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CONTINUATION METHOD FOR α-SUBLINEAR MAPPINGS

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ABSTRACT. Let B be a real Banach space partially ordered by a closed convex cone P with nonempty interior $\overset{\circ}{P}$. We study the continuation method for the monotone operator $A:\overset{\circ}{P}\to\overset{\circ}{P}$ which satisfies

$$A(tx)\geq t^{\alpha(a,b)}\,A(x),$$

for all $x\in\overset{\circ}{P}$, $t\in[a,\,b]\subset(0,\,1)$, where $\alpha(a,b)\in(0,\,1)$. Thompson's metric is among the main tools we are using.

1. Introduction

Let B be a real Banach space partially ordered by a closed convex cone P with nonempty interior, which is denoted by $\overset{\circ}{P}$. Suppose $A:\overset{\circ}{P}\to\overset{\circ}{P}$ is monotone, i.e., $Ax\geq Ay$ when $x\geq y$, and satisfies

$$A(tx)\;\geq\;\varphi(t)\,A(x), \tag{1}$$

where $t\in(0,\,1)$ and φ is a positive function on $(0,\,1)$. The fixed points of this type of operator were much discussed under various assumptions on φ. Among them, M. A. Krasnosel'skiĭ studied u_0-concave operator ([5]), where $\varphi(t)=[1+\eta(x,t)]t$ with $\eta(x,t)>0$, D. Guo established the existence of the unique fixed point for α-concave operators ([4]), where $\varphi(t)=t^\alpha$ with $\alpha\in(0,\,1)$, and U. Krause proved fixed point theorems for ascending operators ([6]), where $\varphi:[0,\,1]\to[0,\,1]$ is continuous and $\lambda<\varphi(\lambda)$ for $\lambda\in(0,\,1)$. In [1], we investigated the mixed monotone counterpart of the monotone operator A which satisfies

$$A(tx)\;\geq\;t^{\alpha(a,b)}\,A(x), \tag{2}$$

for all $x\in\overset{\circ}{P}$, $t\in[a,\,b]\subset(0,\,1)$, where $\alpha(a,\,b)\in(0,\,1)$. This class of operator includes Guo's α-concave operator and U. Krause's ascending operator (see [1, Corollary 3.2]). We say that a monotone operator is α-sublinear if it satisfies (2).

One important method for solving an operator equation $F(x)=0$ is the continuation method, i.e., to continuously deform F to a simpler operator G such that $G(x)=0$ is easily solved. In the present paper, we intend to discuss the continuation method for α-sublinear mappings. Our work is motivated by a paper of A. Granas ([3]).

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$x, y \in P - \{0\}$ are called comparable if there exist positive numbers λ and μ such that $\lambda x \le y \le \mu x$. This defines an equivalent relationship, and splits $P - \{0\}$ into disjoint components of P. $\overset{\circ}{P}$ is a component of P if $\overset{\circ}{P} \ne \emptyset$.

Unless specified otherwise, throughout this paper, we assume that the norm is monotone, i.e., $0 < x \le y$ implies that $\|x\| \le \|y\|$. Hence all the cones in this paper are normal, since P is normal iff B has an equivalent norm which is monotone.

Let C be a component of P and $x, y \in C$. Put

$$M(x/y) \;=\; \inf\{\lambda :\; x \le \lambda y\} \quad \text{and} \quad M(y/x) \;=\; \inf\{\mu :\; y \le \mu x\}.$$

Thompson's metric is defined by

$$\bar{d}(x, y) \;=\; \ln\{\max[M(x/y), M(y/x)]\}.$$

$\bar{d}(x, y)$ is a metric on C and C is complete with respect to $\bar{d}$ under our assumption on P ([7, Lemma 3]).

The following theorem is just the monotone operator version of Theorem 3.1 in [1], which was proved by appealing to Thompson's metric.

Theorem 1.1. *Let C be a component of P, and $A : C \to C$ be α-sublinear. Then A has exactly one fixed point x^* in C, and for any point $x_0 \in C$, we have $A^n(x_0) \to x^*$ as $n \to \infty$.*

We also need the following two lemmas.

Lemma 1.2 (Thompson [7])**.** *If the norm is monotone, then*

$$\|x - y\| \;\le\; 3\, b\, e^{\bar{d}(x,y) - 1}$$

for all $x, y \in P$ with $\|x\| \le b$ and $\|y\| \le b$.

Lemma 1.3. *Let $u \in \overset{\circ}{P}$ and $B(u,r) \subset P$, where $B(u,r) \;=\; \{x \in B :\; \|x-u\| \;<\; r\}$. Then*

$$\bar{d}(x, u) \;\le\; \ln\{\max(\frac{r + \|x-u\|}{r}, \frac{r}{r - \|x-u\|})\}$$

for all $x \in B(u,r)$.

Proof. Without loss of generality, we assume $x \ne u$. Then $x \in B(u,r)$ implies that $u \pm \frac{r(x-u)}{\|x-u\|} \in P$. It follows that

$$x \;\le\; \frac{r + \|x-u\|}{r}\, u \quad \text{and} \quad u \;\le\; \frac{r}{r - \|x-u\|}\, x.$$

Hence

$$\bar{d}(x, u) \;\le\; \ln\{\max(\frac{r + \|x-u\|}{r}, \frac{r}{r - \|x-u\|})\}.$$

□

Let (X, d) be a complete metric space and $D \subset X$ a closed subset. We say that $T : D \to X$ is a generalized contraction if for each $(a, b) \subset (0, \infty)$, there exists $L(a,b) \in (0, 1)$ such that

$$d(Tx, Ty) \;\le\; L(a,b)\, d(x, y), \tag{3}$$

where $x, y \in D$ and $a \le d(x,y) \le b$. The following theorem is due to M. A. Krasnosel'skiĭ ([5, Theorem 34.5], see also [2, Theorem (1.3.3)]).

Theorem 1.4 (Generalized Contraction Principle)**.** *If $T : X \to X$ is a generalized contraction, then there exists a unique fixed point x^* of T, and for any point $x \in X$, we have $\lim_{n\to\infty} x_n = x^*$, where $x_n = T^n x$, $n = 1, 2, \cdots$.*

This paper is organized as follows. In Section 2, we generalize A. Granas's main theorem in [3] to generalized contraction mappings. Section 3 discusses the continuation method for α-sublinear mappings. An example of application is given in Section 4.

2. Topological transversality for generalized contraction mappings

In this section, U stands for a bounded open set of X. Let $G(U)$ be the set of all generalized contraction mappings $T : \overline{U} \to X$, and $G_0(U) = \{T \in G(U) : (\operatorname{Fix} T) \cap \partial U = \emptyset\}$, where $\operatorname{Fix} T = \{x \in \overline{U} : x = Tx\}$. We denote $\operatorname{diam} U = \sup\{\|x - y\| : x, y \in U\}$ and $\operatorname{dist}(A_1, A_2) = \inf\{\|x_1 - x_2\| : x_1 \in A_1, x_2 \in A_2\}$, where A_1 and A_2 are subsets of X.

We say $T \in G_0(U)$ is traverse or essential (cf. [2, pp. 58-60] and [3]) if T has a fixed point, i.e., the graph of T crosses or traverses the diagonal of $U \times X$. The following theorem discusses the topological transversality for operators in $G_0(U)$.

Theorem 2.1. *Suppose $\{H_t\} \subset G_0(U)$, $t \in [0, 1]$, satisfy:*

(H1) For each $(a, b) \subset (0, \infty)$, there exists $L(a, b) \in (0, 1)$ such that

$$d(H_t(x_1), H_t(x_2)) \le L(a, b)\, d(x_1, x_2) \tag{4}$$

for all $t \in [0, 1]$ and $x_1, x_2 \in \overline{U}$ with $a \le d(x_1, x_2) \le b$, where $L(a, b)$ is independent of t.

(H2) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(H_{t_1}(x), H_{t_2}(x)) \le \varepsilon$ for all $x \in \overline{U}$ and $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, where δ is independent of x.

If H_0 has a fixed point in U, then so does H_t for each $t \in [0, 1]$.

Proof. Let $\Lambda = \{\lambda \in [0, 1] : x = H_\lambda(x) \text{ for some } x \in U\}$. $\Lambda \ne \emptyset$ since $0 \in \Lambda$.

(i) Λ is closed in $[0, 1]$.

Let $\lambda_n \to \lambda_0$ with $\lambda_n \in \Lambda$ and $x_n \in U$ such that $x_n = H_{\lambda_n} x_n$. Then

$$d(x_n, x_m) \le d(H_{\lambda_n}(x_n), H_{\lambda_m}(x_n)) + d(H_{\lambda_m}(x_n), H_{\lambda_m}(x_m)). \tag{5}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Otherwise, for any $k > 0$, there exist $n_k, m_k > k$ such that $d(x_{n_k}, x_{m_k}) \ge \delta$, where δ is a positive constant. Let $M = \operatorname{diam} U$. (5) leads to

$$d(x_{n_k}, x_{m_k}) \le d(H_{\lambda_{n_k}}(x_{n_k}), H_{\lambda_{m_k}}(x_{n_k})) + L(\delta, M)\, d(x_{n_k}, x_{m_k}),$$

and so

$$\delta \le d(x_{n_k}, x_{m_k}) \le \frac{d(H_{\lambda_{n_k}}(x_{n_k}), H_{\lambda_{m_k}}(x_{n_k}))}{1 - L(\delta, M)}. \tag{6}$$

By (H2), $d(H_{\lambda_{n_k}}(x_{n_k}), H_{\lambda_{m_k}}(x_{n_k})) \to 0$ as $k \to \infty$. We reach a contradiction from (6). Hence there exists $x_0 \in \overline{U}$ such that $x_n \to x_0$.

On the other hand,

$$\begin{aligned} d(x_n, H_{\lambda_0}(x_0)) &\le d(H_{\lambda_n}(x_n), H_{\lambda_0}(x_n)) + d(H_{\lambda_0}(x_n), H_{\lambda_0}(x_0)) \\ &\le d(H_{\lambda_n}(x_n), H_{\lambda_0}(x_n)) + d(x_n, x_0) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $x_0 = H_{\lambda_0}(x_0)$. Since $(\operatorname{Fix} H_{\lambda_0}) \cap \partial U = \emptyset$, $x_0 \in U$ and $\lambda_0 \in \Lambda$.

(ii) Λ is open in [0, 1].

Let $\lambda_0 \in \Lambda$ and $x_0 = H_{\lambda_0}(x_0)$, where $x_0 \in U$. Choose $r > 0$ such that $r < \operatorname{dist}(x_0, \partial U)$. There exists $\varepsilon_1 > 0$ so that $d(H_\lambda(x_0), H_{\lambda_0}(x_0)) < \frac{r}{2}$ when $\|\lambda - \lambda_0\| < \varepsilon_1$. Hence for $\lambda \in [0, 1] \cap (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1)$ and $x \in B(x_0, \frac{r}{2})$,

$$\begin{aligned} d(H_\lambda(x), x_0) &\leq d(H_\lambda(x), H_\lambda(x_0)) + d(H_\lambda(x_0), H_{\lambda_0}(x_0)) \\ &\leq d(x, x_0) + \frac{r}{2} \\ &\leq r. \end{aligned}$$

For $(1 - L(\frac{r}{2}, r))\, r$, there exists $\varepsilon_2 > 0$ such that $d(H_\lambda(x_0), H_{\lambda_0}(x_0)) < (1 - L(\frac{r}{2}, r))\, r$ when $\|\lambda - \lambda_0\| < \varepsilon_2$. Then for $\lambda \in [0, 1] \cap (\lambda_0 - \varepsilon_2, \lambda_0 + \varepsilon_2)$ and $x \in U$ with $\frac{r}{2} \leq d(x, x_0) \leq r$,

$$\begin{aligned} d(H_\lambda(x), x_0) &\leq d(H_\lambda(x), H_\lambda(x_0)) + d(H_\lambda(x_0), H_{\lambda_0}(x_0)) \\ &\leq L(\frac{r}{2}, r)\, d(x, x_0) + (1 - L(\frac{r}{2}, r))\, r \\ &\leq r. \end{aligned}$$

Put $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. For all $\lambda \in [0, 1] \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, we have $H_\lambda : \overline{B}(x_0, r) \to \overline{B}(x_0, r)$. By the Generalized Contraction Principle, there exists $x \in \overline{B}(x_0, r) \subset U$ such that $H_\lambda(x) = x$. We conclude that $\lambda \in \Lambda$, and $[0, 1] \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subset \Lambda$.

Therefore $\Lambda \neq \emptyset$ is both open and closed, and consequently $\Lambda = [0, 1]$. □

The following example illustates that Theorem 2.1 is indeed more general than Theorem 3.1 in [3].

Example 2.2. Let $X = [0, \infty)$, $d(x, y) = \|x - y\|$ and $G_t(x) = \frac{x}{1+x} + t$, where $x, y \in X$ and $t \in [0, 1]$. Consider $U = [0, 2)$. Then $\partial U = \{2\}$. For each $t \in [0, 1]$, we have

$$d(G_t(x), G_t(y)) = \frac{\|x - y\|}{(1 + x)(1 + y)} \leq \frac{\|x - y\|}{1 + \|x - y\|} = \frac{1}{1 + d(x, y)}\, d(x, y).$$

For $0 < a \leq d(x, y) \leq b < \infty$, we can put $L(a, b) = \frac{1}{1+a}$. Hence G_t is a generalized contraction, however it is not a contraction in the usual sense. Since 2 is not a fixed point for any G_t and G_0 has a fixed point $0 \in U$, we apply Theorem 2.1 to conclude that G_t has a fixed point in $U = [0, 2)$ for each $t \in [0, 1]$.

3. Continuation method for α-sublinear mappings

In this section, we will use Thompson's metric and Theorem 2.1 as tools to study α-sublinear mappings.

Theorem 3.1. *Let $S_t : \overset{\circ}{P} \to \overset{\circ}{P}$ be monotone for each $t \in [0, 1]$, and satisfy:*

(H1) For each $[a, b] \subset (0, 1)$, there exists $\alpha(a, b) \in (0, 1)$ such that

$$S_t(cx) \geq c^{\alpha(a,b)} S_t(x) \tag{7}$$

for all $x \in \overset{\circ}{P}$ and $c \in [a, b]$, where $\alpha(a, b)$ is independent of $t \in [0, 1]$.

(H2) There exists a bounded open set U with $\overline{U} \subset \overset{\circ}{P}$ and dist$(\overline{U}, \partial P) = r > 0$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|S_{t_1}(x) - S_{t_2}(x)\| < \varepsilon$$

for all $x \in \overline{U}$ and $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, where δ is independent of x.

Suppose $(\mathrm{Fix\,S_t}) \cap \partial \mathrm{U} = \emptyset$ *for all* $t \in [0, 1]$. *If* S_0 *has a fixed point in* U, *then so does* S_t *for each* $t \in [0, 1]$, *and the sequence* $\{S_t^n(x)\}$ *converges to the unique fixed point of* S_t *for any* $x \in \overset{\circ}{P}$.

Proof. Let $x,\, y \in \overset{\circ}{P}$ with $\bar{d}(x, y) \in [-\ln b,\, -\ln a]$, where $[a,\, b] \subset (0,\, 1)$. Without loss of generality, assume $M(x/y) \geq M(y/x)$. Then $\bar{d}(x, y) = \ln M(x/y)$ and $\frac{1}{b} \leq M(x/y) \leq \frac{1}{a}$. Now

$$\begin{aligned} S_t(x) &\geq S_t(\, M(y/x)^{-1} y\,) \\ &\geq S_t(\, M(x/y)^{-1} y\,) \\ &\geq M(x/y)^{-\alpha(a,b)} S_t(y). \end{aligned}$$

Thus $M(\, S_t(y)/S_t(x)\,) \leq M(x/y)^{\alpha(a,b)}$. On the other hand,

$$\begin{aligned} S_t(y) &\geq S_t(\, M(x/y)^{-1} x\,) \\ &\geq M(x/y)^{-\alpha(a,b)} S_t(x) \end{aligned}$$

implies that $M(\, S_t(x)/S_t(y)\,) \leq M(x/y)^{\alpha(a,b)}$. Hence

$$\begin{aligned} \bar{d}(S_t(x),\, S_t(y)) &\leq \ln[M(x/y)^{\alpha(a,b)}] \\ &= \alpha(a,b)\, \ln M(x/y) \\ &= L(-\ln b,\, -\ln a)\, \bar{d}(x,y), \end{aligned}$$

where $L(-\ln b,\, -\ln a) = \alpha(a,b)$.

Lemma 1.2 and Lemma 1.3 imply that U is also open in Thompson's metric, and its closure $\overline{U}$ and boundary ∂U are identical in both the norm topology and Thompson's metric topology.

Let $\varepsilon > 0$ be given. There exists $\varepsilon_1 \in (0,\, r)$ such that

$$\ln\{\, \max(\frac{r+\beta}{r},\, \frac{r}{r-\beta})\,\} < \varepsilon$$

for all $\beta \in [0,\, \varepsilon_1)$. By (H2), there exists $\delta > 0$ such that $\|S_{t_1}(x) - S_{t_2}(x)\| < \varepsilon_1$ for all $x \in \overline{U}$ and $t_1,\, t_2 \in [0,\, 1]$ with $|t_1 - t_2| < \delta$. Using Lemma 1.3, we have

$$\bar{d}(S_{t_1}(x),\, S_{t_2}(x)) \;\leq\; \ln\{\max(\frac{r+\beta}{r},\, \frac{r}{r-\beta})\} \;<\; \varepsilon$$

for all $x \in \overline{U}$ and $t_1,\, t_2 \in [0,\, 1]$ with $|t_1 - t_2| < \delta$. If $(\mathrm{Fix}\, S_t) \cap \partial U = \emptyset$ for all $t \in [0,\, 1]$ and S_0 has a fixed point in U, then we can apply Theorem 2.1 to conclude that S_t has a fixed point in U for each $t \in [0,\, 1]$.

(H1) implies that the sequence $\{S_t^n(x)\}$ converges to the unique fixed point of S_t for any $x \in \overset{\circ}{P}$ by Theorem 1.1. □

The following is a nonlinear alternative theorem for α-sublinear mappings.

Theorem 3.2. *Let* $A : \overset{\circ}{P} \to \overset{\circ}{P}$ *be an* α*-sublinear mapping and* U *be a nonempty open bounded subset with* $\overline{U} \subset \overset{\circ}{P}$ *and* $\mathrm{dist}(\bar{U}, \partial P) > 0$. *If* $A(\overline{U})$ *is bounded, then* A *has at least one of the following properties:*

(i) A *has a unique fixed point in* $\overline{U}$, *and the sequence* $A^n(x)$ *converges to that fixed point for any* $x \in \overset{\circ}{P}$.

(ii) $A(\partial U)$ *contains a point of some exterior ray, i.e., there exists* $x_0 \in U$ *such that* $Ay_0 = x_0 + \tau\,(y_0 - x_0)$ *for some* $\tau > 1$ *and* $y_0 \in \partial U$.

Proof. Let $x_0 \in U$ and consider $S_t(x) = t\,Ax + (1-t)\,x_0$. By the definition of α-sublinear mapping, for each $[a,b] \subset (0,1)$, there exists $\alpha(a,b) \in (0,\,1)$ such that for all $x \in \overset{\circ}{P}$ and $c \in [a,b]$,

$$\begin{aligned} S_t(cx) &= t\,A(cx) + (1-t)\,x_0 \\ &\geq t\,c^{\alpha(a,b)}\,Ax + (1-t)\,x_0 \\ &\geq c^{\alpha(a,b)}\,(t\,Ax + (1-t)\,x_0) \\ &= c^{\alpha(a,b)}\,S_t(x). \end{aligned}$$

Let $M = \sup\{\|y\| :\, y \in A(\overline{U})\}$. For any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2\,\max\{M, \|x_0\|\}}$. Then for $x \in \overline{U}$ and $t_1,\, t_2 \in [0,\,1]$ with $|t_1 - t_2| < \delta$,

$$\begin{aligned} \|S_{t_1}(x) - S_{t_2}(x)\| &= \|(t_1 - t_2)\,Ax - (t_1 - t_2)\,x_0\| \\ &\leq |t_1 - t_2|\,\|Ax\| + |t_1 - t_2|\,\|x_0\| \\ &< \varepsilon. \end{aligned}$$

Note that S_0 has a fixed point $x_0 \in U$. Assume that A does not have a fixed point in $\overline{U}$, then by Theorem 3.1, there exists $y_0 \in \partial U$ and $t \in (0,\,1)$ such that $S_t(y_0) = y_0$, i.e., $t\,Ay_0 + (1-t)\,x_0 = y_0$. It follows that $Ay_0 = x_0 + \tau(y_0 - x_0)$, where $\tau = \frac{1}{t} > 1$. □

Remark. The distinction between cases (i) and (ii) in Theorem 3.2 cannot be sharpened to a proper alternative. Let's consider the so-called square root version of Fibonacci's rabbit population model:

$$\overset{\circ}{P} = \overset{\circ}{R}{}^2_+, \qquad A(a,b) = (\sqrt{a} + \sqrt{b}, \sqrt{a}), \quad (a,b) \in \overset{\circ}{R}{}^2_+.$$

Suppose $U = (3,4) \times (1,2) \subset \overset{\circ}{R}{}^2_+$. It is easy to check that A has a fixed point $(a^*, b^*) \approx (3.08, 1.75) \in U$ and $A(\overline{U}) = [1 + \sqrt{3},\ 2 + \sqrt{2}] \times [\sqrt{3}, 2]$. Take $y_0 = (3, 1.44) \in \partial U$; then $A(y_0) = (1.2 + \sqrt{3},\ \sqrt{3})$. Now there exists $x_0 = (4.8 - \sqrt{3},\ 2.88 - \sqrt{3}) \in U$ such that $A(y_0) = x_0 + \tau(y_0 - x_0)$ with $\tau = 2$. Hence cases (i) and (ii) in Theorem 3.2 are not mutually exclusive.

4. Example

The following example illustrates the application of Theorem 3.2 to the Dirichlet problem for a uniformly elliptic differential operator.

Let Ω be a bounded convex domain in R^n $(n \geq 2)$, whose boundary $\partial\Omega$ belongs to $C^{2+\mu}$ $(0 < \mu < 1)$ and consider the Dirichlet problem

$$\begin{cases} Lu = f(x,u), \\ u|_{\partial\Omega} = 0, \end{cases} \tag{8}$$

where

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)\,u$$

is a uniformly elliptic differential operator, i.e., there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \geq \nu\,|\xi|^2, \quad x \in \overline{\Omega}, \ \ \xi = (\xi_1, \cdots, \xi_n) \in R^n,$$

and $a_{ij}(x) = a_{ji}(x)$, $c(x) \geq 0$, all coefficients a_{ij} , b_i , $c \in C^{\mu}(\overline{\Omega})$.

Suppose $f(x,u) > 0$ is continuous for all $x \in \overline{\Omega}$ and $u \geq 0$. The solution of (8) is equivalent to the fixed point of the integral operator

$$Au(x) = \int_{\overline{\Omega}} G(x,y)\, f(y,u(y))\, dy, \tag{9}$$

where $G(x,y)$ is the corresponding Green function which satisfies

$$0 < G(x,y) < \begin{cases} k_0\, |x-y|^{2-n}, & n > 2, \\ k_0\, |\ln|x-y||, & n = 2, \end{cases}$$

where $x,\, y, \in \Omega$ and $x \neq y$.

It is well known that A is monotone and completely continuous from P into P (see [4, pp. 60-62]), where $P = \{\, u \in C(\overline{\Omega} \mid u(x) \geq 0,\ \forall x \in \overline{\Omega}\}$. The sup norm of $C(\overline{\Omega})$ is monotone in the partial order introduced by cone P. Note that $\mathring{P} = \{\, u \in C(\overline{\Omega} \mid u(x) > 0,\ \forall x \in \overline{\Omega}\}$, and it is easy to see $A : \mathring{P} \to \mathring{P}$. Let $U = \{\, u \in C(\overline{\Omega} \mid m < u(x) < M,\ \forall x \in \overline{\Omega}\}$, where m and M are positive constants. Then $\overline{U} \subset \mathring{P}$ and $A(\overline{U})$ is bounded due to the complete continuity of A.

If there exists a lower semicontinuous function $\phi :\ (0,\,1) \to (0,\,1)$ such that $\phi(r) > r$ and

$$f(x,tu) \geq \phi(t)\, f(x,u),$$

then $A(tu) \geq \phi(t)\, A(u)$. This implies that A is α-linear by observing $\phi(t) = t^{\log_t \phi(t)}$ and $\log_t \phi(t)$ attains its maximum $\alpha(a,b)$ on each $[a,\, b] \subset (0,\,1)$ due to the lower semicontinuity of ϕ. Applying Theorem 3.2, we have at least one of the following:

(i) A has a unique fixed point $u_0 \in \overline{U}$ and the sequence

$$u_{n+1}(x) = \int G(x,y)\, f(y,u_n(y))\, dy, \quad n = 1,\, 2,\, \cdots,$$

converges to $u_0(x)$ in sup norm for any initial function $u_1 \in C(\overline{\Omega})$ with $u_1(x) > 0$ for all $x \in \overline{\Omega}$.

(ii) $A(\partial U)$ contains a point of some exterior ray, i.e., there exists $u_0 \in C(\overline{\Omega})$ with $m < u(x) < M$, $x \in \overline{\Omega}$, such that

$$\int G(x,y)\, f(y,v_0(y))\, dy = u_0 + \tau\,(v_0(x) - u_0(x))$$

for some $\tau > 1$ and $v_0 \in \partial U$.

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