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ON Δ–GOOD MODULE CATEGORIES WITHOUT SHORT CYCLES

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(Communicated by Ken Goodearl)

To our teacher Shaoxue Liu on the occasion of his 70th birthday

ABSTRACT. Let A be a quasi–hereditary algebra, and $\mathcal{F}(\Delta)$ the Δ–good module category consisting of A–modules which have a filtration by standard modules. An indecomposable module M in $\mathcal{F}(\Delta)$ is said to be on a short cycle in $\mathcal{F}(\Delta)$ if there exist an indecomposable module N in $\mathcal{F}(\Delta)$ and a chain of two nonzero noninvertible maps $M \to N \to M$. It is shown that two indecomposable modules in $\mathcal{F}(\Delta)$ are isomorphic if they are not on short cycles in $\mathcal{F}(\Delta)$ and have the same composition factors. Moreover, if there is no short cycle in $\mathcal{F}(\Delta)$, we show that $\mathcal{F}(\Delta)$ is finite, that is, there are only finitely many isomorphism classes of indecomposables in $\mathcal{F}(\Delta)$. This is an analogue to a result in a complete module category proved by Happel and Liu.

INTRODUCTION AND PRELIMINARIES

Let A be an artin algebra over a commutative artin ring R, mod A the category of finitely generated left A–modules. A cycle in mod A is a sequence $X_0 \to X_1 \to \cdots \to X_n = X_0$ of nonzero noninvertible maps between indecomposable A–modules. Such a cycle is said to be short if $n = 2$. Cycles in mod A have been widely studied (see, for example, [8], [1], [11], [6], [5]). In particular, in [5] it is proved that A is of finite type if mod A contains no short cycles.

In the study of a quasi–hereditary algebra A, instead of the complete module category mod A, one is mainly interested in the category $\mathcal{F}(\Delta)$ consisting of A–modules which have a filtration by standard modules. It is proved by Ringel in [9] that $\mathcal{F}(\Delta)$ has almost split sequences. Thus the usual techniques of representation theory can be adapted. In this paper we consider the notion of a short cycle in $\mathcal{F}(\Delta)$, i.e. a chain of nonzero noninvertible maps $M \to N \to M$ with M and N indecomposable in $\mathcal{F}(\Delta)$. We will show that an analogous result to the one of Happel and Liu in [5] holds for $\mathcal{F}(\Delta)$, namely, if $\mathcal{F}(\Delta)$ contains no short cycles, then $\mathcal{F}(\Delta)$ is finite.

We recall the definition of a quasi–hereditary algebra. Let A be an artin algebra. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of all A–modules which have a Θ–filtration, that is, a filtration $0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$ such that each factor M_{i-1}/M_i is isomorphic to one object in Θ

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for $1 \leq i \leq t$. The modules in $\mathcal{F}(\Theta)$ are called Θ-good modules, and the category $\mathcal{F}(\Theta)$ is called the Θ-good module category.

Let $E(\lambda)$, $\lambda \in \Lambda$, be a complete set of simple A–modules, where Λ is a finite partially ordered set. For each $\lambda \in \Lambda$, let $P(\lambda)$ (or $P_A(\lambda)$) be a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let Δ (respectively, ∇) be the full subcategory consisting of all $\Delta(\lambda)$, $\lambda \in \Lambda$ (respectively, all $\nabla(\lambda)$, $\lambda \in \Lambda$). We call modules in Δ standard modules and ones in ∇ costandard modules.

The algebra A, or better the pair (A, Λ), is called quasi–hereditary if for each $\lambda \in \wedge$ we have

(i) $\mathrm{End}_A(\Delta(\lambda))$ is a division ring;

(ii) $P(\lambda) \in \mathcal{F}(\Delta)$, and moreover, $P(\lambda)$ has a Δ–filtration with factors $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

In case A is quasi–hereditary, the poset $\wedge$ is called the weight poset of A.

Now we review some basic facts from [9] which will be needed later on. First, the Δ–good module category $\mathcal{F}(\Delta)$ of a quasi–hereditary algebra A admits the following description:

$$\begin{aligned} \mathcal{F}(\Delta) &= \{X \in \mathrm{mod}\ A | \mathrm{Ext}^1_A(X, \nabla) = 0\} \\ &= \{X \in \mathrm{mod}\ A | \mathrm{Ext}^i_A(X, \nabla) = 0 \ \text{ for all } i \geq 1\}. \end{aligned}$$

Dually, one has that

$$\begin{aligned} \mathcal{F}(\nabla) &= \{Y \in \mathrm{mod}\ A | \mathrm{Ext}^1_A(\Delta, Y) = 0\} \\ &= \{X \in \mathrm{mod}\ A | \mathrm{Ext}^i_A(\Delta, Y) = 0 \ \text{ for all } i \geq 1\}. \end{aligned}$$

Thus $\mathcal{F}(\Delta)$ is closed under kernels of surjective maps, and $\mathcal{F}(\nabla)$ is closed under cokernels of injective maps.

Further, for each $\lambda \in \Lambda$, there exist an indecomposable module $T(\lambda)$ and exact sequences

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0 \quad \text{and} \quad 0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0,$$

where $X(\lambda)$ is filtered with factors $\Delta(\mu)$, $\mu < \lambda$, and $Y(\lambda)$ is filtered with factors $\nabla(\mu)$, $\mu < \lambda$. The module $T := \bigoplus_{\lambda \in \Lambda} T(\lambda)$ is a generalized tilting and cotilting modules such that $\mathrm{add}\, T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Moreover, the Ext–projective modules in $\mathcal{F}(\Delta)$ are the projective A–modules and the Ext–injective modules in $\mathcal{F}(\Delta)$ are the A–modules in $\mathrm{add}\, T$.

Throughout the paper A will denote a quasi–hereditary algebra, and $\mathcal{F}(\Delta)$ the Δ–good module category. For a non–projective A–module X in $\mathcal{F}(\Delta)$, by $\tau_\Delta X$ we denote the left-hand term in an almost split sequence $0 \to \tau_\Delta X \to E \to X \to 0$ in $\mathcal{F}(\Delta)$. (Note that by τ_A we will denote the usual Auslander–Reiten translate DTr in the complete module category mod A.) Finally, by $\Gamma_{\mathcal{F}(\Delta)}$ we denote the Auslander–Reiten quiver of $\mathcal{F}(\Delta)$ (see the definition in [10]), and by $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ the stable Auslander–Reiten quiver of $\mathcal{F}(\Delta)$, i.e. the full translation subquiver of $\Gamma_{\mathcal{F}(\Delta)}$ obtained by deleting all vertices of the form $\tau_\Delta^{-t} p$, or of the form $\tau_\Delta^t g$ $(t \geq 0)$, where p is a projective vertex, and g an injective vertex.

1. Δ–GOOD MODULES DETERMINED BY THEIR COMPOSITION FACTORS

Let A be a quasi–hereditary algebra with a weight poset Λ and $\mathcal{F}(\Delta)$ the Δ–good module category of A. For every pair of A–modules X and Y, by $\langle X, Y\rangle$ we denote the length as an R–module of $\mathrm{Hom}_A(X, Y)$.

Lemma 1.1. *Let M, N be two modules in $\mathcal{F}(\Delta)$. Then the following are equivalent:*
(i) *M and N have the same composition factors,*
(ii) *$\langle P(\lambda), M\rangle = \langle P(\lambda), N\rangle$ for all $\lambda \in \Lambda$,*
(iii) *$\langle M, I(\lambda)\rangle = \langle N, I(\lambda)\rangle$ for all $\lambda \in \Lambda$,*
(iv) *$\langle M, T(\lambda)\rangle = \langle N, T(\lambda)\rangle$ for all $\lambda \in \Lambda$,*
(v) *$\langle M, \nabla(\lambda)\rangle = \langle N, \nabla(\lambda)\rangle$ for all $\lambda \in \Lambda$,*
(vi) *$\langle M, X\rangle = \langle N, X\rangle$ for all $X \in \mathcal{F}(\nabla)$.*

Proof. (i)⇔(ii)⇔(iii) are well–known. (v)⇒(vi)⇒(iv) are clear.

(iv)⇒(iii): Suppose that $\langle M, T(\lambda)\rangle = \langle N, T(\lambda)\rangle$ for all $\lambda \in \Lambda$. Then $\langle M, T'\rangle = \langle N, T'\rangle$ for all $T' \in \operatorname{add} T$. Since T is a cotilting module, for each $I(\lambda)$, there is an exact sequence

$$0 \longrightarrow T_s \xrightarrow{f_s} \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} I(\lambda) \longrightarrow 0,$$

where $T_i \in \operatorname{add} T$ for all $0 \le i \le s$. By applying $\mathrm{Hom}_A(M, -)$ and $\mathrm{Hom}_A(N, -)$, one obtains the exact sequences (since all $\ker f_i \in \mathcal{F}(\nabla)$ and $\mathrm{Ext}^1(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0$):

$$0 \to \mathrm{Hom}(M, T_s) \to \cdots \to \mathrm{Hom}(M, T_1) \to \mathrm{Hom}(M, T_0) \to \mathrm{Hom}(M, I(\lambda)) \to 0,$$

$$0 \to \mathrm{Hom}(N, T_s) \to \cdots \to \mathrm{Hom}(N, T_1) \to \mathrm{Hom}(N, T_0) \to \mathrm{Hom}(N, I(\lambda)) \to 0.$$

This implies that $\langle M, I(\lambda)\rangle = \langle N, I(\lambda)\rangle$.

(iii)⇒(v) follows from the fact that the global dimension of a quasi–hereditary algebra is always finite. □

Lemma 1.2. *Let M, N be modules in $\mathcal{F}(\Delta)$. Assume that M and N have the same composition factors. Then, for each indecomposable module $X \in \mathcal{F}(\Delta)$,*

$$\langle X, M\rangle - \langle M, \tau_\Delta X\rangle = \langle X, N\rangle - \langle N, \tau_\Delta X\rangle.$$

(If X is projective, we simply set $\tau_\Delta X = 0$.)

Proof. By [1, Theorem 1.4], it holds that

$$\begin{aligned}\langle X, M\rangle - \langle M, \tau_A X\rangle &= \langle P_0, M\rangle - \langle P_1, M\rangle \\ &= \langle P_0, N\rangle - \langle P_1, N\rangle = \langle X, N\rangle - \langle N, \tau_A X\rangle,\end{aligned}$$

where $P_1 \to P_0 \to X \to 0$ is a minimal projective resolution of X.

Let $f : Y \to \tau_A X$ be a minimal right $\mathcal{F}(\Delta)$–approximation of $\tau_A X$, which is surjective since $\mathcal{F}(\Delta)$ contains all projective modules. By [4, Proposition 9.10], one has $Y \cong \tau_\Delta X \oplus T_X$ for some $T_X \in \operatorname{add} T$. From Wakamatsu's Lemma [12] it follows that $K := \ker f$ is in $\mathcal{F}(\nabla)$. By applying $\mathrm{Hom}_A(M, -)$ and $\mathrm{Hom}_A(N, -)$ to the exact sequence $0 \to K = \ker f \to Y \xrightarrow{f} \tau_A X \to 0$, we obtain the following exact sequences:

$$0 \longrightarrow \mathrm{Hom}_A(M, K) \longrightarrow \mathrm{Hom}_A(M, Y) \longrightarrow \mathrm{Hom}_A(M, \tau_A X) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}_A(N, K) \longrightarrow \mathrm{Hom}_A(N, Y) \longrightarrow \mathrm{Hom}_A(N, \tau_A X) \longrightarrow 0.$$

Thus we get the following formulae:

$$\langle M, \tau_\Delta X\rangle + \langle M, T_X\rangle = \langle M, Y\rangle = \langle M, K\rangle + \langle M, \tau_A X\rangle,$$

$$\langle N, \tau_\Delta X\rangle + \langle N, T_X\rangle = \langle N, Y\rangle = \langle N, K\rangle + \langle N, \tau_A X\rangle.$$

By Lemma 1.1, it then follows that

$$\langle X, M\rangle - \langle M, \tau_\Delta X\rangle = \langle X, M\rangle - \langle M, \tau_A X\rangle - \langle M, K\rangle + \langle M, T_X\rangle$$

$$= \langle X, N\rangle - \langle N, \tau_A X\rangle - \langle N, K\rangle + \langle N, T_X\rangle = \langle X, N\rangle - \langle N, \tau_\Delta X\rangle.$$

This finishes the proof of the lemma. □

Proposition 1.3. *Let M and N be indecomposable modules in $\mathcal{F}(\Delta)$. Assume that M and N have the same composition factors and are not on short cycles in $\mathcal{F}(\Delta)$. Then M and N are isomorphic.*

Proof. Note that $\langle M, \tau_\Delta M\rangle = 0$. Indeed, suppose $\langle M, \tau_\Delta M\rangle \neq 0$. There is a non–zero map $\phi : M \to \tau_\Delta M$. Consider the almost split sequence $0 \to \tau_\Delta M \xrightarrow{f} E \xrightarrow{g} M \to 0$ in $\mathcal{F}(\Delta)$. Since f is injective, it holds that $f\phi \neq 0$. Thus E contains an indecomposable summand E_i such that $\mathrm{rad}(M, E_i) \neq 0$. This implies the existence of a chain of non–zero non–invertible maps $M \to E_i \to M$, i.e. M is on a short cycle. This gives a contradiction.

By Lemma 1.2 we have the formula

$$\langle X, M\rangle - \langle M, \tau_\Delta X\rangle = \langle X, N\rangle - \langle N, \tau_\Delta X\rangle$$

for each indecompoable modules X in $\mathcal{F}(\Delta)$. Letting $X = M$, one obtains that $\langle M, M\rangle = \langle M, N\rangle - \langle N, \tau_\Delta M\rangle$. This implies that $\langle M, N\rangle \neq 0$, i.e. $\mathrm{Hom}_A(M, N) \neq 0$.

Similarly, one has that $\mathrm{Hom}_A(N, M) \neq 0$. If M and N are not isomorphic, there would be a short cycle $M \to N \to M$, which contradicts the assumption. Hence M and N are isomorphic. □

Corollary 1.4. *Suppose that there is no short cycle in $\mathcal{F}(\Delta)$. Then the indecomposable modules in $\mathcal{F}(\Delta)$ are determined by their composition factors.*

Corollary 1.5. *Suppose Γ is a preprojective component of $\Gamma_{\mathcal{F}(\Delta)}$. Then the modules in Γ are determined by their composition factors.*

Proof. Since Γ is preprojective, each module in Γ does not lie in a cycle in $\mathcal{F}(\Delta)$. This implies particularly that all modules in Γ are not on short cycles in $\mathcal{F}(\Delta)$. Hence, by Proposition 1.3, the modules in Γ are determined by their composition factors. □

2. Subspace categories

In this section we recall from [8, 2.5] the notion of a subspace category and some basic results. Further, we review a result in [2] which interprets a Δ–good module category in terms of a subspace category. However, the formulation is more general. The whole section serves as a tool used in the next section.

Let $\mathcal{K}$ be a Krull–Schmidt category over a commutative artin ring R, D a division ring over R which is finitely generated as an R–module, and $|\cdot| : \mathcal{K} \longrightarrow \mathrm{mod}\, D$ an additive functor. We call the pair $(\mathcal{K}, |\cdot|)$ a vectorspace category and denote by $\check{\mathcal{U}}(\mathcal{K}, |\cdot|) =: \mathcal{X}$, and call it a subspace category of $(\mathcal{K}, |\cdot|)$, the category of all triples

$V = (V_0, V_\omega, \gamma_V)$, where $V_\omega \in \text{mod } D$, $V_0 \in \mathcal{K}$ and $\gamma_V : V_\omega \to |V_0|$ is a D–linear map. A morphism from $V \to V'$ by definition is a pair (f_0, f_ω), where $f_0 : V_0 \to V'_0$ and $f_\omega : V_\omega \to V'_\omega$, such that $f_\omega \gamma_{V'} = \gamma_V |f_0|$.

Let $\mathcal{S}(\mathcal{K})$ be the class of exact sequences

$$0 \longrightarrow U \xrightarrow{\mu} V \xrightarrow{\varepsilon} W \longrightarrow 0$$

in $\mathcal{X} = \check{\mathcal{U}}(\mathcal{K}, |\cdot|)$ such that the sequence

$$0 \longrightarrow U_0 \xrightarrow{\mu_0} V_0 \xrightarrow{\varepsilon_0} W_0 \longrightarrow 0$$

is split exact in $\mathcal{K}$ and the sequence

$$0 \longrightarrow U_\omega \xrightarrow{\mu_\omega} V_\omega \xrightarrow{\varepsilon_\omega} W_\omega \longrightarrow 0$$

is exact in mod D. It is easy to see that $(\mathcal{X}, \mathcal{S}(\mathcal{K}))$ is an exact category (see, for example, [4, Chapter 9]). Then, for every pair of objects V, W in $\mathcal{X}$, we can then form the abelian group $\text{Ext}^1_{\mathcal{X}}(V, W)$ in a usual way. Clearly, each D–linear map $\delta : V_\omega \to |W_0|$ gives rise to an exact sequence in $\mathcal{S}(\mathcal{K})$ as follows:

$$\begin{tikzcd}
0 \arrow[r] & W_\omega \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[d, "\gamma_W"] & W_\omega \oplus V_\omega \arrow[r, "\begin{bmatrix}0 & 1\end{bmatrix}"] \arrow[d, "\begin{bmatrix}\gamma_W & \delta\\ 0 & \gamma_V\end{bmatrix}"] & V_\omega \arrow[r] \arrow[d, "\gamma_V"] & 0 \\
0 \arrow[r] & |W_0| \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"'] & |W_0| \oplus |V_0| \arrow[r, "\begin{bmatrix}0 & 1\end{bmatrix}"'] & |V_0| \arrow[r] & 0
\end{tikzcd}$$

We denote the equivalence class of the exact sequence above by $\bar{\delta}$. It is easy to check that, for $\delta, \delta' : V_\omega \to |W_0|$, it holds that $\bar{\delta} = \bar{\delta}'$ if and only if $\delta - \delta' = |\beta_0|\gamma_V - \gamma_W \beta_\omega$ for some $\beta_0 : V_0 \to W_0$ in $\mathcal{K}$ and some D–linear map $\beta_\omega : V_\omega \to W_\omega$. Thus, for each pair of objects V and W in $\mathcal{X}$, we obtain the following exact sequence:

$$\begin{aligned}0 \longrightarrow \text{Hom}(V, W) \xrightarrow{\nu_1} \text{Hom}_{\mathcal{K}}(V_0, W_0) \times \text{Hom}_D(V_\omega, W_\omega) \xrightarrow{\nu_2} \text{Hom}_D(V_\omega, |W_0|)\\ \xrightarrow{\nu_3} \text{Ext}^1_{\mathcal{K}}(V, W) \longrightarrow 0\end{aligned}$$

where ν_1 is the inclusion, ν_2 maps (β_0, β_ω) to $|\beta_0|\gamma_V - \gamma_W \beta_\omega$, and ν_3 maps δ to $\bar{\delta}$.

Now assume that $\mathcal{K}$ is finite, i.e. there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{K}$. Given $V = (V_0, V_\omega, \gamma_V)$ in $\mathcal{X}$, we define its dimension vector to be $\underline{\dim}_{\mathcal{X}} V := ([V_0], \dim_D V_\omega) \in G(\mathcal{K}) \times \mathbb{Z}$, where $G(\mathcal{K})$ is the free abelian group with basis the set of isomorphism classes, say $[X_1], [X_2], \cdots, [X_n]$, of indecomposable objects in $\mathcal{K}$. In $G(\mathcal{K}) \times \mathbb{Z}$, by $e(\omega)$ we denote the additional basis vector $(0, 1)$. We define on $G(\mathcal{K}) \times \mathbb{Z}$ a bilinear form $\langle -, - \rangle_{\mathcal{X}}$ as follows:

$$\begin{aligned}
&\langle [X_i], [X_j] \rangle_{\mathcal{X}} = l_R(\text{Hom}_{\mathcal{K}}(X_i, X_j)),\\
&\langle [X_i], e(\omega) \rangle_{\mathcal{X}} = 0,\\
&\langle e(\omega), [X_i] \rangle_{\mathcal{X}} = -l_R(|X_i|),\\
&\langle e(\omega), e(\omega) \rangle_{\mathcal{X}} = l_R(D).
\end{aligned}$$

where l_R denotes the length as an R–module. The corresponding quadratic form will be denoted by $q_{\mathcal{X}}$ with $q_{\mathcal{X}}(x) = \langle x, x \rangle_{\mathcal{X}}$. Note that $q_{\mathcal{X}}$ is a quadratic form with integral coefficents. We identify $G(\mathcal{K}) \times \mathbb{Z}$ with $\mathbb{Z}^{n+1}$.

Proposition 2.1. *For every V, W in $\mathcal{X}$, it holds that*

$$\langle \underline{\dim}_{\mathcal{X}} V, \underline{\dim}_{\mathcal{X}} W \rangle_{\mathcal{X}} = l_R(Hom_{\mathcal{X}}(V, W)) - l_R(Ext^1_{\mathcal{X}}(V, W)).$$

For the proof of this proposition we refer to [8, 2.5(3″)].

Now let A be a quasi–hereditary algebra with a weight poset Λ, and $\mathcal{F}(\Delta_A)$ (or simply $\mathcal{F}(\Delta)$) the Δ–good module category of A. Suppose that $\omega \in \Lambda$ is a maximal element. Thus the standard module $\Delta_A(\omega)$ corresponding to ω is the indecomposable projective module $P(\omega) = Ae_\omega$, and its endomorphism algebra $\mathrm{End}_A(P(\omega))$ is a division algebra, denoted by D. Let us denote by A_0 the factor algebra of A by the hereditary ideal $Ae_\omega A$. Then A_0 is automatically a quasi-hereditary algebra with the standard modules $\Delta_A(\lambda)$, $\lambda \in \Lambda\backslash\{\omega\}$. The projective module $P(\omega)$ defines an additive functor $\mathrm{Ext}^1_A(-, P(\omega)) : (\mathcal{F}(\Delta_{A_0}))^{op} \to \mathrm{mod}\ D$. Thus we obtain a vectorspace category $((\mathcal{F}(\Delta_{A_0}))^{op}, \mathrm{Ext}^1_A(-, P(\omega)))$.

For each module $M \in \mathcal{F}(\Delta)$, we have an exact sequence

$$0 \longrightarrow P(\omega)^m \xrightarrow{\alpha_M} M \xrightarrow{\pi_M} M_0 \longrightarrow 0,$$

where α_M denotes the canonical inclusion and where π_M is the canonical surjection from M onto the factor module $M_0 \in \mathcal{F}(\Delta_{A_0})$. Applying $\mathrm{Hom}_A(-,\, P(\omega))$ to this sequence, we get the following long exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_A(M_0,\, P(\omega)) \longrightarrow \mathrm{Hom}_A(M,\, P(\omega)) \longrightarrow \mathrm{Hom}_A(P(\omega)^m,\, P(\omega)) \\ \xrightarrow{\delta_M} \mathrm{Ext}^1_A(M_0,\, P(\omega)) \longrightarrow \mathrm{Ext}^1_A(M,\, P(\omega)) \longrightarrow 0\,. \end{aligned}$$

We define $\eta(M) = (M_0, \mathrm{Hom}_A(P(\omega)^m, P(\omega)), \delta_M) \in \check{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}}, \mathrm{Ext}^1_A(-,\, P(\omega)))$ since $\mathrm{Hom}_A(P(\omega)^m,\, P(\omega)) \cong D^m$. For each $f \in \mathrm{Hom}_A(N,\, M)$ we define $\eta(f) = (f_0,\, f_\omega)$ by the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & P(\omega)^m \arrow[r, "\alpha_M"] & M \arrow[r, "\pi_M"] & M_0 \arrow[r] & 0 \\ 0 \arrow[r] & P(\omega)^n \arrow[r, "\alpha_N"] \arrow[u, "f'_\omega"'] & N \arrow[r, "\pi_N"] \arrow[u, "f"'] & N_0 \arrow[r] \arrow[u, "f_0"'] & 0 \end{tikzcd}$$

(Here the existence of f'_ω follows from the fact $\mathrm{Hom}_A(P(\omega), M_0) = 0$ and hence f'_ω is the restriction of f onto the submodule $P(\omega)^n$.) Put $f_\omega = \mathrm{Hom}_A(f'_\omega, P(\omega))$. Then $\eta(f)$ is a morphism from $\eta(M)$ to $\eta(N)$ because we have the desired commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}_A(P(\omega)^m,\, P(\omega)) \arrow[r, "\delta_M"] \arrow[d, "\mathrm{Hom}_A(f'_\omega,\, P(\omega))"] & \mathrm{Ext}^1_A(M_0,\, P(\omega)) \arrow[r] \arrow[d, "\mathrm{Ext}^1_A(f_0,\, P(\omega))"] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}_A(P(\omega)^n,\, P(\omega)) \arrow[r, "\delta_N"] & \mathrm{Ext}^1_A(N_0,\, P(\omega)) \arrow[r] & \cdots \end{tikzcd}$$

As a conclusion, we obtain a functor

$$\eta : \mathcal{F}(\Delta_A)^{\mathrm{op}} \longrightarrow \check{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}},\, \mathrm{Ext}^1_A(-,\, P(\omega))).$$

Proposition 2.2. *The functor η is full and dense such that the kernel of η (i.e. the ideal formed by the morphisms f with $\eta(f) = 0$) is contained in the radical of $\mathcal{F}(\Delta_A)^{op}$.*

We refer to [2] for the proof of the proposition (see also [7, 3.2]).

3. Δ–GOOD MODULE CATEGORIES WITHOUT SHORT CYCLES

This section is devoted to the proof of the fact that a Δ–good module category without short cycles is finite. We keep notation from Section 2.

Let A be a quasi–hereditary algebra with a weight poset Λ, $\mathcal{F}(\Delta)$ the Δ–good module category of A. Let ω be a maximal element of Λ and A_0 the factor algebra $A/Ae_\omega A$ of A.

From now on, we suppose that $\mathcal{F}(\Delta)$ contains no short cycles. Thus the Δ–good module category $\mathcal{F}(\Delta_{A_0})$ of A_0 does not contain short cycles either. Using an inductive argument on the cardinality of the weight poset Λ, we may suppose that $\mathcal{F}(\Delta_{A_0})$ is finite. For convenience, in the following we simply write $\mathcal{K}$ for $\mathcal{F}(\Delta_{A_0})^{\mathrm{op}}$, and $\mathcal{X}$ for $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}}, \mathrm{Ext}^1_A(-, P(\omega)))$.

Proposition 3.1. (1) *Let V be indecomposable in $\mathcal{X}$. Then $End(V)$ is a division ring, and $Ext^1_{\mathcal{X}}(V,V) = 0$; thus $q_{\mathcal{X}}(\underline{\dim}_{\mathcal{X}} V) = l_R(End(V))$.*

(2) *Let V and V' be indecomposable objects in $\mathcal{X}$. If $\underline{\dim}_{\mathcal{X}} V' = \underline{\dim}_{\mathcal{X}} V$, then $V \cong V'$.*

(3) *The quadratic form $q_{\mathcal{X}}$ is weakly positive.*

Proof. (1) By Proposition 2.2, there is a full and dense functor $\eta : \mathcal{F}(\Delta_A)^{\mathrm{op}} \to \breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}}, \mathrm{Ext}^1_A(-, P(\omega))) = \mathcal{X}$ such that the kernel of η is contained in the radical of $\mathcal{F}(\Delta_A)^{\mathrm{op}}$. Thus that $\mathcal{F}(\Delta)$ contains no short cycles implies that $\mathcal{X}$ contains no short cycles.

Observe that any non-zero noninvertible endomorphism f of V gives a short cycle $V \xrightarrow{f} V \xrightarrow{f} V$, and that any non–split exact sequence (in $\mathcal{S}(\mathcal{K})$)

$$0 \longrightarrow V \longrightarrow W \longrightarrow V \longrightarrow 0$$

gives a short cycle $V \to W' \to V$, where W' is any indecomposable summand of W. This shows that $\mathrm{End}(V)$ is a division ring and $\mathrm{Ext}^1_{\mathcal{X}}(V,V) = 0$. Thus, by Proposition 2.1, it holds that $q_{\mathcal{X}}(\underline{\dim}_{\mathcal{X}} V) = l_R(\mathrm{End}(V))$.

(2) Take M, M' in $\mathcal{F}(\Delta)$ such that $\eta(M) \cong V$ and $\eta(M') \cong V'$. Then M and M' are indecomposable. By the construction of η, $\underline{\dim}_{\mathcal{X}} \eta(M) = \underline{\dim}_{\mathcal{X}} \eta(M')$ implies that M and M' have the same composition factors; thus they are isomorphic by Proposition 1.3. Hence V and V' are isomorphic.

(3) For each $0 < x = (x_1, \cdots, x_{n+1}) \in \mathbb{Z}^{n+1}$, i.e. $x \neq 0$ and $x_i \geq 0$ for $1 \leq i \leq n+1$, we choose an object V in $\mathcal{X}$ with $\underline{\dim}_{\mathcal{X}}(V) = x$ and smallest possible $l_R(\mathrm{End}(V))$. Assume that $V = \bigoplus_{i=1}^{s} V(i)$ with the $V(i)$ indecomposable. Then by (1) we have that $\mathrm{Ext}^1_{\mathcal{X}}(V(i), V(i)) = 0$ for all $1 \leq i \leq s$, and by [8, Lemma 1. (2.3)] we have that $\mathrm{Ext}^1_{\mathcal{X}}(V(i), V(j)) = 0$ for all $i \neq j$. Thus

$$q_{\mathcal{X}}(x) = q_{\mathcal{X}}(\underline{\dim}_{\mathcal{X}} V) = l_R(\mathrm{End}(V)) > 0,$$

that is, $q_{\mathcal{X}}$ is weakly positive. □

Corollary 3.2. (1) *The correspondence $M \longmapsto \underline{\dim}_{\mathcal{X}} \eta(M)$ induces an injection from the set of isomorphism classes of indecomposables in $\mathcal{F}(\Delta)$ to the set of positive vectors in $\mathbb{Z}^{n+1}$ such that $q_{\mathcal{X}}(x) > 0$. In particular, if A is a quasi–hereditary algebra over an algebraically closed field, there is a bijection between the set of isomorphism classes of indecomposables in $\mathcal{F}(\Delta)$ and the set of positive roots of $q_{\mathcal{X}}$.*

(2) *Each τ_Δ–orbit in $\Gamma_{\mathcal{F}(\Delta)}$ contains only finitely many modules.*

Proof. (1) The first assertion is a direct implication of Proposition 3.1. The second assertion follows from a similar argument in [8, 2.4(9)].

(2) Let M be an indecomposable non–projective module in $\mathcal{F}(\Delta)$. By [4, Corollary 9.6], we conclude that $\overline{\mathrm{End}(\tau_\Delta M)} \cong \underline{\mathrm{End}(M)}$, where $\overline{\mathrm{End}(\tau_\Delta M)}$ denotes the factor algebra of $\mathrm{End}(\tau_\Delta M)$ by the ideal formed by endomorphisms factoring through Ext–injectives, and $\underline{\mathrm{End}(M)}$ the factor algebra of $\mathrm{End}(M)$ by the ideal formed by endomorphisms factoring through Ext–projectives. Since both $\mathrm{End}(\tau_\Delta(M)$ and $\mathrm{End}(M)$ are division rings ($\mathcal{F}(\Delta)$ contains no short cycles), we infer that $\mathrm{End}(\tau_\Delta M) \cong \overline{\mathrm{End}(\tau_\Delta M)} \cong \underline{\mathrm{End}(M)} \cong \mathrm{End}(M)$. This implies that

$$\begin{aligned} q_{\mathcal{X}}(\underline{\dim}_{\mathcal{X}}\eta(M)) = {} & l_R(\mathrm{End}(\eta(M))) = l_R(\mathrm{End}(M)) \\ = l_R(\mathrm{End}(\tau_\Delta M)) = {} & l_R(\mathrm{End}(\eta(\tau_\Delta M))) = q_{\mathcal{X}}(\underline{\dim}_{\mathcal{X}}\eta(\tau_\Delta M)). \end{aligned}$$

For each $t \in \mathbb{Z}$, by applying an argument in [8, 1.1(2)], one can easily see that there are only finitely many positive vectors in $\mathbb{Z}^{n+1}$ with $q_{\mathcal{X}}(x) = t$. This together with (1) shows that each τ_Δ–orbit in $\Gamma_{\mathcal{F}(\Delta)}$ contains only finitely many modules. □

Theorem 3.3. *Assume that $\mathcal{F}(\Delta)$ contains no short cycles. Then $\mathcal{F}(\Delta)$ is finite.*

Proof. Suppose that $\mathcal{F}(\Delta)$ is infinite. Since, by Corollary 3.2, each τ_Δ–orbit in $\Gamma_{\mathcal{F}(\Delta)}$ contains only finitely many modules, we infer that $\Gamma_{\mathcal{F}(\Delta)}$ has infinitely many τ_Δ–orbits. By [10, Sect. 3] and using a similar argument in [5, Lemma 3], one can see that $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ has an infinite stable component, say Γ. Then it is clear that Γ contains only τ_Δ–periodic modules. By [10, Sect. 4], Γ is a stable tube. Let $r \geq 1$ be the rank of Γ and M a module in Γ with quasi–length $r+1$. Then there are two sectional paths in Γ (thus also sectional paths in $\Gamma_{\mathcal{F}(\Delta)}$):

$$M_1 \xrightarrow{f_1} M_2 \to \cdots \to M_r \xrightarrow{f_r} M_{r+1} = M,$$

$$M = N_{r+1} \xrightarrow{g_1} N_r \to \cdots \to N_2 \xrightarrow{g_r} N_1$$

such that M_i and N_i have quasi–length i for all $1 \leq i \leq r+1$. Further, we have that $M_1 = \tau_\Delta^{r+1} N_1 = N_1$. Since none of M_i and N_i is projective, we have, by the proposition in Section 2 in [10], that $f_r \cdots f_1 \neq 0$ and $g_r \cdots g_1 \neq 0$. This gives a short cycle $M_1 \to M \to N_1 = M_1$, a contradiction. Therefore, $\mathcal{F}(\Delta)$ is finite. □

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