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A PALEY-WIENER THEOREM FOR THE SPHERICAL LAPLACE TRANSFORM ON CAUSAL SYMMETRIC SPACES OF RANK 1

NILS BYRIAL ANDERSEN AND GESTUR ÓLAFSSON

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Abstract. We formulate and prove a topological Paley-Wiener theorem for the normalized spherical Laplace transform defined on the rank 1 causal symmetric spaces $\mathcal{M} = SO_o(1,n)/SO_o(1,n-1)$, for $n \geq 2$.

Introduction

The spherical Laplace transform on causal symmetric spaces was introduced in [FHO, §8] as a generalization of the spherical Fourier transform on Riemannian symmetric spaces defined by Helgason (see [H1, Chapter 4]). Both transforms can be expressed in terms of (integrating against) spherical functions. It was furthermore shown in [O1, §5] that the spherical functions on the Riemannian dual of a causal symmetric space can be written as an expansion in spherical functions on the causal symmetric space. The inversion formula for the spherical Laplace transform easily follows (see [O1, §6]).

One of the most important results on the spherical Fourier transform is the (topological) Paley-Wiener theorem (see [H1, Chapter 4, §7] and [H2, Chapter 3, §5] for details) generalizing the classical Paley-Wiener theorem on Euclidean spaces. In this paper we generalize these results to the normalized spherical Laplace transform on causal symmetric spaces $\mathcal{M}$ of rank 1, thereby partially solving an open problem posed by the second author in [O2, §5].

The paper is divided into two sections: in the first section we recall some results on the spherical Fourier transform on the Riemannian dual $\mathcal{M}^d$ of $\mathcal{M}$, and in the second we consider the spherical Laplace transform defined on $\mathcal{M}$. We define the Paley-Wiener space, the supposed image space of spherical Laplace transform, according to the growth and symmetry conditions satisfied by the spherical functions on $\mathcal{M}$. The Paley-Wiener theorem for the normalized spherical Laplace transform follows by using Cauchy's theorem to change the path of integration in the inversion formula and from the Paley-Wiener theorem for the spherical Fourier transform on $\mathcal{M}^d$.

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We have tried to keep notations and proofs to a minimum in order to make the presentation as clear as possible; we refer the reader to [FHO], [HO], [O1] and [O2] for more details on spherical functions and the spherical Laplace transform defined on causal symmetric spaces. The spherical Laplace transform in the rank 1 case can be considered as a Laplace-Jacobi transform (see [M] for a detailed analysis of the latter transform) but we note that the Paley-Wiener theorem is new even in the rank one case.

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Notation and preliminaries

In the following we consider the causal symmetric (real hyperbolic) space $\mathcal{M} = G/H$ with $G = SO_o(1,n)$ and $H = SO_o(1,n-1)$ and its Riemannian dual $\mathcal{M}^d = G/K$, where $K = SO_o(n)$. Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathfrak{a}$ be the abelian subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{a} = \left\{ X_t = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}.$$

We choose the unique positive root $\alpha \in \mathfrak{a}^*$ as $\alpha(X_t) = t$. Let $\mathfrak{a}^+ = \{X_t \in \mathfrak{a} | t > 0\}$. We identify the complex dual $\mathfrak{a}^*_{\mathbb{C}}$ of $\mathfrak{a}$ with $\mathbb{C}$ via the map $\mathbb{C} \ni z \mapsto z\alpha \in \mathfrak{a}_{\mathbb{C}}$. Let $\mathfrak{n} = \mathfrak{g}_\alpha$ and $\overline{\mathfrak{n}} = \mathfrak{g}_{-\alpha}$ denote the positive and negative root space respectively. Let $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, $N = \exp \mathfrak{n}$ and $\overline{N} = \exp \overline{\mathfrak{n}}$, where exp is the exponential mapping from $\mathfrak{g}$ to G. We also consider the open semigroup $S^o = HA^+H$ in G. Let finally $a_t = \exp X_t \in A$.

Let $\eta : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}^d)$ denote the Flensted-Jensen isomorphism between the commutative algebras of invariant differential operators on $\mathcal{M}$ and $\mathcal{M}^d$ respectively (mapping the Laplace-Beltrami operator Δ on $\mathcal{M}$ onto the Laplace-Beltrami operator $\Delta^d = \eta(\Delta)$ on $\mathcal{M}^d$). Let $\Pi(D)$ and $\Pi^d(D^d)$ denote the radial part (on A^+) of $D \in \mathcal{D}(\mathcal{M})$ and $D^d \in \mathcal{D}(\mathcal{M}^d)$ respectively. There exists a unique map $C_c^\infty(H\backslash S^o/H) \ni f \mapsto f^d \in C_c^\infty(K\backslash G/K)$ such that $f_{|A^+} = f^d_{|A^+}$ and $\Pi(D) f_{|A^+} = \Pi^d(\eta(D)) f^d_{|A^+}$ (see [HO] or [O1, §4] for more details).

The spherical Fourier transform on $\mathcal{M}^d$

In this section we recall some well-known definitions and results for the spherical Fourier transform on a Riemannian symmetric space (see e.g. [H1, Chapter 4] and [H2, Chapter 3]).

Let $\lambda \in \mathbb{C}$. Define the Poisson kernel for $\mathcal{M}^d$ by

$$KAN \ni kan = x \mapsto a^{-(\lambda+\rho)} =: p^d_\lambda(x),$$

where $\rho = (n-1)/2$. The spherical functions on $\mathcal{M}^d$ can be written as

$$\psi^d_\lambda(a_t) = \int_K p^d_{-\lambda}(a_t k)dk = {}_2F_1\left(\frac{1}{2}(\lambda+\rho), \frac{1}{2}(-\lambda+\rho); \rho + \frac{1}{2}; -\sinh^2 t\right),$$

for $a_t \in A$. The spherical functions are bi-K-invariant and $\Delta^d \psi^d_\lambda = (\lambda^2 - \rho^2)\psi^d_\lambda$ for all $\lambda \in \mathbb{C}$. They satisfy the following inequality, for all $\lambda \in \mathbb{C}$ and all $a_t \in A$:

$$\left|\psi^d_\lambda(a_t)\right| \le c(1+|t|)e^{(|\mathrm{Re}\lambda| - \rho)|t|},$$

for some constant c, and they are invariant under sign change, i.e. $\psi^d_{-\lambda} = \psi^d_\lambda$.

The spherical Fourier transform $\mathcal{F}$ on $\mathcal{M}^d$ is defined as

$$\mathcal{F}(f)(\lambda) = \int_G f(x)\psi^d_\lambda(x)dx = \int_{A^+} f(a)\psi^d_\lambda(a)\delta(a)da,$$

for $f \in C_c^\infty(K\backslash G/K)$, where $\delta(a_t) = \sinh^{2\rho} t$. Let

$$c^d(\lambda) := \int_{\overline{N}} p^d_\lambda(\overline{n})d\overline{n} = \frac{\Gamma(2\rho)}{\Gamma(\rho)}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\rho)}$$

denote the Harish-Chandra c-function for $\mathcal{M}^d$. We note that $|c^d(\lambda)|^2 = c^d(\lambda)c^d(-\lambda)$ for $\lambda \in i\mathbb{R}$. The inversion formula for $\mathcal{F}$ reads (after normalizing $d\lambda$ suitably):

$$f(x) = \int_{i\mathbb{R}} \mathcal{F}(f)(\lambda)\psi^d_{-\lambda}(x)|c^d(\lambda)|^{-2}d\lambda,$$

for all $f \in C_c^\infty(K\backslash G/K)$ and $x \in G$.

Let $R > 0$. Let $C_R^\infty(K\backslash G/K) := \{f \in C_c^\infty(K\backslash G/K) | f(a_t) = 0 \text{ for } t > R\}$ and define the Paley-Wiener space $\mathcal{H}_R(\mathbb{C})$ as the space of even holomorphic functions g on $\mathbb{C}$ of exponential type R, i.e. satisfying the estimate

$$\sup_{\lambda\in\mathbb{C}} e^{-R|Re\lambda|}(1+|\lambda|)^N |g(\lambda)| < \infty,$$

for all $N \in \mathbb{N} \cup \{0\}$. Furthermore denote by $\mathcal{H}(\mathbb{C})$ the union of the spaces $\mathcal{H}_R(\mathbb{C})$, $R > 0$.

Theorem 1 (The Paley-Wiener Theorem)**.** *The Fourier transform is a topological linear isomorphism from $C_c^\infty(K\backslash G/K)$ onto $\mathcal{H}(\mathbb{C})$. More precisely it is a topological linear isomorphism from $C_R^\infty(K\backslash G/K)$ onto $\mathcal{H}_R(\mathbb{C})$ for all $R > 0$.*

The spherical Laplace transform on $\mathcal{M}$

We define spherical functions on $\mathcal{M}$ according to [O1, Definition 4.1]:

Definition 2. An H-bi-invariant continuous function $\varphi : S^o \to \mathcal{C}$ is called a spherical function if there exists a character χ of $\mathcal{D}(\mathcal{M})$ such that (in the sense of distributions) $D\varphi = \chi(D)\varphi$ for all $D \in \mathcal{D}(\mathcal{M})$.

Define the Poisson kernel for $\mathcal{M}$ (and the open orbit HAN) by

$$HAN \ni han = x \mapsto a^{-(\lambda+\rho)} =: p_\lambda(x)$$

and $p_\lambda \equiv 0$ on $G\backslash HAN$. We can construct spherical functions φ_λ as follows:

$$\varphi_\lambda(x) := \int_H p_{-\lambda}(xh)dh,$$

for $x \in S^o$, whenever this integral exists (see [O1, Theorem 4.10]). Using the calculations in [FHO, §10] we see that the integral converges for $x \in S^o$ and $\mathrm{Re}\lambda < 1-\rho$ and we get the following explicit formula for φ_λ:

$$\varphi_\lambda(a_t) = c(\lambda)(2\cosh t)^{\lambda-\rho}{}_2F_1\left(\frac{1}{2}(-\lambda+\rho), \frac{1}{2}(-\lambda+\rho+1); 1-\lambda; \cosh^{-2} t\right),$$

where

$$c(\lambda) := \int_{\overline{N}} p_\lambda(\overline{n})d\overline{n} = 2^{2\rho-1}\Gamma(\rho)\frac{\Gamma(-\lambda-\rho+1)}{\Gamma(-\lambda+1)}.$$

We note that $\Delta\varphi_\lambda = (\lambda^2-\rho^2)\varphi_\lambda$ when defined.

Let φ^o_λ denote the normalized spherical function $\varphi^o_\lambda(x) = c(\lambda)^{-1}\varphi_\lambda(x)$. We have the following uniform growth estimate on φ^o_λ (due to Helgason (rank one) and Gangolli):

Lemma 3. *Fix $\sigma > 0$. There exists a constant c_σ such that*

$$|\varphi^o_\lambda(a_t)| \leq c_\sigma e^{(Re\lambda-\rho)t},$$

for $Re\lambda \leq 0$ and all $t \in]\sigma, \infty[$.

Sketch of the proof. Consider a formal power series solution to the differential equation $\Pi(\Delta)\varphi^o_\lambda = \delta(t)^{-1}\frac{\partial}{\partial t}(\delta(t)\frac{\partial \varphi^o_\lambda}{\partial t}) = (\lambda^2 - \rho^2)\varphi^o_\lambda$ of the form $e^{(\lambda-\rho)t}\sum_{n=0}^\infty e^{-nt}\Gamma_n(\lambda)$, with $\Gamma_n(\lambda)$ to be determined (and $\Gamma_0 \equiv 1$). Substitution into the differential equation gives a recurrence formula that defines $\Gamma_n(\lambda)$ uniquely for $\lambda \notin \frac{1}{2}\mathbb{N}$. We can estimate $\Gamma_n(\lambda)$ as follows: There exist constants $c, \varkappa > 0$ such that

$$|\Gamma_n(\lambda)| < c(1+n)^\varkappa,$$

for $Re\lambda \leq 0$ and all $n \in \mathbb{N}$. The above estimate on φ^o_λ follows easily. See [BS, §7-9] for a complete proof in a more general setup. □

We define the normalized spherical Laplace transform $\mathcal{L}^o$ on $\mathcal{M}$ as

$$\mathcal{L}^o(f)(\lambda) = \int_{A^+} f(a)\varphi^o_\lambda(a)\delta(a)da,$$

for $f \in C^\infty_c(A^+) \cong C^\infty_c(H\backslash S^o/H)$ and $\mathrm{Re}\lambda < 1 - \rho$. From the explicit formula for φ^o_λ, we see that the function $\lambda \mapsto \mathcal{L}^o(f)(\lambda)$ extends to a meromorphic function on $\mathbb{C}$ with at most poles for $\lambda \in \mathbb{N}$.

Let $R > r > 0$ and define $C^\infty_{r,R}(A^+) := \{f \in C^\infty_c(A^+) | f(a_t) = 0 \text{ for } 0 < t < r \text{ and } t > R\}$. We equip $C^\infty_{r,R}(A^+)$ with the natural Fréchet space topology and $C^\infty_c(A^+)$ with the inductive limit topology. From Lemma 3 we get the following uniform growth estimate on the normalized spherical Laplace transform acting on $C^\infty_{r,R}(A^+)$:

Lemma 4. *Let $R > r > 0$ and let $N \in \mathbb{N} \cup \{0\}$. There exists a constant $c > 0$, only depending on r and R, such that, for all $f \in C^\infty_{r,R}(A^+)$*

$$\sup_{\mathrm{Re}\lambda \leq 0} e^{-r\mathrm{Re}\lambda}(1+|\lambda|^2)^N |\mathcal{L}^o f(\lambda)| \leq c\sum_{n=0}^N ||(\Delta+\rho^2)^n f||_\infty < \infty.$$

Proof. Since $\mathcal{L}^o(\Delta f)(\lambda) = (\lambda^2 - \rho^2)\mathcal{L}^o(f)(\lambda)$ and $\lambda^2 = \lambda^2 - \rho^2 + \rho^2$, we easily get

$$(1+|\lambda|^2)^N |\mathcal{L}^o(f)(\lambda)| = \sum_{n=0}^N \binom{N}{n}|\lambda|^{2n} |\mathcal{L}^o(f)(\lambda)|$$

$$= \sum_{n=0}^N \binom{N}{n} \left|\mathcal{L}^o((\Delta+\rho^2)^n f)(\lambda)\right| \leq ce^{r\mathrm{Re}\lambda}\sum_{n=0}^N \binom{N}{n}||(\Delta+\rho^2)^n f||_\infty,$$

for $Re\lambda \leq 0$ and all $f \in C^\infty_{r,R}(A^+)$, where $c > 0$ is a constant only depending on r and R. □

Using the correspondence between (the radial parts of) invariant differential operators on $\mathcal{M}$ respectively on $\mathcal{M}^d$ (see also [O1, Theorem 5.9]), we get

$$\psi^d_\lambda(a) = c^d(\lambda)\varphi^o_\lambda(a) + c^d(-\lambda)\varphi^o_{-\lambda}(a),$$

for $a \in A^+$ and $\lambda \notin \mathbb{Z}\backslash\{0\}$ (or use [O1, p.63, Eq (17) & p.64, Eq (22)] and the hypergeometric expressions for the spherical functions), which is the Harish-Chandra expansion formula for ψ^d_λ. Let $f \in C_c^\infty(A^+)$. We see that $\mathcal{L}^o f$ satisfies the following functional equation:

$$c^d(\lambda)\mathcal{L}^o(f)(\lambda) + c^d(-\lambda)\mathcal{L}^o(f)(-\lambda) = \mathcal{F}(f^d)(\lambda),$$

for $\lambda \notin \mathbb{Z}\backslash\{0\}$. The inversion formula for the normalized spherical Laplace transform is now an easy consequence of the inversion formula for the spherical Fourier transform (see also [M, p. 993]):

Theorem 5 (The Inversion Formula)**.** *Let $f \in C_c^\infty(A^+)$. Then*

$$f(a) = 2\int_{i\mathbb{R}} \mathcal{L}^o(f)(\lambda)\psi^d_{-\lambda}(a)c^d(-\lambda)^{-1}d\lambda,$$

for all $a \in A^+$.

All the above suggests the following definition of the Paley-Wiener space, the supposed image space of the normalized spherical Laplace transform:

Definition 6. Let $R > r > 0$. We define the Paley-Wiener space $PW_{r,R}(\mathbb{C})$ as the space of meromorphic functions g on $\mathbb{C}$, with at most poles for $\lambda \in \mathbb{N}$, such that

(i)
$$\sup_{\mathrm{Re}\lambda \leq 0} e^{-r\mathrm{Re}\lambda}(1+|\lambda|)^N\,|g(\lambda)| < \infty,$$
for all $N \in \mathbb{N}\cup\{0\}$, and

(ii) the c^d-weighted average $\mathrm{P}^{\mathrm{av}}g(\lambda) := c^d(\lambda)g(\lambda) + c^d(-\lambda)g(-\lambda)$ extends to a function in $\mathcal{H}_R(\mathbb{C})$.

Furthermore denote by $PW(\mathbb{C})$ the union of the spaces $PW_{r,R}(\mathbb{C})$ over all $R > r > 0$.

We define a Fréchet space topology on $PW_{r,R}(\mathbb{C})$ by means of the seminorms

$$\sigma_{r,N}(g) = \sup_{\mathrm{Re}\lambda\leq 0} e^{-r\mathrm{Re}\lambda}(1+|\lambda|)^N\,|g(\lambda)|$$

and

$$\sigma_{R,N}(g) = \sup_{\lambda\in\mathbb{C}} e^{-R|\mathrm{Re}\lambda|}(1+|\lambda|)^N\,|\mathrm{P}^{\mathrm{av}}g(\lambda)|\,.$$

We furthermore equip the space $PW(\mathbb{C})$ with the inductive limit topology.

We remark that $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o$ acts injectively on $C_c^\infty(A^+)$, since $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o(f) = \mathcal{F}(f^d) = 0$ implies $f = f^d = 0$ on A^+ for any $f \in C_c^\infty(A^+)$, by the injectivity of the spherical Fourier transform is injective. But we will need that P^{av} is injective on $PW(\mathbb{C})$. The following lemma and its proof was communicated to us by H. Schlichtkrull.

Lemma 7. *Let g be a meromorphic function on $\mathbb{C}$ that satisfies item* (i) *of Definition* 6*. Assume that $P^{av}g = 0$. Then $g = 0$.*

Proof. Let $g^1(\lambda) = g(\lambda)/c^d(-\lambda)$. Then $\mathrm{P}^{\mathrm{av}}g(\lambda) = 2c^d(\lambda)c^d(-\lambda)\mathrm{avg}^1(\lambda)$, where

$$\mathrm{avg}^1(\lambda) := \frac{1}{2}[g^1(\lambda) + g^1(-\lambda)]$$

is the average of g^1 over the Weyl group ± 1. It follows from the assumption $\mathrm{P}^{\mathrm{av}} g = 0$ that $\mathrm{av} g^1 = 0$. Let

$$\gamma(s) = \int_{\mathbb{R}} g^1(i\lambda) e^{-is\lambda}\, d\lambda, \qquad s \in \mathbb{R},$$

denote the Euclidean Fourier transform of $g^1(i\,\cdot\,)$. It follows from (i) and [H1, Proposition IV.7.2] that (i) is satisfied by g^1 as well. In particular, $g^1(i\,\cdot\,) \in L^1(\mathbb{R})$. The condition (i) implies that g is holomorphic in an open set containing $\{z \in \mathbb{C} \mid \mathrm{Re} z \leq 0\}$. Moreover, the standard argument with Cauchy's theorem shows that γ is supported on $[r, \infty[$. On the other hand, the average $\mathrm{av}\gamma$ of γ is the Fourier transform of $\mathrm{av} g^1(i\cdot)$, which vanishes; hence $\mathrm{av}\gamma$ vanishes as well. The support condition now implies that $\gamma = 0$. Since the Euclidean Fourier transform is injective on $L^1(\mathbb{R})$, we conclude that g^1, and therefore also g, vanishes. □

Theorem 8 (The Paley-Wiener Theorem)**.** *The normalized spherical Laplace transform $\mathcal{L}^o$ is a topological linear isomorphism from $C_c^\infty(A^+)$ onto $PW(\mathbb{C})$. More precisely it is a topological linear isomorphism from $C_{r,R}^\infty(A^+)$ onto $PW_{r,R}(\mathbb{C})$ for all $R > r > 0$.*

Proof. By the Paley-Wiener theorem for the spherical Fourier transform, Lemma 4 and the open mapping theorem for Fréchet spaces, it only remains to show that the normalized spherical Laplace transform maps $C_{r,R}^\infty(A^+)$ onto $PW_{r,R}(\mathbb{C})$ for all $R > r > 0$.

Consider the wave packet $\mathcal{I}g \in C^\infty(A^+)$ of $g \in PW_{r,R}(\mathbb{C})$ defined by

$$\mathcal{I}g(a) = 2\int_{i\mathbb{R}} g(\lambda)\psi_{-\lambda}^d(a) c^d(-\lambda)^{-1} d\lambda,$$

for $a \in A^+$. By Cauchy's theorem we get, for $0 < t < r$ and $\mu < 0$,

$$\begin{aligned}
\mathcal{I}g(a_t) &= 2\int_{i\mathbb{R}} g(\lambda)\psi_{-\lambda}^d(a_t) c^d(-\lambda)^{-1} d\lambda \\
&= 2\int_{i\mathbb{R}} g(\lambda+\mu)\psi_{-\lambda-\mu}^d(a_t) c^d(-\lambda-\mu)^{-1} d\lambda \\
&\to 0 \quad \text{for} \quad \mu \to -\infty,
\end{aligned}$$

since $\left|\psi_\lambda^d(a_t) c^d(\lambda)^{-1}\right| \leq c(1+|\lambda|)^\rho (1+t) e^{(|\mathrm{Re}\lambda| - \rho)t}$ for $\mathrm{Re}\lambda \geq 0$, for some constant $c > 0$. An easy calculation shows that

$$\begin{aligned}
\mathcal{I}g(a_t) &= 2\int_{i\mathbb{R}} g(\lambda)\psi_{-\lambda}(a_t) c^d(-\lambda)^{-1} d\lambda \\
&= \sum_{w=\pm 1} \int_{i\mathbb{R}} g(w\lambda)\psi_{-w\lambda}^d(a_t) c^d(-w\lambda)^{-1} d\lambda \\
&= \sum_{w=\pm 1} \int_{i\mathbb{R}} c^d(w\lambda) g(w\lambda)\psi_{-w\lambda}^d(a_t) \left|c^d(\lambda)\right|^{-2} d\lambda \\
&= \int_{i\mathbb{R}} \left(\sum_{w=\pm 1} c^d(w\lambda) g(w\lambda)\right) \psi_{-\lambda}^d(a_t) \left|c^d(\lambda)\right|^{-2} d\lambda \\
&= \int_{i\mathbb{R}} \mathrm{P}^{\mathrm{av}} g(\lambda)\psi_{-\lambda}^d(a_t) \left|c^d(\lambda)\right|^{-2} d\lambda,
\end{aligned}$$

which we recognize as the inverse Fourier transform of $\mathrm{P}^{\mathrm{av}}g \in \mathcal{H}_R(\mathbb{C})$; whence $\mathcal{I}g(a_t) = 0$ for all $t > R$ by the Paley-Wiener theorem for the spherical Fourier transform on $\mathcal{M}^d$.

Since $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o f = \mathcal{F}f^d$ for all $f \in C_c^\infty(A^+)$, the above also yields

$$\mathrm{P}^{\mathrm{av}}\mathcal{L}^o\mathcal{I}g = \mathcal{F}(\mathcal{I}g)^d = \mathrm{P}^{\mathrm{av}}g,$$

for all $g \in PW(\mathbb{C})$; hence Lemma 7 implies that $\mathcal{L}^o\mathcal{I}g = g$ for all $g \in PW(\mathbb{C})$ and we conclude that $\mathcal{L}^o$ maps $C_c^\infty(A^+)$ onto $PW(\mathbb{C})$. □

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Institut de Mathématiques, Analyse Algèbrique, Université Pierre et Marie Curie, Case 82, Tour 46-0, 3e étage, 4, place Jussieu, F-75252 Paris Cedex 05, France

E-mail address: byrial@math.jussieu.fr

Current address: Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

E-mail address: byrial@math.lsu.edu

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

E-mail address: olafsson@math.lsu.edu