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TRACE-CLASS PERTURBATION AND STRONG CONVERGENCE: WAVE OPERATORS REVISITED

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Abstract. We give a new construction of wave operators for a self-adjoint operator under trace-class perturbation. This construction requires no quantitative estimates.

The well-known theorem of Kato and Rosenblum [2], [5] asserts that if A and A' are self-adjoint operators and if $A - A'$ belongs to $\mathcal{C}_1$, the trace class, then the absolutely continuous parts of A and A' are unitarily equivalent. Because of the importance of the problem of trace-class perturbation, many improvements and generalizations of this theorem have appeared in the literature. [3] and [4] contain a reasonably complete account of the development between [2], [5] and the late 1970's. In these works the unitary equivalence between A_{ac} and A'_{ac} was established through the existence of the wave operators

$$W_\pm(A', A) = \text{s-}\lim_{\lambda\to\pm\infty} e^{-i\lambda A'} e^{i\lambda A} P_{\mathrm{ac}}(A).$$

In fact, as mentioned in [5], the original idea of using the operator $e^{-i\lambda A'}e^{i\lambda A}$ dates back to Friedrichs [1]. The first generalization of the Kato-Rosenblum theorem to the setting of operator tuples was made by Voigt [8]. The use of the exponential function $e_\lambda(x) = \exp(i\lambda x)$ is fundamental to these "time-dependent" constructions of wave operators.

In [6], Voiculescu generalized wave operators to the setting of commuting tuples under perturbation by norm ideals of compact operators. He showed that, if $T = (T_1, \dots, T_N)$ and $T' = (T'_1, \dots, T'_N)$ are commuting tuples of self-adjoint operators such that $T_j - T'_j \in \mathcal{C}^{(0)}$, $j = 1, \dots, N$, and if the norm ideal $\mathcal{C}$ has the property

$$\lim_{n\to\infty} n^{-1/2} \|\omega_1 \otimes \omega_1 + \cdots + \omega_n \otimes \omega_n\|_{\mathcal{C}} = 0,$$

where $\{\omega_n\}_{n=1}^\infty$ is any orthonormal set, then the wave operator for the perturbation problem $T \to T'$ exists in the strong operator topology and is *unique* [6, Theorem 1.5]. This uniqueness is in sharp contrast with the problem of trace-class perturbation for single operator; in general, the two wave operators W_+ and W_- do not necessarily coincide. While essentially a time-dependent approach, Voiculescu's

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work shows that, for perturbations other than that by the trace class, it is possible to construct wave operators without explicitly using the exponential function $e_\lambda(x) = \exp(i\lambda x)$.

The treatment of the original theorem of Kato-Rosenblum in the framework of [6], however, proved to be elusive. In fact this was one of the problems Voiculescu raised during the 1983 ICM [7, page 1043]. Also, the reason for the fact that W_+ and W_- may differ or, equivalently, that the scattering operator $S = W_-^* W_+$ is not necessarily $P_{\mathrm{ac}}(A)$, has never been made clear in the previous constructions of wave operators; one usually gets $S \neq P_{\mathrm{ac}}(A)$ from explicit computations.

The purpose of this note is to give a proof of the Kato-Rosenblum theorem within the context suggested by Voiculescu [6], [7]. Indeed, what we will prove is slightly stronger than the original version. Moreover, this new proof has the following three distinct features: (a) It identifies the cause for $W_+ \neq W_-$. (b) It only uses certain limited properties satisfied by the exponential functions $e_\lambda(x) = \exp(i\lambda x)$, but not the exponential functions themselves. (c) Unlike previous ones, our proof involves no quantitative estimates. Indeed, our proof is surprisingly soft.

We start by recalling a few well-known facts. For the rest of the paper, H and M will denote the Hilbert transform and the multiplication by the coordinate function on $\mathbf{R}$. That is, for $f \in L^2(\mathbf{R})$ or for $f \in L^2(\mathbf{R}, \mathcal{M}) = L^2(\mathbf{R}) \otimes \mathcal{M}$, where $\mathcal{M}$ is a Hilbert space, we write

$$(Hf)(x) = \frac{1}{\pi i}\text{p.v.}\int \frac{f(y)}{y-x}dy \qquad \text{and} \qquad (Mf)(x) = xf(x).$$

Recall that $e^{i\lambda M}He^{-i\lambda M} = \chi_{(0,\infty)}(D-\lambda) - \chi_{(-\infty,0)}(D-\lambda)$, where D is the differential operator $(1/i)d/dx$. Hence

$$\text{s-}\lim_{\lambda\to\infty} e^{i\lambda M}He^{-i\lambda M} = -1 \qquad \text{and} \qquad \text{s-}\lim_{\lambda\to-\infty} e^{i\lambda M}He^{-i\lambda M} = 1.$$

Also recall that if A and A' are self-adjoint operators, $z \in \mathbf{C}\backslash\mathbf{R}$ and $\lambda \in \mathbf{R}$, then

$$(A'-z)^{-1}(e^{i\lambda A'} - e^{i\lambda A})(A-z)^{-1} = i\int_0^\lambda e^{isA'}\{(A-z)^{-1} - (A'-z)^{-1}\}e^{i(\lambda-s)A}ds.$$

Therefore if $(A-z)^{-1} - (A'-z)^{-1}$ belongs to the trace class, then so does $(A'-z)^{-1}(e^{i\lambda A'} - e^{i\lambda A})(A-z)^{-1}$. As it turns out, these are the only properties of the exponential function which are relevant to the construction of wave operators.

A sequence of Borel functions $\{\varphi_n\}$ on $\mathbf{R}$ is said to be of class Ω_+ if

(i) $|\varphi_n(t)| = 1$ for all $t \in \mathbf{R}$ and $n \in \mathbf{N}$.

(ii) If A and A' are self-adjoint operators such that $(A-z)^{-1} - (A'-z)^{-1} \in \mathcal{C}_1$ for some $z \in \mathbf{C}\backslash\mathbf{R}$, then $(A'-z)^{-1}(\varphi_n(A') - \varphi_n(A))(A-z)^{-1} \in \mathcal{C}_1$ for every $n \in \mathbf{N}$.

(iii$^+$) s-$\lim_{n\to\infty} \varphi_n(M)H\varphi_n^*(M) = -1$.

A sequence of Borel functions $\{\varphi_n\}$ on $\mathbf{R}$ is said to be of class Ω_- if it satisfies (i), (ii) and

(iii$^-$) s-$\lim_{n\to\infty} \varphi_n(M)H\varphi_n^*(M) = 1$.

It is elementary that (ii) and (i) imply

(II) If A and A' are self-adjoint operators such that $(A-z)^{-1} - (A'-z)^{-1} \in \mathcal{C}_1$ for some $z \in \mathbf{C}\backslash\mathbf{R}$, then $(\varphi_n(A') - \varphi_n(A))(A-z)^{-2} \in \mathcal{C}_1$ for every $n \in \mathbf{N}$.

Because $\|(H\pm1)\varphi_n^*(M)f\| = \|\varphi_n(M)(H\pm1)\varphi_n^*(M)f\|$, (iii$^+$) and (iii$^-$) respectively imply

(III$^+$) s-$\lim_{n\to\infty}(H+1)\varphi_n^*(M) = 0$.

(III$^-$) s-$\lim_{n\to\infty}(H-1)\varphi_n^*(M) = 0$.

Thus, if $\{\lambda_n\}$ are positive numbers such that $\lim_{n\to\infty}\lambda_n = \infty$, then the sequence $\{\exp(i\lambda_n x)\}$ (resp. $\{\exp(-i\lambda_n x)\}$) is of class Ω_+ (resp. Ω_-). As we will see, the dichotomy between (iii$^+$) and (iii$^-$) is the cause for $W_+ \neq W_-$.

Theorem. *Let A and A' be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that*

$$(A-z)^{-1} - (A'-z)^{-1} \in \mathcal{C}_1 \quad \textit{for some } z \in \mathbf{C}\backslash\mathbf{R}.$$

Then there are partial isometries $W_+(A', A)$ and $W_-(A', A)$ such that

$$s\text{-}\lim_{n\to\infty} \varphi_n^*(A')\varphi_n(A)P_{ac}(A) = W_+(A', A) \qquad \textit{and}$$
$$s\text{-}\lim_{n\to\infty} \psi_n^*(A')\psi_n(A)P_{ac}(A) = W_-(A', A),$$

where $\{\varphi_n\}$ is any sequence of class Ω_+ and $\{\psi_n\}$ any sequence of class Ω_-.

Proof. We only need to establish the strong convergence; that the limits $W_+(A', A)$ and $W_-(A', A)$ are independent of the choices of $\{\varphi_n\}$ and $\{\psi_n\}$ follows from an observation borrowed from [6]: If one mixes two sequences of a given class, then one obtains a new sequence of the same class. Moreover, we will only consider the case of Ω_+; the case of Ω_- differs only in one detail, which will be pointed out in due course.

We may assume that $\mathcal{H} = (\bigoplus_{j\in J} L^2(\Delta_j)) \oplus \mathcal{H}_{\mathrm{s}}$, where each summand is invariant under A, $A|\mathcal{H}_{\mathrm{s}}$ is purely singular, and $A|(\bigoplus_{j\in J} L^2(\Delta_j)) = M$, the multiplication by the coordinate function. Here, each Δ_j is a Borel set in $\mathbf{R}$ and, as usual, $L^2(\Delta_j) = \chi_{\Delta_j} L^2(\mathbf{R})$. Define $W_n = \varphi_n^*(A')\varphi_n(A)$ and $T_{k,n} = W_k^* W_n - 1$ for n, $k \in \mathbf{N}$.

Pick a $j_0 \in J$ and let $\xi \in L^2(\Delta_{j_0})$ be a *bounded* function whose support is contained in a *finite* interval I. Define the operator $\tilde{M}_\xi$ on $\mathcal{H}$ by the formula $\tilde{M}_\xi f = \xi f_{j_0}$ for $f = (\bigoplus_{j\in J} f_j) \oplus h$, where $h \in \mathcal{H}_{\mathrm{s}}$ and $f_j \in L^2(\Delta_j)$, $j \in J$. Define

$$Y_\xi = \pi i(A-z)\tilde{M}_\xi(H+1)\tilde{M}_\xi^*(A-z).$$

(For the case of Ω_-, replace the operator $H+1$ above by $H-1$.) Because $\operatorname{supp}\xi \subset I$, Y_ξ is a bounded operator. Denote $K = (A-z)^{-1} - (A'-z)^{-1}$. We have $[T_{k,n}, (A-z)^{-1}] = W_k^*\varphi_n^*(A')K\varphi_n(A) - \varphi_k^*(A)K\varphi_k(A')W_n \in \mathcal{C}_1$ and, therefore,

$$|\mathrm{tr}([T_{k,n}, (A-z)^{-1}]Y_\xi)| \le \|K\varphi_n(A)Y_\xi\|_1 + \|Y_\xi\varphi_k^*(A)K\|_1.$$

It follows from (III$^+$), the identity $\tilde{M}_\xi^*\varphi_n^*(A) = \varphi_n^*(M)\tilde{M}_\xi^*$, and the assumptions on ξ that $\text{s-}\lim_{n\to\infty} Y_\xi^*\varphi_n^*(A) = 0 = \text{s-}\lim_{k\to\infty} Y_\xi\varphi_k^*(A)$, which leads to $\|Y_\xi\varphi_k^*(A)K\|_1 \to 0$ and $\|K\varphi_n(A)Y_\xi\|_1 = \|Y_\xi^*\varphi_n^*(A)K^*\|_1 \to 0$ as $\min\{k,n\}\to\infty$. That is,

$$\lim_{\min\{k,n\}\to\infty} \mathrm{tr}([T_{k,n}, (A-z)^{-1}]Y_\xi) = 0. \tag{1}$$

By (II), $(W_n - 1)(A-z)^{-2} \in \mathcal{C}_1$ and $(W_k^* - 1)(A-z)^{-2} \in \mathcal{C}_1$. Since $T_{k,n} = (W_k^*-1)(W_n-1)+(W_k^*-1)+(W_n-1)$, we have $T_{k,n}(A-z)^{-2} \in \mathcal{C}_1$. Since $(A-z)^2 Y_\xi$ is bounded, $T_{k,n}Y_\xi = \{T_{k,n}(A-z)^{-2}\}\{(A-z)^2 Y_\xi\} \in \mathcal{C}_1$. Thus, $\mathrm{tr}((A-z)^{-1}T_{k,n}Y_\xi) = \mathrm{tr}(T_{k,n}Y_\xi(A-z)^{-1})$. Now Y_ξ was designed so that $[(A-z)^{-1}, Y_\xi] = \xi\otimes\xi$. Hence

$$\mathrm{tr}([T_{k,n}, (A-z)^{-1}]Y_\xi) = \mathrm{tr}(T_{k,n}[(A-z)^{-1}, Y_\xi]) = \mathrm{tr}(T_{k,n}\xi\otimes\xi) = \langle T_{k,n}\xi, \xi\rangle. \tag{2}$$

Because $(W_k - W_n)^*(W_k - W_n) = -T_{k,n} - T_{n,k}$, it follows from (1) and (2) that

$$\lim_{\min\{k,n\}\to\infty} \|(W_k - W_n)\xi\|^2 = - \lim_{\min\{k,n\}\to\infty} \langle (T_{k,n} + T_{n,k})\xi, \xi\rangle = 0.$$

Since the *linear span* of such ξ's is dense in $\bigoplus_{j\in J} L^2(\Delta_j)$, this completes the proof. $\square$

Remark. To deduce that $(A'-w)^{-1}W_+(A',A) = W_+(A',A)(A-w)^{-1}$ for $w \in \mathbf{C}\backslash\mathbf{R}$, which is necessary for establishing the *unitary equivalence* of A_{ac} and A'_{ac}, one needs *one* sequence of class Ω_+ which has the additional property

$$\text{w-}\lim_{n\to\infty} \varphi_n(A)P_{\mathrm{ac}}(A) = 0. \tag{3}$$

For example, use the sequence $\{\exp(inx)\}$. But, nowhere in our proof did we need anything like (3) for the *strong convergence* of $\{\varphi_n^*(A')\varphi_n(A)P_{\mathrm{ac}}(A)\}$. In fact it is not even clear that Ω_+ would imply (3), although it is difficult to imagine that (iii$^+$) and (3) are completely unrelated.

References

1. K. Friedrichs, *On the perturbation of continuous spectra*, Commun. on Appl. Math. **1** (1948), 361-401. MR **10:**547e
2. T. Kato, *Perturbation of continuous spectra by trace class operators*, Proc. Japan Acad. **33** (1957), 260-264. MR **19:**1068d
3. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1976. MR **53:**11389
4. M. Reed and B. Simon, *Methods of modern mathematical physics*, III *Scattering theory*, Academic Press, New York, 1979. MR **80m:**81085
5. M. Rosenblum, *Perturbations of continuous spectrum and unitary equivalence*, Pacific J. Math. **7** (1957), 997-1010. MR **19:**756i
6. D. Voiculescu, *Some results on norm-ideal perturbations of Hilbert space operators*. II, J. Operator Theory **5** (1981), 77-100. MR **83f:**47014
7. D. Voiculescu, *Hilbert space operators modulo normed ideals*, Proc. International Congress of Mathematicians (Warsaw 1983) Vol. 2, 1041-1047, PWN, Warsaw, 1984. MR **87b:**47049
8. J. Voigt, *Perturbation theory for commutative m-tuples of self-adjoint operators*, J. Funct. Anal. **25** (1977), 317-334. MR **56:**9301

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