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THE SORGENFREY LINE HAS A LOCALLY PATHWISE CONNECTED CONNECTIFICATION

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Abstract. We answer a question of Alas, Tkačenko, Tkachuk and Wilson by constructing a connected locally pathwise connected Hausdorff space in which the Sorgenfrey line can be densely embedded.

A connectification of a T_2-space X is a connected Hausdorff space Y in which X can be densely embedded. In such a case Y is called a connectification of X, and X is said to be connectifiable. In 1977 Emeryk and Kulpa showed, in response to a question of van Douwen, that the Sorgenfrey line cannot be densely embedded in a connected T_3-space, although it is connectifiable (see [2]). In a recent paper Alas, Tkačenko, Tkachuk and Wilson asked if the Sorgenfrey line has a locally connected connectification ([1]).

The aim of this paper is to give a strongly positive answer to the above question.

We refer the reader to [3] for notations and terminology not explicitly given.

Recall that a collection of pairwise disjoint non-empty open subsets of a space X is called a cellular family.

Theorem 1. *The Sorgenfrey line has a locally pathwise connected connectification.*

Proof. Let $S = (\mathbb{R}, \tau)$ be the Sorgenfrey line and let $D = \{d_n : n \in \mathbb{N}\}$ be a countable dense subset of S, and for every $n, i \in \mathbb{N}$ set $B(n,i) = [d_n, d_n + \frac{1}{i}[$ and $C(n,i) = [d_n + \frac{1}{i+1}, d_n + \frac{1}{i}[$.

Let Ω be the subset of $\mathbb{N}^3$ consisting of all $\omega = (n,m,i)$ with $n < m$ and $B(n,i) \cap B(m,i) = \emptyset$. Set $\Lambda = \Omega \times (\mathbb{Z} \setminus \{0\})$.

By induction we can choose, for every $\lambda = (n,m,i,k) \in \Lambda$, a free open filter $\mathcal{F}_\lambda$ on S with a countable base such that:

i) if $k > 0$, then $C(n, i+k-1) \in \mathcal{F}_\lambda$;

ii) if $k < 0$, then $C(m, i-k-1) \in \mathcal{F}_\lambda$;

iii) the family $\Phi = \{\mathcal{F}_\lambda : \lambda \in \Lambda\}$ is Hausdorff separated (i.e., if $\lambda \neq \lambda'$, then there are $F \in \mathcal{F}_\lambda$ and $F' \in \mathcal{F}_{\lambda'}$ such that $F \cap F' = \emptyset$).

Note that $B(n,i) \cup B(m,i) \in \mathcal{F}_\lambda$.

Now let $H =]0,1[\cap Q \setminus (\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}\})$, and set $\Gamma = \Omega \times H$.

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For each $q \in H$ set

$$k_q = \begin{cases} n-1, & \text{if } \frac{1}{n+1} < q < \frac{1}{n} \leq \frac{1}{2}, \\ 1-n, & \text{if } \frac{1}{2} \leq 1 - \frac{1}{n} < q < 1 - \frac{1}{n+1}. \end{cases}$$

For each $\gamma = (\omega, q) \in \Gamma$ let $\lambda_\gamma = (\omega, k_q) \in \Lambda$. Clearly, for every $\lambda \in \Lambda$, there are countably many $\gamma \in \Gamma$ such that $\lambda_\gamma = \lambda$.

Now for every $\gamma \in \Gamma$ choose a countably generated open filter $\mathcal{G}_\gamma$ on S finer than $\mathcal{F}_{\lambda_\gamma}$ so that, for every $\lambda \in \Lambda$, the family $\{\mathcal{G}_\gamma : \gamma \in \Gamma, \lambda_\gamma = \lambda\}$ is totally Hausdorff separated, i.e., for every γ such that $\lambda_\gamma = \lambda$, there is a $A_\gamma \in \mathcal{G}_\gamma$ such that $\{A_\gamma : \gamma \in \Gamma, \lambda_\gamma = \lambda\}$ is a cellular family (an easy proof of the existence of the family $\{\mathcal{G}_\gamma : \gamma \in \Gamma, \lambda_\gamma = \lambda\}$ can be found in [4]).

Set $Y = \Omega \times]0,1[$, and for every $y = (\omega, r) \in Y$ and any $\epsilon > 0$, let us denote by $]y-\epsilon, y+\epsilon[$ the set $\{(\omega, s) : |s-r| < \epsilon, \ s \in]0,1[\}$.

Moreover $[y-\epsilon, y+\epsilon]$ will denote the set $\{(\omega, s) : |s-r| \leq \epsilon, \ s \in]0,1[\}$.

Put $\mathbb{T} = \mathbb{R} \cup Y$ and let τ^* be the topology on $\mathbb{T}$ generated by the base $\{A^* : A \in \tau\}$, where A^* is the subset of $\mathbb{T}$ characterized by the following properties:

1) $A^* \cap \mathbb{R} = A$;

2) for every $y \in Y$, $y \in A^* \Leftrightarrow (\exists \epsilon > 0 : \gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma \Rightarrow A \in \mathcal{G}_\gamma)$.

Observe that for every $y \in A^* \cap Y$ there is some $\epsilon > 0$ such that $[y-\epsilon, y+\epsilon] \subset A^*$.

We claim that $\mathbb{T}$, endowed with the topology τ^*, is a connected locally pathwise connected (and hence pathwise connected) T_2-space in which S is densely embedded.

Clearly S is a dense subspace of $\mathbb{T}$.

To check that $\mathbb{T}$ is connected and locally pathwise connected it is enough to show that A^* is pathwise connected for every $A \in \tau$.

First let us show that for every $\omega = (n, m, i) \in \Omega$ the function $f_\omega : [0,1] \to \mathbb{T}$ defined by

1) $f_\omega(0) = d_n$, $f_\omega(1) = d_m$;

2) $f_\omega(t) = (\omega, t)$ for every $t \in]0,1[$

is a path.

Let $t \in]0,1[$, $y = f_\omega(t) = (\omega, t)$ and take a $G \in \tau$ such that G^* is a neighbourhood of y in $\mathbb{T}$. Then there is a positive ϵ such that $G \in \mathcal{G}_\gamma$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$. Hence $f_\omega([t-\epsilon, t+\epsilon] \cap]0,1[) = [y-\epsilon, y+\epsilon] \subset G^*$. Therefore f_ω is continuous on $]0,1[$.

Now let us prove that f_ω is continuous at 0. Let $G \in \tau$ such that $d_n \in G^*$. Since $d_n \in G$, there is a $k > 0$ such that $B(n, i+k-1) \subset G$. Observe that $G \in \mathcal{F}_{(\omega,h)}$ for every $h \geq k$ (note that $\mathcal{F}_{(\omega,h)} \ni C(n, i+h-1) \subset B(n, i+h-1) \subset B(n, i+k-1)$ for every $h \geq k$). Therefore $G \in \mathcal{G}_{(\omega,t)}$ for every $t \in]0, \frac{1}{1+k}[\cap H$ (in fact let h be such that $t \in]\frac{1}{2+h}, \frac{1}{1+h}[\cap H$; then $\mathcal{G}_{(\omega,t)}$ is finer than $\mathcal{F}_{(\omega,h)}$).

Now let $t \in]0, \frac{1}{1+k}[$, $y = f_\omega(t) = (\omega, t)$ and take $\epsilon > 0$ such that $[t-\epsilon, t+\epsilon] \subset]0, \frac{1}{1+k}[$ and $G \in \mathcal{G}_\gamma$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$. Then $y \in G^*$. Hence $f_\omega([0, \frac{1}{1+k}[) \subset G^*$, and f_ω is continuous at 0. Similarly it can be shown that f_ω is continuous at 1. Therefore $f_\omega$is a path.

Now let us show that for every $A \in \tau$ and $x, y \in A^*$ there is a path in A^* between x and y.

i) $x = d_n$ and $y = d_m$. Let i be such that $B(n,i) \cup B(m,i) \subset A$; then $A \in \mathcal{G}_\gamma$ for every $\gamma = (\omega, q) \in \Gamma$, where $\omega = (n, m, i)$ and $n < m$. Now set $z = (\omega, t)$ with

$t \in]0,1[$ and let us show that $z \in A^*$. Take a positive ϵ; then $A \in \mathcal{G}_\gamma$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$. Hence $z \in A^*$. Therefore f_ω is a path between x and y in A^*.

ii) $x = d \in A \cap D$ and $y \in A$. Let $\{B_n : n \in \mathbb{N}\}$ be a decreasing base for S in y such that $B_1 = A$, $C_n = B_n \setminus cl_X(B_{n+1}) \neq \emptyset$ and $d \in C_1$. Put $d_{n_1} = d$ and take $d_{n_k} \in C_k \cap D$ with $n_k < n_{k+1}$, for every $k \geq 2$. Now choose, for every $k \in \mathbb{N}$, i_k so that $B(n_k, i_k) \cup B(n_k, i_{k-1}) \subset C_k$. Set $\omega_k = (n_k, n_{k+1}, i_k)$ and observe that f_{ω_k} is a path between d_{n_k} and $d_{n_{k+1}}$ contained in $(B(n_k, i_k) \cup B(n_{k+1}, i_k))^*$ (and hence in B_k^*). Since $\{B_n^* : n \in \mathbb{N}\}$ is a base for $\mathbb{T}$ at y and, for every $n \in \mathbb{N}$, all but finitely many f_{ω_k} are contained in B_n^*, it is possible to find a path in A^* between x and y.

iii) $x = d \in A$ and $y = (\omega, t) \in A^* \setminus A$. Take $\epsilon > 0$ such that $[y-\epsilon, y+\epsilon] \subset A^*$. Since $[y-\epsilon, y+\epsilon]$ is a path, we may assume that $y \in \Gamma$. If B^* is a neighbourhood of y, then there is $\epsilon > 0$ such that $B \in \mathcal{G}_\gamma$ for each $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$, in particular $B \in \mathcal{G}_y$.

Now if $\mathcal{B}$ is a countable base of $\mathcal{G}_y$, then $\{B^* : B \in \mathcal{B}\}$ is a local π-base of $\mathbb{T}$ at y . Arguing as in the previous case we can find a path in A^* between x and y.

Now it remains to show that $\mathbb{T}$ is a Hausdorff space. So let us consider two distinct points x and y in $\mathbb{T}$.

1) $x, y \in \mathbb{R}$. Choose $A, B \in \tau$ such that $A \cap B = \emptyset$, $x \in A$ and $y \in B$. Clearly A^* and B^* are open in $\mathbb{T}$, $x \in A^*$ and $y \in B^*$. We claim that $A^* \cap B^* = \emptyset$. Suppose not, and take $p \in A^* \cap B^*$. Since $p \in Y$, there is some $\epsilon > 0$ such that $A, B \in \mathcal{G}_\gamma$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$, a contradiction.

2) $x \in \mathbb{R}$ and $y = (\omega, r) \in Y$. It is easy to see that there are a sufficiently small $\epsilon > 0$ and $\mathcal{F}_1, \mathcal{F}_2 \in \Phi$ such that, for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$, $\mathcal{F}_{\lambda_\gamma} = \mathcal{F}_1$ or $\mathcal{F}_{\lambda_\gamma} = \mathcal{F}_2$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are free filters, there are $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$ and a neighbourhood B of x such that $A_i \cap B = \emptyset$, $i \in \{1, 2\}$. Set $A = A_1 \cup A_2$ and observe that $A^* \cap B^* = \emptyset$. Clearly $x \in B^*$; moreover $y \in A^*$ (since $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$, it follows that $A \in \mathcal{F}_{\lambda_\gamma} \subset \mathcal{G}_\gamma$ for every $\gamma \in [y-\epsilon, y+\epsilon] \cap \Gamma$).

3) $x = (\omega, r), y = (\omega', r') \in Y$, with $\omega \neq \omega'$. Take $\epsilon > 0$ and $\mathcal{F}_i \in \Phi$ for every $i \in \{1, 2, 3, 4\}$ so that:

i) $\mathcal{F}_i \neq \mathcal{F}_j$ whenever $i \neq j$;

ii) $\mathcal{F}_1 \subset \mathcal{G}_\gamma$ or $\mathcal{F}_2 \subset \mathcal{G}_\gamma$ for every $\gamma = (\omega, t) \in [x-\epsilon, x+\epsilon] \cap \Gamma$;

iii) $\mathcal{F}_3 \subset \mathcal{G}_\gamma$ or $\mathcal{F}_4 \subset \mathcal{G}_\gamma$ for every $\gamma = (\omega', t) \in [y-\epsilon, y+\epsilon] \cap \Gamma$.

Since $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4\}$ is Hausdorff separated, we can choose $A_i \in \mathcal{F}_i$ for every $i \in \{1, 2, 3, 4\}$ in such a way that the family $\{A_1, A_2, A_3, A_4\}$ is cellular.

Put $A = A_1 \cup A_2$, $B = A_3 \cup A_4$. It is clear that A^* and B^* are open sets of $\mathbb{T}$ such that $x \in A^*$ and $y \in B^*$. Since $A \cap B = \emptyset$, it follows that A^* and B^* are disjoint.

4) $y_1 = (\omega, r_1), y_2 = (\omega, r_2) \in Y$, with $r_1 < r_2$.

If there is some $n \geq 2$ such that $\frac{1}{n} \in]r_1, r_2[$ or $1 - \frac{1}{n} \in]r_1, r_2[$, then there are $\epsilon > 0$ and $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ as in the previous case. So let us suppose that there is $n \geq 1$ such that $\frac{1}{n+1} \leq r_1 < r_2 \leq \frac{1}{n}$ (the case $1 - \frac{1}{n} \leq r_1 < r_2 \leq 1 - \frac{1}{n-1}$ is similar). Then there are $\epsilon > 0$ and $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \in \Phi$ such that

i) $[y_1 - \epsilon, y_1 + \epsilon] \cap [y_2 - \epsilon, y_2 + \epsilon] = \emptyset$;

ii) $\mathcal{F} \subset \mathcal{G}_\gamma$ or $\mathcal{F}_1 \subset \mathcal{G}_\gamma$ for every $\gamma \in [y_1 - \epsilon, y_1 + \epsilon] \cap \Gamma$.

iii) $\mathcal{F} \subset \mathcal{G}_\gamma$ or $\mathcal{F}_2 \subset \mathcal{G}_\gamma$ for every $\gamma \in [y_2 - \epsilon, y_2 + \epsilon] \cap \Gamma$.

Choose $F \in \mathcal{F}$, $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ so that the family $\{F, F_1, F_2\}$ is cellular.

Since the family $\{\mathcal{G}_\gamma : \gamma \in \Gamma, \mathcal{F} \subset \mathcal{G}_\gamma\}$ is totally Hausdorff separated, take $A_\gamma \in \mathcal{G}_\gamma$ such that $A_\gamma \subset F$ and the family $\{A_\gamma : \gamma \in \Gamma, \mathcal{F} \subset \mathcal{G}_\gamma\}$ is cellular.

Set $C_i = F_i \cup \bigcup\{A_\gamma : \gamma \in [y_i - \epsilon, y_i + \epsilon] \cap \Gamma, \mathcal{F} \subset G_\gamma\}$ and note that $C_1 \cap C_2 = \emptyset$.

Now it is enough to observe that C_1^* and C_2^* are disjoint open subsets of $\mathbb{T}$ such that $y_i \in C_i^*$, $i \in \{1, 2\}$. $\square$

Remark 1. A space X is called feebly compact if every countable open cover of X has a finite subfamily whose union is dense. Observe that if G is an open subset of a space X whose closure is not feebly compact, then G is an element of a free open filter on X with a countable base (see Lemma 3.7 in [6]). It is easy to see that in the proof of Theorem 1 we use only the fact that the Sorgenfrey line is a first countable separable Hausdorff space in which every non-empty open subset has non feebly compact closure.

Therefore we have the following more general result.

Theorem 2. *Every first countable separable Hausdorff space with no non-empty open subsets with feebly compact closure has a locally pathwise connected connectification.*

A space is called pathwise connectifiable if it can be densely embedded in a pathwise connected Hausdorff space ([4]). Since every non-empty countable Hausdorff space without isolated points is not feebly compact ([5], Theorem 5.2), we have the following

Corollary 1 ([4]). *Every countable first countable Hausdorff space without isolated points is pathwise connectifiable.*

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