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A NOTE ON HAMILTON SEQUENCES FOR EXTREMAL BELTRAMI COEFFICIENTS

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ABSTRACT. F. P. Gardiner gave a sufficient condition for a sequence to be a Hamilton sequence for an extremal Beltrami coefficient. In this note, we shall consider the converse problem, proving that the condition is not necessary.

1. INTRODUCTION

Given a hyperbolic Riemann surface R covered by the unit disk Δ, we denote by $M(R)$ the unit ball of the space $L^\infty(R)$ of all essentially bounded Beltrami differentials on R. We also denote by $SQ(R)$ the unit sphere of the space $Q(R)$ of all integrable holomorphic quadratic differentials on R. Let $\Phi : M(R) \to T(R)$ denote the canonical projection from $M(R)$ to the Teichmüller space $T(R)$ of R.

We first recall the following

Theorem 1. *Suppose $\mu \in M(R)$ is extremal and (ϕ_n) is a sequence in $SQ(R)$. If $\Phi(k_n|\phi_n|/\phi_n)$ converges in the Teichmüller metric to $\Phi(\mu)$ for some sequence (k_n), then (ϕ_n) is a Hamilton sequence for μ.*

Theorem 1 was proved by F. P. Gardiner [1] (see also [5]). In this note, we shall consider the converse problem, proving that the converse of Theorem 1 is not true. In fact, we can prove the following stronger result.

Theorem 2. *Let R be of conformal infinite type. Any non-zero extremal Beltrami coefficient $\mu \in M(R)$ possesses a Hamilton sequence (ϕ_n) such that, for any sequence (k_n) in $(0, 1)$, $\Phi(k_n|\phi_n|/\phi_n)$ does not converge to $\Phi(\mu)$ in the Teichmüller metric.*

2. PRELIMINARIES

In this section, we will recall some basic definitions and notations from Teichmüller theory. For more details see the book [2].

For a given $\mu \in M(R)$, denote by f^μ the quasiconformal mapping with domain R and Beltrami coefficient μ, which is uniquely determined up to a conformal mapping on $f^\mu(R)$. Two elements μ and ν in $M(R)$ are equivalent, which is denoted by $\mu \sim \nu$, if f^μ and f^ν are Teichmüller equivalent, meaning as usual that there

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exists a conformal mapping g from $f^\mu(R)$ onto $f^\nu(R)$ such that f^ν and $g \circ f^\mu$ are homotopic (mod ∂R). Then $T(R) = M(R)/\sim$ is the Teichmüller space of R. Recall that $\Phi : M(R) \to T(R)$ denotes the canonical projection.

For any Beltrami coefficient $\mu \in M(R)$, define

$$k_0(\mu) = \inf\{\|\nu\|_\infty : \nu \sim \mu\},$$

and set

$$K_0(\mu) = \frac{1 + k_0(\mu)}{1 - k_0(\mu)}.$$

Similarly, we define

$$h(\mu) = \inf\{\|\nu|R - E\|_\infty : \nu \sim \mu, E \subset R \text{ compact}\},$$

and let

$$H(\mu) = \frac{1 + h(\mu)}{1 - h(\mu)}.$$

Now the Teichmüller distance between points $\Phi(\mu_1)$ and $\Phi(\mu_2)$ is defined as

$$d(\Phi(\mu_1), \Phi(\mu_2)) = \frac{1}{2} \log K_0(\mu),$$

where μ is the Beltrami coefficient of the mapping $f^{\mu_1} \circ (f^{\mu_2})^{-1}$.

We say that $\mu \in M(R)$ is extremal if $\|\mu\|_\infty = k_0(\mu)$. It is well known (see [3], [4], [6] or Chapter 6 in [2]) that μ is extremal iff μ satisfies the Hamilton-Krushkal condition, that is, there exists a sequence (ϕ_n) in $SQ(R)$ such that

$$\lim_{n\to\infty} Re \iint_R \mu\phi_n = \|\mu\|_\infty.$$

Such a sequence (ϕ_n) is called a Hamilton sequence for μ. It is called degenerate if $\phi_n \to 0$ locally uniformly in R.

We also need two fundamental Reich-Strebel inequalities (see [6] or Chapter 6 in [2]). They are

$$\frac{1}{K_0(\mu)} \le \iint_R |\phi| \frac{|1 - \mu\frac{\phi}{|\phi|}|^2}{1 - |\mu|^2} \qquad \text{for all } \phi \in SQ(R) \tag{1}$$

and

$$K_0(\mu) \le \iint_R |\phi_0| \frac{|1 + \mu\frac{\phi_0}{|\phi_0|}|^2}{1 - |\mu|^2}, \tag{2}$$

where $\phi_0 \in SQ(R)$ satisfies $k_0(\mu)|\phi_0|/\phi_0 \sim \mu$.

3. Proof of Theorem 2

Let μ be an extremal Beltrami coefficient with $k = \|\mu\|_\infty > 0$. If $\Phi(\mu)$ is a Strebel point (for the definition of Strebel point, see, for example, [5]), then there exists some ϕ in $SQ(R)$ such that $\mu = k|\phi|/\phi$. By the density of non-Strebel differentials (see [5]), there exists a sequence (ϕ_n) in $SQ(R)$ such that $\|\phi_n - \phi\| \to 0$ as $n \to \infty$, but each point $\Phi(k_n|\phi_n|/\phi_n)$ is not a Strebel point for $k_n \in (0, 1)$. Since $\lim_{n\to\infty} \|\phi_n - \phi\| = 0$, it is obvious that (ϕ_n) is a Hamilton sequence for μ. On the other hand, since $\Phi(\mu)$ is a Strebel point, by the openness of Strebel points, $\Phi(k_n|\phi_n|/\phi_n)$ cannot converge in the Teichmüller metric to $\Phi(\mu)$ for any sequence (k_n) in (0, 1).

Now let μ be an extremal Beltrami coefficient with $k = \|\mu\|_\infty > 0$ such that $\Phi(\mu)$ is not a Strebel point, that is, μ possesses a degenerate Hamilton sequence. Let (ϕ_n) in $SQ(R)$ be such a sequence; then

$$\lim_{n\to\infty} Re \iint_R \mu\phi_n = k. \tag{3}$$

Choose a sequence $\{E_n\}$ of compact subsets of R such that

$$\iint_{E_n} |\phi_n| = 1 + o(1) \qquad \text{as } n \to \infty. \tag{4}$$

Define $\mu_n = \mu\chi_{E_n}$, where χ denotes the characteristic function of a set.

By the fundamental inequality (1), noting (3) and (4), we get, as $n \to \infty$, that

$$\begin{aligned}
\frac{1}{K_0(\mu_n)} &\le \iint_R |\phi_n| \frac{|1-\mu_n \frac{\phi_n}{|\phi_n|}|^2}{1-|\mu_n|^2} \\
&= \iint_{E_n} |\phi_n| \frac{|1-\mu \frac{\phi_n}{|\phi_n|}|^2}{1-|\mu|^2} + \iint_{R-E_n} |\phi_n| \\
&= \iint_{E_n} |\phi_n| \frac{|1-\mu \frac{\phi_n}{|\phi_n|}|^2}{1-|\mu|^2} + o(1) \\
&= \iint_{E_n} |\phi_n| \frac{|1-\mu \frac{\phi_n}{|\phi_n|}|^2}{1-|\mu|^2} + \iint_{R-E_n} |\phi_n| \frac{|1-\mu \frac{\phi_n}{|\phi_n|}|^2}{1-|\mu|^2} + o(1) \\
&= \iint_{R} |\phi_n| \frac{|1-\mu \frac{\phi_n}{|\phi_n|}|^2}{1-|\mu|^2} + o(1) \\
&= \frac{1-k}{1+k} + o(1),
\end{aligned}$$

which implies

$$K_0(\mu_n) = \frac{1+k}{1-k} + o(1) \qquad \text{as } n \to \infty. \tag{5}$$

Noting that the boundary dilatation $H(\mu_n) = 1$, we conclude by Strebel's Frame Mapping Criterion (see Chapter 6 in [2]) that $\Phi(\mu_n)$ is a Strebel point when n is sufficiently large, so there is a ψ_n in $SQ(R)$ such that $\mu_n \sim k_0(\mu_n)|\psi_n|/\psi_n$. By the fundamental inequality (2), we get

$$\begin{aligned}
K_0(\mu_n) &\le \iint_R |\psi_n| \frac{|1+\mu_n \frac{\psi_n}{|\psi_n|}|^2}{1-|\mu_n|^2} \\
&= \iint_{E_n} |\psi_n| \frac{|1+\mu \frac{\psi_n}{|\psi_n|}|^2}{1-|\mu|^2} + \iint_{R-E_n} |\psi_n| \\
&\le \frac{1+k}{1-k} \iint_{E_n} |\psi_n| + \iint_{R-E_n} |\psi_n|, \qquad (6)
\end{aligned}$$

from which along with (5) it follows that

$$\iint_{E_n} |\psi_n| = 1 + o(1) \qquad \text{as } n \to \infty. \tag{7}$$

Consequently, by (6), (7) we have, as $n \to \infty$, that

$$\begin{aligned} K_0(\mu_n) &\le \iint_{E_n} |\psi_n| \frac{|1+\mu\frac{\psi_n}{|\psi_n|}|^2}{1-|\mu|^2} + o(1) \\ &= \iint_{E_n} |\psi_n| \frac{|1+\mu\frac{\psi_n}{|\psi_n|}|^2}{1-|\mu|^2} + \iint_{R-E_n} |\psi_n| \frac{|1+\mu\frac{\psi_n}{|\psi_n|}|^2}{1-|\mu|^2} + o(1) \\ &= \iint_{R} |\psi_n| \frac{|1+\mu\frac{\psi_n}{|\psi_n|}|^2}{1-|\mu|^2} + o(1), \end{aligned}$$

which, by (5), forces that

$$\lim_{n\to\infty} Re \iint_R \mu\psi_n = k,$$

that is, (ψ_n) is a Hamilton sequence for μ.

Now we prove that, for any sequence (k_n) in (0, 1), $\Phi(k_n|\psi_n|/\psi_n)$ does not converge to $\Phi(\mu)$ in the Teichmüller metric.

In fact, by definition, the Beltrami coefficient ν_n of the mapping $f^{\mu_n} \circ (f^\mu)^{-1}$ is $\nu\chi_{f^\mu(R-E_n)}$, where ν is the Beltrami coefficient of the inverse mapping $(f^\mu)^{-1}$, that is

$$\nu = -\mu\frac{\overline{\partial f^\mu}}{\partial f^\mu} \circ (f^\mu)^{-1}.$$

Since μ is extremal which possesses a degenerate Hamilton sequence, so does ν. Noting that $f^\mu(E_n)$ is compact in $f^\mu(R)$, we conclude that $\nu_n = \nu\chi_{f^\mu(R-E_n)}$ is also extremal. Consequently,

$$d(\Phi(\mu_n), \Phi(\mu)) = \frac{1}{2}\log K_0(\nu_n) = \frac{1}{2}\log\frac{1+k}{1-k}.$$

So $\Phi(k_0(\mu_n)|\psi_n|/\psi_n) = \Phi(\mu_n)$ does not converge to $\Phi(\mu)$ in the Teichmüller metric. Noting that $k_0(\mu_n) = k + o(1)$ as $n \to \infty$, we conclude that, for any (k_n) with $k_n \in (0,1)$, $\Phi(k_n|\psi_n|/\psi_n)$ does not converge to $\Phi(\mu)$ in the Teichmüller metric.

Now the proof of Theorem 2 is complete.

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Theorem 2 says that the converse of Theorem 1 is not true. On the other hand, by the density of Strebel points (see [5]), for any point τ in the Teichmüller space $T(R)$, there exist a sequence (k_n) in (0, 1) and a sequence (ϕ_n) in $SQ(R)$ such that $\Phi(k_n|\phi_n|/\phi_n)$ converges in the Teichmüller metric to τ. Now Theorem 1 implies that (ϕ_n) is a Hamilton sequence for any extremal Beltrami coefficient μ in the class τ. We state this as

Proposition 3. *Any point τ in the Teichmüller space $T(R)$ possesses a sequence (ϕ_n) in $SQ(R)$ such that (ϕ_n) is a Hamilton sequence for any extremal Beltrami coefficient μ in the class τ and $\Phi(k_n|\phi_n|/\phi_n)$ converges in the Teichmüller metric to $\Phi(\mu)$ for some sequence (k_n) in $(0,1)$.*

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References

1. F. P. Gardiner, *Approximation of infinite dimensional Teichmüller spaces*, Tran. Amer. Math. Soc. 282(1984), 367-383. MR **85f:**30082
2. F. P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley-Interscience, New York, 1987. MR **88m:**32044
3. R. S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*, Tran. Amer. Math. Soc. 138(1969), 399-406. MR **39:**7093
4. S. L. Krushkal, *Extremal quasiconformal mappings*, Siberian Math. J. 10(1969), 573–583. MR **39:**2972
5. N. Lakic, *Strebel points*, Proceedings of the Bers' Colloquium, AMS 1997. MR **99c:**32027
6. E. Reich and K. Strebel, *Extremal quasiconformal mappings with given boundary values*, In: Contributions to Analysis, A collection of papers dedicated to Lipman Bers. Academic Press, New York, 1974, pp. 375-391. MR **50:**13511

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