c\tau_A = 0$ on $[A, A]$. Hence, $\sigma = c\tau_A$. Therefore, equating norms, we get $c = 1$. Hence, $\sigma = \tau_A$. This proves the theorem. □

The hypotheses of the preceding theorem applies if I is a non-unital, separable, matroid C*-algebra with a unique tracial state. For, such an algebra is necessarily simple, implying the tracial state is faithful. Also, if A is the multiplier algebra of I, then A/I is infinite and simple [6]. Hence, A/I admits no tracial states.

We give an example now of a non-unital, separable, matroid C*-algebra with a unique tracial state. This example was shown to me by M. Rørdam. I would like to thank him for this and also for an example—not included here—of a non-simple, unital C*-algebra with a unique, faithful tracial state. Rørdam's example of the latter was a little more complicated than that given here, but the idea of its construction inspired the example given here.

Let J be the C*-tensor product $B \otimes K$, where B is an infinite-dimensional UHF algebra and K is the C*-algebra of all compact operators on a separable, infinite-dimensional, Hilbert space H with an orthonormal basis $(e_n)_{n=1}^{\infty}$. Let e_{mn} be the operator on H that maps a vector x onto $(x \mid e_n)e_m$. If a is an element of J, then we may write $a = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \otimes e_{mn}$ for unique elements a_{mn} in B.

Since B is an infinite-dimensional UHF algebra, we may choose a sequence $(p_n)_{n=1}^{\infty}$ of non-zero projections of B for which $\sum_{n=1}^{\infty} \operatorname{tr}(p_n) < \infty$ and $p_1 = 1$, where tr is the unique tracial state of B. Set $q_m = \sum_{i=1}^{m} p_i \otimes e_{ii}$ and set $I = (\bigcup_{m=1}^{\infty} q_m J q_m)^{-}$. Clearly, I is a separable, matroid C*-algebra. If I were unital, then the approximate unit (q_m) of I would converge to the unit. Hence, the sequence of non-zero projections (p_m) would converge to zero. Since this is clearly impossible, I cannot be unital. A tracial state on I is obtained by rescaling the positive linear functional defined by setting $\tau(a) = \sum_{m=1}^{\infty} \operatorname{tr}(p_m a_{mm} p_m)$ for $a \in I$. Using the uniqueness of the tracial state on B, it is straightforward to verify that I has no other tracial state.

To sum up, we have shown that I is a non-unital, separable, matroid C*-algebra with a unique tracial state, as required.

Our example of a non-simple, unital C*-algebra with a unique, faithful tracial state is not separable. We do not know of a separable example.

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