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# COMPOSITION OPERATORS ON DIRICHLET-TYPE SPACES

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Abstract. The Dirichlet-type space $D^p$ $(1 \leq p \leq 2)$ is the Banach space of functions analytic in the unit disc with derivatives belonging to the Bergman space $A^p$. Let $\Phi$ be an analytic self-map of the disc and define $C_\Phi(f) = f \circ \Phi$ for $f \in D^p$. The operator $C_\Phi : D^p \to D^p$ is bounded (respectively, compact) if and only if a related measure $\mu_p$ is Carleson (respectively, compact Carleson). If $C_\Phi$ is bounded (or compact) on $D^p$, then the same behavior holds on $D^q$ $(1 \leq q < p)$ and on the weighted Dirichlet space $D_{2-p}$. Compactness on $D^p$ implies that $C_\Phi$ is compact on the Hardy spaces and the angular derivative exists nowhere on the unit circle. Conditions are given which, together with the angular derivative condition, imply compactness on the space $D^p$. Inner functions which induce bounded composition operators on $D^p$ are discussed briefly.

## Introduction

Let $D = \{z : |z| < 1\}$ and let $A$ denote normalized area measure on $D$. The Bergman space $A^p$ $(1 \leq p < \infty)$ is the Banach space of functions $f$ analytic in the disc $D$ such that $\| f \|_{A^p}^p = \int_D | f(z) |^p \, dA(z) < \infty$. The Dirichlet-type space $D^p$ is the set of functions analytic in $D$ with derivatives belonging to $A^p$. The set $D^p$ is a Banach space, with norm given by

$$\| f \|_{D^p} = | f(0) | + \| f' \|_{A^p} . \tag{1}$$

Note that point evaluation is continuous on $D^p$ and $D^p \subset D^q$ if $1 \leq q < p$.

Throughout this paper, $\Phi$ denotes an analytic self-map of $D$. The composition operator $C_\Phi$ is defined by $C_\Phi(f) = f \circ \Phi$ for $f \in D^p$. If $C_\Phi(f) \in D^p$ for every $f \in D^p$, then $C_\Phi$ is bounded, by the Closed Graph Theorem.

Interest in the spaces $D^p$ is motivated by the work of R. Roan [19] and B. D. MacCluer [14], who studied composition operators on $S^p$, the space of functions with derivatives in the Hardy space $H^p$ for $p \geq 1$. Other related work appears in [15], where MacCluer and J. H. Shapiro studied composition operators on the weighted Dirichlet spaces $D_\alpha$.

## Section 1

Let $\mu$ be a finite positive Borel measure on $D$. For $|\zeta| = 1$ and $0 < \delta \leq 2$, $S(\zeta, \delta)$ is the Carleson set $\{z \in D : |z - \zeta| < \delta\}$. The measure $\mu$ is said to be

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Carleson if there is a constant $C$ such that $\mu(\ S(\zeta,\delta)\ )\ \leq C\delta^2$ for all $\zeta$ and $\delta$. The measure is said to be compact Carleson if

$$\lim_{\delta\to 0}\sup_{|\zeta|\ =\ 1}\frac{\mu(S(\zeta,\delta))}{\delta^2}\ =\ 0.$$

Carleson measures have been useful in the study of composition operators in several settings [11, 13, 14, 15, 27, 28].

For $w\in D$, let $N_2(\Phi,w)$ denote the number of zeroes (counting multiplicities) of the equation $\Phi(z)-w$. For $1\leq p<2$ and $w\in D$, $N_p(\Phi,w)$ is defined to be the modified counting function

$$N_p(\Phi,w)\ =\ \sum\frac{1}{|\ \Phi'(z)\ |^{2-p}}$$

where the sum extends over the zeroes of $\Phi-w$, repeated by multiplicity. In particular, $N_p(\Phi,w)=0$ for $w\notin\Phi(D)$. Let $\mu_p$ be the measure defined on $D$ by $d\mu_p(w)\ =\ N_p(\Phi,w)\ dA(w)$, $1\leq p\leq 2$. Note that $\mu_p$ is a finite measure if and only if $\Phi\in D^p$.

The proofs of the first two theorems are standard. A sketch of the proofs is provided in Section 3.

**Theorem 1.** *The operator $C_\Phi$ is bounded on $D^p$ if and only if $\mu_p$ is a Carleson measure.*

**Theorem 2.** *The operator $C_\Phi$ is compact on $D^p$ if and only if the measure $\mu_p$ is compact Carleson.*

The case $p=2$ in Theorems 1 and 2 was first proven by MacCluer and Shapiro in their study of the weighted Dirichlet spaces $D_\alpha$ [15]. An analytic function $f$ belongs to $D_\alpha(\alpha>-1)$ if $\int_D\ |\ f'(z)\ |^2\ (1-\ |\ z\ |^2)^\alpha\ dA(z)<\infty$. Note that $D_0=D^2$, the classical Dirichlet space. For $p=2$ the norm (1) is equivalent to the norm used in [15] for $D_0$.

Throughout the remainder of this work $C$ will denote a positive constant, the exact value of which will vary from one appearance to the next.

**Theorem 3.** *If $C_\Phi$ is bounded (respectively, compact) on $D^p$ and $1\leq q<p$, then $C_\Phi$ is bounded (respectively, compact) on $D^q$.*

*Proof.* The hypotheses imply that $\Phi\in D^q$ for $1\leq q\leq p$. Thus $\mu_q$ is a finite measure on the disc. Since $q<p$, Hölder's Inequality implies that

$$\mu_q(S(\zeta,\delta))\ =\ \int_{\Phi^{-1}(S(\zeta,\delta))}|\ \Phi'\ |^q\ \ dA$$

$$\leq\ (\int_{\Phi^{-1}(S(\zeta,\delta))}|\ \Phi'\ |^p\ \ dA)^{q/p}\ (\int_{\Phi^{-1}(S(\zeta,\delta))}1\ dA)^{(p-q)/p}$$

$$=\mu_p(S(\zeta,\delta))^{q/p}\ A\Phi^{-1}(S(\zeta,\delta))^{(p-q)/p}. \tag{2}$$

Since $C_\Phi$ is bounded on the Bergman spaces, Theorem 4.3 [15] implies that the measure $A\Phi^{-1}$ is Carleson. Thus (2) yields

$$\mu_q(S(\zeta,\delta))\ \leq C\ \delta^{2(p-q)/p}\ \mu_p(S(\zeta,\delta))^{q/p}. \tag{3}$$

If $C_\Phi$ is bounded on $D^p$, Theorem 1 implies that there is a constant $C$ such that $\mu_p(S(\zeta,\delta))\ \leq\ C\delta^2$ for all $\zeta$ and $\delta$ as described above. Thus (3) shows that $\mu_q(S(\zeta,\delta))\ \leq C\delta^2$. By a second application of Theorem 1, $C_\Phi$ is bounded on $D^q$.

If $C_\Phi$ is compact on $D^p$, then $\mu_p(S(\zeta,\delta))/\delta^2 \to 0$ uniformly on $\{\zeta : \mid \zeta \mid = 1\}$ as $\delta \to 0$. Relation (3) shows that $\mu_q$ is a compact Carleson measure. By Theorem 2, $C_\Phi$ is compact on $D^q$. □

The following corollary is an immediate consequence of Theorems 2 and 3.

**Corollary 1.** 1. *If $\Phi$ is an analytic self-map of $D$ with bounded multiplicity, then $C_\Phi$ is bounded on $D^p$ for $1 \leq p \leq 2$.*
2. *If $\Phi \in D^p$ and $\parallel \Phi \parallel_\infty < 1$, then $C_\Phi$ is compact on $D^q$ for $1 \leq q \leq p$.*

In [15, p. 892] MacCluer and Shapiro defined a measure $\nu_\alpha$ on the disc by $d\nu_\alpha(z) = \mid \Phi'(z) \mid^2 (1- \mid z \mid^2)^\alpha dA(z)$ $(\alpha > -1)$. The operator $C_\Phi$ is bounded (respectively, compact) on the weighted Dirichlet space $D_\alpha$ if and only if the measure $\nu_\alpha \Phi^{-1}$ is $\alpha$-Carleson (respectively, compact $\alpha$-Carleson). Details can be found in [15].

The space $D_\alpha$ is said to be heavily weighted if $-1 < \alpha < 0$. If $C_\Phi$ is bounded (respectively, compact) on a heavily weighted space, then $C_\Phi$ is bounded (compact) on $D_0 = D^2$ [15]. Theorem 3 now implies that $C_\Phi$ is bounded (compact) on $D^p$ for $1 \leq p < 2$. Shapiro has shown that compactness on $D_\alpha$ for $-1 < \alpha < 0$ is equivalent to the conditions $\Phi \in D_\alpha$ and $\parallel \Phi \parallel_\infty < 1$ [21].

**Theorem 4.** *For $1 \leq p \leq 2$, let $\alpha = 2-p$. If $C_\Phi$ is bounded (respectively, compact) on $D^p$, then $C_\Phi$ is bounded (compact) on the weighted Dirichlet space $D_\alpha$.*

*Proof.* It is enough to consider $1 \leq p < 2$.

Suppose that $C_\Phi$ is bounded on $D^p$ and let $\alpha = 2-p$. Then

$$\nu_\alpha(D) = \int_D \mid \Phi'(z) \mid^p \mid \Phi'(z) \mid^\alpha (1- \mid z \mid^2)^\alpha \, dA(z). \tag{4}$$

Since $\Phi$ is an analytic self-map of the unit disc, the Schwarz-Pick Lemma guarantees that

$$\frac{1- \mid z \mid^2}{1- \mid \Phi(z) \mid^2} \mid \Phi'(z) \mid \leq 1.$$

Thus (4) implies that

$$\nu_\alpha(D) \leq \int_D \mid \Phi'(z) \mid^p (1- \mid \Phi(z) \mid^2)^\alpha \, dA(z) \leq \int_D \mid \Phi' \mid^p \, dA.$$

Since $\Phi \in D^p$, it follows that $\nu_\alpha$ is a finite measure on $D$.

It remains to prove that $\nu_\alpha \Phi^{-1}(S(\zeta,\delta)) \leq C\delta^{2+\alpha}$ for all $\mid \zeta \mid = 1$ and $0 < \delta \leq 2$. By the argument given in the previous paragraph,

$$\nu_\alpha \Phi^{-1}(S(\zeta,\delta)) \leq \int_{\Phi^{-1}(S(\zeta,\delta))} \mid \Phi'(z) \mid^p (1- \mid \Phi(z) \mid^2)^\alpha \, dA(z).$$

For $z \in \Phi^{-1}(S(\zeta,\delta))$, $1- \mid \Phi(z) \mid < \delta$. Thus

$$\nu_\alpha \Phi^{-1}(S(\zeta,\delta)) \leq (2\delta)^\alpha \int_{\Phi^{-1}(S(\zeta,\delta))} \mid \Phi' \mid^p \, dA = (2\delta)^\alpha \mu_p(S(\zeta,\delta)).$$

Since $C_\Phi$ is bounded on $D^p$, Theorem 1 implies that $\mu_p(S(\zeta,\delta)) < C\delta^2$. It follows that the measure $\nu_\alpha \Phi^{-1}$ is $\alpha$-Carleson, as required.

The assertion about compactness is proved in a similar way. □

The following theorem was first suggested by Shapiro [23].

**Theorem 5.** *If $C_\Phi$ is compact on $D^p$ for some $1 \le p \le 2$, then $C_\Phi$ is compact on the Hardy spaces.*

*Proof.* Since $C_\Phi$ is compact on $D^p$, Theorem 3 implies that $C_\Phi$ is compact on $D^1$. Theorem 4 now implies that $C_\Phi$ is compact on the weighted Dirichlet space $D_1 = H^2$. As noted in [25], compactness on $H^2$ is equivalent to compactness on $H^q$ for $0 < q < \infty$. □

**Corollary 2.**
1. *If $\Phi$ is an inner function, then $C_\Phi$ is not compact on $D^p$ for any $1 \le p \le 2$.*
2. *If $C_\Phi$ is compact on $D^p$, then the angular derivative exists nowhere on the circle.*
3. *If $C_\Phi$ is compact on $D^p$, then $C_\Phi$ is compact on $A^q_\alpha$ for $0 < q < \infty$ and $\alpha > -1$.*
4. *Suppose that $\Phi'$ is bounded. The operator $C_\Phi$ is compact on $D^p$ for all $1 \le p \le 2$ if and only if $\| \Phi \|_\infty < 1$.*

*Proof.* The first three assertions follow from Theorem 5 and the well-known results about compactness on the Hardy spaces and the weighted Bergman spaces [15].

Suppose that $\Phi'$ is bounded and $\| \Phi \|_\infty = 1$. It follows that the angular derivative is finite at some point on the circle, and thus $C_\Phi$ is not compact on $D^p$.

The remaining implication follows from Corollary 1. □

There exist inner functions $\Phi$ with no finite angular derivative at any point on the circle. Thus the converse of the second assertion in the corollary is false.

Shapiro [21] has shown that there are functions $\Phi$ with $\| \Phi \|_\infty = 1$ such that $C_\Phi$ is compact on $D^2$, the Dirichlet space. Theorem 3 implies that $C_\Phi$ is compact on $D^p$ for $1 \le p \le 2$. Thus the condition $\| \Phi \|_\infty < 1$ is not necessary for $C_\Phi$ to be compact on the Dirichlet-type spaces $D^p$. In comparison, MacCluer proved that $C_\Phi$ is compact on $S^p$ $(1 \le p < \infty)$ if and only if $\Phi \in S^p$ and $\| \Phi \|_\infty < 1$ [14].

**Theorem 6.** *Suppose $\| \Phi \|_\infty = 1$. If $C_\Phi$ is bounded on $D^p$ and the angular derivative of $\Phi$ exists at no point of the circle, then $C_\Phi$ is compact on $D^q$ for $1 \le q < p$.*

*Proof.* As in the proof of Theorem 3, the hypotheses imply that $\mu_q$ is a finite measure and

$$\frac{\mu_q(S(\zeta,\delta))}{\delta^2} \le \left(\frac{\mu_p(S(\zeta,\delta))}{\delta^2}\right)^{q/p} \left(\frac{A\Phi^{-1}(S(\zeta,\delta))}{\delta^2}\right)^{(p-q)/p}. \tag{5}$$

Since the angular derivative of $\Phi$ exists nowhere on the unit circle, the measure $A\Phi^{-1}$ is compact Carleson [15]. Thus the second expression on the right in (5) tends to 0 uniformly on $\{\zeta : |\zeta| = 1\}$ as $\delta \to 0$. Because $C_\Phi$ is bounded on $D^p$, the first expression on the right is bounded. Thus the measure $\mu_q$ is compact Carleson, and Theorem 2 implies the result. □

If $\Phi$ is bounded multiplicity, then $C_\Phi$ is compact on the Hardy spaces if and only if the angular derivative of $\Phi$ exists nowhere on the circle [15]. Theorem 7 states the analogous result for the $D^p$ spaces, $1 \le p < 2$.

**Theorem 7.** *Suppose that $\Phi$ is of bounded multiplicity and $\| \Phi \|_\infty = 1$. The following are equivalent*:

1. *The operator $C_\Phi$ is compact on $D^p$ for all $p, 1 \leq p < 2$.*
2. *$C_\Phi$ is compact on the Hardy spaces.*
3. *The angular derivative of $\Phi$ exists at no point on the circle.*

*Proof.* It suffices to prove that the third assertion implies the first.

Since $\Phi$ has bounded multiplicity, $C_\Phi$ is bounded on $D^2$. Theorem 6 now implies the result. □

In [11], Jovovic and MacCluer point out that a univalent function mapping $D$ onto a non-tangential approach region will not induce a compact composition operator on the Dirichlet space. Thus univalence and non-existence of the angular derivative are not sufficient to imply that $C_\Phi$ is compact on $D^2$.

## Section 2

In this section we present a brief discussion of inner functions which induce bounded composition operators on the space $D^p$.

If $\Phi$ is a finite Blaschke product, then $\Phi'$ is bounded, and thus $\Phi$ induces a composition operator on $D^p$ for $1 \leq p \leq 2$. D. J. Newman and H. S. Shapiro proved that if $\Phi$ is inner and $\Phi \in D^2$, then $\Phi$ is a finite Blaschke product [16]. If $\Phi$ is an infinite Blaschke product or a singular inner function, then $\Phi$ need not belong to $D^1$ [3, 20]. Since $D^p \subset D^1$ for $1 \leq p \leq 2$, this shows that such an inner function may fail to induce a bounded composition operator on any of the Dirichlet-type spaces.

As a positive example, let $\Lambda(z) = e^{-(1+z)/(1-z)}$ for $z \in D$. It is clear that $\Lambda \notin D^2$. Let $w \in \Lambda(D) = D\backslash\{0\}$ and denote the preimages of $w$ by $z_n(w)$, $n \in \mathbf{Z}$. For $1 \leq p < 2$,

$$N_p(\Lambda, w) = \sum_{n \in \mathbf{Z}} \frac{| 1 - z_n(w) |^{2(2-p)}}{(2 | w |)^{2-p}}.$$

There is a natural number $N$ and a positive constant $C_1$ independent of $w$ such that $C_1 < | n | | 1 - z_n(w) |$ for $| w | > 1/2$ and $| n | > N$. It follows that $N_{3/2}(\Lambda, w) = \infty$ for all $| w | > 1/2$, and thus $\Lambda \notin D^p$ for $3/2 \leq p \leq 2$.

By similar reasoning there is a positive constant $C_2$ and a natural number $N$ such that $| n | | 1 - z_n(w) | < C_2$ for all $| n | > N$ and all $w \in \Lambda(D)$. It follows that $N_p(\Lambda, w) < C | w |^{p-2}$ for $1 \leq p < 3/2$. Thus $\mu_p$ is a finite measure. A calculation shows that $\mu_p$ is Carleson. By Theorem 1 $C_\Lambda$ is bounded on $D^p$ if and only if $1 \leq p < 3/2$.

As a generalization of this example let $F$ be defined by

$$F(z) = \int_{\{x:|x|=1\}} \frac{1 + xz}{1 - xz}\, d\mu(x) \tag{6}$$

where $\mu$ is a convex combination of finitely many point masses on the circle. For $z \in D$, define $\Psi(z) = e^{-F(z)}$ and $h(z) = (F(z) - 1)/(F(z) + 1)$. Then $h$ is an analytic self-map of the disc and $\Psi = \Lambda \circ h$. Since $h$ is rational and $| h(e^{i\theta}) | = 1$ for $0 \leq \theta < 2\pi$, $h$ is a finite Blaschke product. It follows that $C_\Psi$ is a bounded operator on $D^p$ for $1 \leq p < 3/2$.

We close this section by noting a connection between the Besov spaces $B_\gamma$ and the spaces $D^p$. A function $f$ analytic in $D$ belongs to $B_\gamma$ $(0 < \gamma < 1)$ if

$$\int_0^{2\pi} \int_0^1 \mid f(re^{i\theta}) \mid \ (1-r)^{1/\gamma \ -2} \ dr \ d\theta < \infty.$$

A result of Hardy and Littlewood [9, p. 84] implies that $B_\gamma \subset A^{2\gamma}$. Several authors have studied conditions on inner functions $\Phi$ sufficient to imply that $\Phi' \in B_\gamma$, and thus, $\Phi \in D^{2\gamma}$ [1, 2, 3, 8, 18, 20].

## Section 3

The proofs of Theorems 1 and 2 will be presented in this section. The proofs depend upon Theorem 4.3 in [15], which states that for a finite positive measure $\nu$, the identity map $I : A^p \to L^p(\nu)$ is bounded (respectively, compact) if and only if $\nu$ is Carleson (respectively, compact Carleson).

We may assume that $\Phi$ is not constant. Unless otherwise indicated, all integrals in this section are taken over the unit disc $D$.

For the proof of Theorem 1, assume that $C_\Phi$ is bounded on $D^p$. Let $f \in D^p$ with $f(0) = 0$. The hypothesis implies that

$$\int \mid f' \circ \Phi \mid^p \mid \Phi' \mid^p \ dA \ \leq \ C \int \mid f' \mid^p \ dA.$$

By a change of variable (see [7], p. 36) and the definition of $\mu_p$,

$$\int \mid f' \circ \Phi \mid^p \mid \Phi' \mid^p \ dA \ = \ \int \mid f' \mid^p \ d\mu_p.$$

Let $g \in A^p$ and define $f(z) \ = \ \int_0^z g(w) \ dw$. The argument above shows that

$$\int \mid g \mid^p \ d\mu_p \ \leq \ C \ \int \mid g \mid^p \ dA \tag{7}$$

and it follows that $\mu_p$ is a Carleson measure.

For the converse, suppose that $\mu_p$ is a Carleson measure. Then (7) holds for every $g \in A^p$. It follows that

$$\int \mid (f \circ \Phi)' \mid^p \ dA \ = \ \int \mid f' \mid^p \ d\mu_p \ \leq \ C \ \int \mid f' \mid^p \ dA \ \leq \ C \parallel f \parallel_{D^p}^p$$

for every $f \in D^p$. Since evaluation at $\Phi(0)$ is a bounded linear functional on $D^p$, the argument shows that $C_\Phi$ is a bounded operator on $D^p$. The proof of Theorem 1 is complete. □

For the proof of Theorem 2, note that $C_\Phi : D^p \to D^p$ is compact if and only if $\parallel f_n \circ \Phi \parallel_{D^p} \to 0$ for any bounded sequence $(f_n)$ in $D^p$ with $f_n \to 0$ uniformly on compact subsets.

Suppose that $C_\Phi$ is compact on $D^p$. Let $(g_n)$ be a bounded sequence in $A^p$ with $g_n \to 0$ uniformly on compact subsets, and define $f_n(z) = \int_0^z g_n(w) dw$. The hypothesis implies $\parallel g_n \parallel_{L^p(\mu_p)} = \parallel (f_n \circ \Phi)' \parallel_{A^p} \to 0$ as $n \to \infty$ and thus $\mu_p$ is compact Carleson.

For the converse, assume that $\mu_p$ is compact Carleson. Suppose $\parallel f_n \parallel_{D^p} \leq 1$ and $f_n \to 0$ uniformly on compact subsets. Since $I : A^p \to L^p(\mu_p)$ is compact, it follows that $\parallel (f_n \circ \Phi)' \parallel_{A^p} = \parallel f_n' \parallel_{L^p(\mu_p)} \to 0$. Since $\mid f_n(\Phi(0)) \mid \to 0$, the argument yields $\parallel f_n \circ \Phi \parallel_{D^p} \to 0$. Thus $C_\Phi$ is compact on $D^p$. □

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