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A CHARACTERIZATION OF ALGEBRAS WITH POLYNOMIAL GROWTH OF THE CODIMENSIONS

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ABSTRACT. Let A be an associative algebras over a field of characteristic zero. We prove that the codimensions of A are polynomially bounded if and only if any finite dimensional algebra B with $Id(A) = Id(B)$ has an explicit decomposition into suitable subalgebras; we also give a decomposition of the n-th cocharacter of A into suitable S_n-characters.

We give similar characterizations of finite dimensional algebras with involution whose $*$-codimension sequence is polynomially bounded. In this case we exploit the representation theory of the hyperoctahedral group.

§1. INTRODUCTION

Let F be a field of characteristic zero and $F\langle X\rangle = F\langle x_1, x_2, \dots\rangle$ the free algebra of countable rank over F. If A is a PI-algebra over F, that is, an algebra satisfying a polynomial identity, we let $Id(A)$ be the T-ideal of $F\langle X\rangle$ of identities of A. It is well known that $Id(A)$ is completely determined by the multilinear polynomials it contains; if $V_n = \text{Span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ is the space of multilinear polynomials in $x_1, \dots, x_n$, then the sequence $c_n(A) = \dim_F \frac{V_n}{V_n \cap Id(A)}$, $n = 1, 2, \dots$, is called the sequence of codimensions of A and it is an important numerical invariant of $Id(A)$.

It was proved by Regev in [R] that for any PI-algebra A, $c_n(A)$ is exponentially bounded, i.e., there exist constants $a, \alpha > 0$ such that $c_n(A) \leq a\alpha^n$ for all n.

In this paper we study algebras A whose codimension sequence is polynomially bounded i.e., such that for all n, $c_n(A) \leq an^t$ for some constants a, t. Kemer in [K1] gave a characterization of such T-ideals in the language of the representation theory of S_n. It also follows from [K2] that if $c_n(A)$ is polynomially bounded, then $Id(A) = Id(B)$ for a suitable finite dimensional algebra B.

For any finite dimensional algebra A over an algebraically closed field we shall prove that A has polynomial growth of the codimensions if and only if $A = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ where $A_0, A_1, \dots, A_m$ are F-algebras such that 1) for $i = 1, \dots, m$, $A_i = B_i + J_i$, where $B_i \cong F$ and J_i is a nilpotent ideal of A_i, 2) $A_0, J_1, \dots, J_m$ are nilpotent right ideals of A and 3) $A_iA_k = 0$ for all $i, k \in \{1, \dots, m\}, i \neq k$ and $B_iA_0 = 0$.

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Another description of such algebras is given, as in Kemer's paper [K1], in the language of the cocharacters as follows. The symmetric group S_n acts on the left on V_n by $\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),\ \sigma \in S_n,\ f(x_1, \dots, x_n) \in V_n$. This action induces a structure of left S_n-module on $\frac{V_n}{V_n \cap Id(A)}$ and we write $\chi_n(A)$ for its S_n-character; $\chi_n(A)$ is called the n-th cocharacter of A. Let χ_λ denote the irreducible S_n-character associated to the partition $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ and write $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ where $m_\lambda \geq 0$ are the corresponding multiplicities. We shall prove that if $\dim_F A < \infty,\ A$ has polynomial growth of the codimensions if and only if

$$\chi_n(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda \chi_\lambda$$

where J is the Jacobson radical of A and $J^q = 0$.

In the second part of the paper we address ourself to similar questions in the setting of algebras with involution. Let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \dots\rangle$ be the free algebra with involution $*$ and $V_n(*)$ the space of multilinear $*$-polynomials in $x_1, x_1^*, \dots, x_n, x_n^*$. For an algebra with involution A we let $Id(A, *)$ be the ideal of $*$-identities of A; then the sequence of $*$-codimensions is

$$c_n(A, *) = \dim_F \frac{V_n(*)}{V_n(*) \cap Id(A, *)},\ n = 1, 2, \dots .$$

In the last section we characterize finite dimensional algebras A such that $c_n(A, *)$ is polynomially bounded. Moreover by using the representation theory of the hyperoctahedral group $\mathbb{Z}_2 \sim S_n$, we shall obtain a characterization of A in terms of the $*$-cocharacter sequence of A.

§2. Algebras with polynomial growth of the codimensions

In this section we shall characterize algebras with polynomial growth of the codimensions. The following reduction is due to Kemer.

Theorem 1. *Let A be a PI-algebra. If for all n, $c_n(A) \leq an^t$ for some constants a, t, then there exists a finite dimensional algebra B such that $Id(A) = Id(B)$.*

Proof. Let G be the Grassmann algebra of countable dimension over F. By [KR], $c_n(G) = 2^{n-1}$, hence, for n large, $c_n(A) < c_n(G)$. This implies that $Id(A) \not\subseteq Id(G)$ and, by a theorem of Kemer [K2, Theorem 2.3] there exists a finite dimensional algebra B such that $Id(A) = Id(B)$. □

We remark that $c_n(A)$ does not change upon extension of the ground field F. In fact, if K is an extension field of F, then $Id(A) \otimes_F K = Id(A \otimes_F K)$. Therefore in studying properties of $c_n(A)$ we may as well assume that F is an algebraically closed field.

Theorem 2. *Let A be a finite dimensional algebra over an algebraically closed field F. Then the sequence of codimensions $\{c_n(A)\}_{n \geq 1}$ is polynomially bounded if and only if*

1) *$A = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ a vector space direct sum of F-algebras where for $i = 1, \dots, m,\ A_i = B_i + J_i,\ B_i \cong F,\ J_i$ a nilpotent ideal of A_i and $A_0, J_1, \dots, J_m$ are nilpotent right ideals of A;*
2) *for all $i, k \in \{1, \dots, m\}, i \neq k,\ A_i A_k = 0$ and $B_i A_0 = 0$.*

Proof. Let $A = B + J$ be the Wedderburn-Malcev decomposition of A ([CR, Theorem 72.19]) where B is a semisimple subalgebra of A and $J = J(A)$ its Jacobson radical. Write $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \dots, B_m$ simple F-algebras. Since $c_n(A)$ is polynomially bounded, by [GZ], $B_iJB_k = 0$ for all $i \neq k$ and $\dim_F B_i = 1,\ i, k = 1, \dots, m$.

Let $e = e_1 + \cdots + e_m$ be the decomposition of the unit element of B into orthogonal central (in B) idempotents; thus $e_iB = B_i \cong F$. Define for all $i = 1, \dots, m,\ J_i = e_iJ$ and $J_0 = \{x \in J \mid Bx = 0\}$. It is easy to show that $A = B + J = (B_1 + J_1) \oplus \cdots \oplus (B_m + J_m) \oplus J_0$; let $A_i = B_i + J_i$ and $A_0 = J_0$.

For $i \neq k \in \{1, \dots, m\},\ A_iA_k = (B_i + J_i)(B_k + J_k) = 0$ since $e_ie_k = 0$ and $B_iJB_k = 0$. Also, for $i \neq 0,\ B_iA_0 = 0$.

Viceversa, let A satisfy 1) and 2). From the relations $A_iA_k = 0$ and $B_iA_0 = 0$ it follows that $J = A_0 + J_1 + \cdots + J_m$ is a nilpotent two-sided ideal of A and $A = B_1 \oplus \cdots \oplus B_m \oplus J$ where $B_i \cong F$ for all i. Since from the defining relations $A_iA_k = 0$ and $B_iA_0 = 0$ it follows that $B_iJB_k = 0$ for all $i \neq k$, then $c_n(A)$ is polynomially bounded by [GZ]. □

As an immediate consequence of the above result we get

Corollary 1. *Let A be the algebra described in the previous theorem. Let J be the Jacobson radical of A and, for $i = 1, \dots, m$, let $C_i = A_i \oplus A_0$. Then*

$$Id(A) = Id(C_1) \cap \ldots \cap Id(C_m) \cap Id(J).$$

Proof. Let $f \in Id(C_1) \cap \ldots \cap Id(C_m) \cap Id(J)$ and suppose that $f \notin Id(A)$. We may clearly assume that f is multilinear and let $r_1, \dots, r_s \in A$ be such that $f(r_1, \dots, r_s) \neq 0$.

If $r_1, \dots, r_s \in J$, then $f \notin Id(J)$, a contradiction. Hence there exists $r_i \notin J$; by linearity we may assume that $r_i \in B_k$ for some k. Recall that, for all $l,\ B_lA_0 = 0,\ J_l$ is a right ideal of A and, in case $l \neq k,\ A_kA_l = A_lA_k = 0$. From an easy calculation it follows that $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_s \in A_k \cup A_0$. But then $f \notin Id(C_k)$, a contradiction. □

What can be said if F is not algebraically closed?

Let A be a finite dimensional algebra over a field F and write $A = B + J,\ B = B_1 \oplus \cdots \oplus B_m$ with the B_i's simple algebras. If $\overline{F}$ is the algebraic closure of F, we write $\overline{A} = A \otimes_F \overline{F}$; moreover, since $J(\overline{A}) = J(A) \otimes_F \overline{F}$ (see [Ro, Theorem 2.5.36]), we get that

$$\overline{A} \cong \overline{B_1} \oplus \cdots \oplus \overline{B_m} + J(\overline{A})$$

where $\overline{B_i} = B_i \otimes_F \overline{F}$ are semisimple algebras.

Let $Z(B_i)$ be the center of B_i and $t_i = \dim_F Z(B_i)$. Then $\overline{B_i} \cong C_{i1} \oplus \cdots \oplus C_{it_i}$ where $C_{i1} \cong \cdots \cong C_{ik}$ are central simple algebras over $\overline{F}$.

In case $c_n(A) = c_n(\overline{A})$ is polynomially bounded, by [GZ], $C_{ik} \cong \overline{F}$ for all i, k and $C_{ik}J(\overline{A})C_{uv} = 0$ if $(i, k) \neq (u, v)$. It follows that for all $i = 1, \dots, m$, B_i is a field extension of F of degree t_i. Since $\mathrm{char} F = 0$, we write $B_i = F(a_i)$, a simple algebraic extension of F of degree t_i. We have shown that if F is any field and A is an F-algebra with polynomial growth of the codimensions, then

$$A \cong F(a_1) \oplus \cdots \oplus F(a_m) + J(A)$$

and for all $i \neq k,\ F(a_i)J(A)F(a_k) = 0$.

We next give a characterization of polynomial growth in terms of the cocharacter sequence of the algebra.

In the sequel for $\lambda \vdash n$ we also write $\mid \lambda \mid = n$. We write $\chi_\lambda(1) = d_\lambda$ for the degree of the irreducible S_n-character χ_λ and, if T_λ is a tableau of shape λ, we let e_{T_λ} be the corresponding essential idempotent of FS_n. Notice that if $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$, then $\mid \lambda \mid -\lambda_1$ denotes the number of boxes below the first row of the diagram of λ.

Theorem 3. *Let A be a finite dimensional algebra over a field F. Then $\{c_n(A)\}_{n\geq 1}$ is polynomially bounded if and only if*

$$\chi_n(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda \chi_\lambda$$

where $J(A)^q = 0$.

Proof. Note that the decomposition of $\chi_n(A)$ into irreducible components does not change when extending the base field. Therefore, since $J(A \otimes_F \overline{F})^q = 0$, we may assume, without lost of generality, that F is algebraically closed.

Suppose first that the codimensions of A are polynomially bounded. Let λ be a partition of n such that $\mid \lambda \mid -\lambda_1 \geq q$ and suppose by contradiction that $m_\lambda \neq 0$. Then there exists a tableau T_λ such that $e_{T_\lambda}(x_1, \dots, x_n) \notin Id(A)$. Let $\lambda' = (\lambda'_1, \dots, \lambda'_t)$ be the conjugate partition of λ. Then $e_{T_\lambda}(x_1, \dots, x_n)$ is a linear combination of polynomials each alternating on t disjoint sets of $\lambda'_1, \dots, \lambda'_t$ variables, respectively. We shall reach a contradiction by proving that each such polynomial f vanishes in A.

Fix a basis of A which is the union of bases of $B_1, \dots, B_m$ and J respectively. Since $B_iB_k = B_iJB_k = 0$ for all $i \neq k$, in order to get a non-zero value of f we must replace all the variables with elements of J and of one simple component, say, B_i. Also, since $\dim B_i = 1$, we can substitute at most one element of B_i in each alternating set. Hence we can substitute in all at most $t = \lambda_1$ elements from B_i. It follows that in order to get a non-zero value, we must substitute at least $\mid \lambda \mid -\lambda_1 \geq q$ elements from J. Since $J^q = 0$, we get that $f \equiv 0$ and with this contradiction the proof of the first part of the theorem is complete.

Suppose now that $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ and $m_\lambda = 0$ whenever $\mid \lambda \mid -\lambda_1 \geq q$. By [BR] the multiplicities m_λ are polynomially bounded; hence

$$c_n(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda d_\lambda \leq Cn^t \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} d_\lambda$$

and the proof now follows from the hook formula for the degrees d_λ. □

The previous theorem says that A has polynomial growth of the codimensions if and only if all the irreducible characters appearing with non-zero multiplicity in $\chi_n(A)$ have associated diagram with at most $q - 1$ boxes below the first row where $J^q = 0$.

§3. Finite dimensional algebras with involution

In this section we shall prove that if A is a finite dimensional algebra with involution $*$, then in the decomposition $A = B + J$ we can choose B to be invariant under $*$. Beside its own interest, this result will be used in the next section. Throughout we shall assume that $\mathrm{char} F \neq 2$.

Theorem 4. *Let A be a finite dimensional algebra with involution $*$ over F and J its Jacobson radical. Then $J^* = J$ and there exists a maximal semisimple subalgebra B such that $B = B^*$ and $A = B + J$.*

Proof. It is obvious that $J^* = J$. Let $A = B + J$ with B a semisimple subalgebra of A and suppose first that $J^2 = 0$.

Since B^* is also a maximal semisimple subalgebra of A, by the Wedderburn-Malcev theorem there exists $y \in J$ such that $B^* = (1 - y)B(1 + y)$. For $b \in B$ let $\overline{b} \in B$ be such that $b^* = (1 - y)\overline{b}(1 + y)$. Then $b = b^{**} = (1 + y^*)\overline{b}^*(1 - y^*) = (1 + y^*)(1 - y)\overline{\overline{b}}(1 + y)(1 - y^*)$, for a suitable $\overline{\overline{b}} \in B$. It follows that we can write $b = \overline{\overline{b}} + j$ for a suitable $j \in J$; hence $b - \overline{\overline{b}} \in B \cap J = 0$ and $b = \overline{\overline{b}}$ follows. But then, from the above, $b = (1 + y^*)\overline{b}^*(1 - y^*) = (1 + y^*)(1 - y)b(1 + y)(1 - y^*) = (1-y+y^*)b(1+y-y^*)$ since $J^2 = 0$. This says that $y-y^*$ commutes with b. Therefore by writing $y = \frac{y+y^*}{2} + \frac{y-y^*}{2}$, we get $b^* = (1 - y)\overline{b}(1 + y) = (1 - \frac{y+y^*}{2})\overline{b}(1 + \frac{y+y^*}{2})$.

We have proved that $B^* = (1 - s)B(1 + s)$ for a suitable symmetric element $s = s^* \in J$. At this stage it is easy to check that $B' = (1 - \frac{s}{2})B(1 + \frac{s}{2})$ is the desired invariant subalgebra of A.

Suppose now that $J^n = 0, J^{n-1} \neq 0, n > 2$. Set $J^{n-1} = I$. Since $I^* = I$, A/I has an induced involution; also $J(A/I) = J/I$ and $J(A/I)^{n-1} = 0$. Therefore, by induction on n, $A/I = B/I + J/I$ for a suitable semisimple subalgebra $B/I = (B/I)^*$. It follows that we can write $B = C+I$ where C is a semisimple subalgebra of B and, since $B^* = B$, by the first part we may assume that $C^* = C$. By counting dimensions we get that C is a maximal semisimple subalgebra of A and $A = C + J$ is the desired decomposition. □

Recall that an algebra with involution A is $*$-simple if A has no proper $*$-invariant ideals (i.e., ideals I such that $I^* = I$). It is well known and easy to prove that if A is $*$-simple, then either A is simple or $A \cong A_1 \oplus A_1^{\mathrm{op}}$ where A_1 is a simple homomorphic image of A and $*$ on $A_1 \oplus A_1^{\mathrm{op}}$ is the exchange involution $(a, b)^* = (b, a)$ (see [Ro, Proposition 2.13.24]).

Remark 1. If B is a semisimple algebra with involution and $\dim_F B < \infty$, then $B = B_1 \oplus \cdots \oplus B_t$ where $B_1, \ldots, B_t$ are $*$-simple algebras.

Proof. Let $B = C_1 \oplus \cdots \oplus C_m$ be the decomposition of B into simple components. Let $e_1, \ldots, e_m$ be the corresponding orthogonal central idempotents. Let $i \in \{1, \ldots, m\}$; if $e_i^* = e_i$, then $C_i = C_i^* = e_iB$ is $*$-simple. If $e_i^* \neq e_i$, then e_i^*B is still a minimal ideal of B which implies that $e_i^* = e_j$ for some $j \in \{1, \ldots, m\}$. Hence $C_i \oplus C_j$ is $*$-simple. □

§4. $*$-CODIMENSIONS WITH POLYNOMIAL GROWTH

Throughout this section F will be a field of characteristic zero, and A an F-algebra with involution $*$. Let $A^+ = \{a \in A \mid a = a^*\}$ and $A^- = \{a \in A \mid a = -a^*\}$ be the sets of symmetric and skew elements of A respectively.

We consider $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ the free algebra with involution $*$ of countable rank. Recall that $f(x_1, x_1^*, \ldots, x_n, x_n^*) \in F\langle X, *\rangle$ is a $*$-polynomial identity for A if $f(a_1, a_1^*, \ldots, a_n, a_n^*) = 0$ for all $a_1, \ldots, a_n \in A$. The set $Id(A, *)$ of all $*$-polynomial identities of A is a T-ideal of $F\langle X, *\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X,*\rangle$ commuting with the involution. Let

$$V_n(*) = \mathrm{Span}_F\{x_{\sigma(1)}^{a_1}\cdots x_{\sigma(n)}^{a_n} \mid \sigma\in S_n,\ a_i\in\{1,*\}\}$$

be the space of multilinear $*$-polynomials in $x_1, x_1^*, \dots, x_n, x_n^*$.

If we set $s_i = x_i + x_i^*$ and $k_i = x_i - x_i^*$, $i = 1, 2, \dots$, then, since $\mathrm{char} F \neq 2$, we can also write

$$V_n(*) = \mathrm{Span}_F\{w_{\sigma(1)}\cdots w_{\sigma(n)} \mid \sigma\in S_n,\ w_i = s_i \text{ or } w_i = k_i,\ i = 1,\dots,n\}.$$

Let H_n be the hyperoctahedral group. Recall that $H_n = \mathbb{Z}_2 \sim S_n$ is the wreath product of $\mathbb{Z}_2 = \{1,*\}$, the multiplicative group of order 2, and S_n. We write the elements of H_n as $(a_1,\dots,a_n;\sigma)$ where $a_i\in\mathbb{Z}_2$, $\sigma\in S_n$. The action of H_n on $V_n(*)$ defined in [GR] can be rewritten (see [DG]) as follows: for $h = (a_1,\dots,a_n;\sigma)\in H_n$ define $hs_i = s_{\sigma(i)}$, $hk_i = k_{\sigma(i)}^{a_{\sigma(i)}} = \pm k_{\sigma(i)}$ and then extend this action diagonally to $V_n(*)$. Hence $V_n(*)$ becomes a left H_n-module and, since $V_n(*)\cap Id(A,*)$ is a subspace invariant under this action, we can view $V_n(*)/(V_n(*)\cap Id(A,*))$ as an H_n-module. Let $\chi_n(A,*)$ be its character.

The sequence $c_n(A,*) = \chi_n(A,*)(1) = \dim_F \frac{V_n(*)}{V_n(*)\cap Id(A,*)}$, $n = 1,2,\dots$, is called the sequence of $*$-codimensions of A.

Recall that there is a one-to-one correspondence between irreducible H_n-characters and pairs of partitions (λ,μ), where $\lambda\vdash r$, $\mu\vdash n-r$, for all $r = 0,1,\dots,n$. If $\chi_{\lambda,\mu}$ denotes the irreducible H_n-character corresponding to (λ,μ), then we can write

$$\chi_n(A,*) = \sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

where $m_{\lambda,\mu}\geq 0$ are the corresponding multiplicities.

Now, for $r = 0,\dots,n$, we let

$$V_{r,n-r} = \mathrm{Span}\{w_{\sigma(1)}\cdots w_{\sigma(n)} \mid \sigma\in S_n,\ w_i = s_i \text{ for } i = 1,\dots,r \text{ and } w_i = k_i \text{ for } i = r+1,\dots,n\}.$$

Thus $V_{r,n-r}$ is the space of multilinear polynomials in $s_1,\dots,s_r,k_{r+1},\dots,k_n$. It is clear that in order to study $V_n(*)\cap Id(A,*)$ it is enough to study $V_{r,n-r}\cap Id(A,*)$ for all r.

If we let S_r act on the symmetric variables $s_1,\dots,s_r$ and S_{n-r} on the skew variables $k_{r+1},\dots,k_n$, we obtain an action of $S_r\times S_{n-r}$ on $V_{r,n-r}$ and

$$V_{r,n-r}(A,*) = \frac{V_{r,n-r}}{V_{r,n-r}\cap Id(A,*)}$$

becomes a left $S_r\times S_{n-r}$-module. Let $\psi_{r,n-r}(A,*)$ be its character and

$$c_{r,n-r}(A,*) = \psi_{r,n-r}(A,*)(1) = \dim_F V_{r,n-r}(A,*).$$

We write $\psi_{\lambda,\mu}$ for the irreducible $S_r\times S_{n-r}$-character associated to the pair (λ,μ) with $\lambda\vdash r,\mu\vdash n-r$. The following result holds.

Theorem 5 ([DG, Theorem 1.3])**.** *Let A be an algebra with involution; then, for all n,*

$$\chi_n(A,*) = \sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}, \quad \textit{and} \quad \psi_{r,n-r}(A,*) = \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\psi_{\lambda,\mu}.$$

Moreover

$$c_n(A,*) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A,*).$$

We next characterize finite dimensional algebras A with polynomial growth of the $*$-codimensions.

Theorem 6. *Let A be a finite dimensional algebra with involution over an algebraically closed field F. Then the sequence of $*$-codimensions $\{c_n(A,*)\}_{n\geq 1}$ is polynomially bounded if and only if*

1) *the sequence of codimensions $\{c_n(A)\}_{n\geq 1}$ is polynomially bounded;*
2) *$A = B + J$, where B is a maximal semisimple subalgebra of A and $b = b^*$ for all $b \in B$.*

Proof. By [GR, Lemma 4.4], for all n, $c_n(A) \leq c_n(A,*) \leq \alpha n^t$ for some constants α, t, and the sequence of codimensions is polynomially bounded. From [GZ] it follows that $A = B + J$, where $B = B_1 \oplus \cdots \oplus B_m$ and $B_i \cong F$, $B_i J B_k = 0$ for all $i \neq k$. By Theorem 4 we may also assume that $B^* = B$.

Suppose by contradiction that $*$ is not the identity map on B; then, by Remark 1, there exist B_i, B_k such that $C = B_i \oplus B_k \cong F \oplus F$ is $*$-simple with involution $(a,b)^* = (b,a)$. Notice that, for all $\sigma \in S_n$ and $a_1, \ldots, a_n \in \{1,*\}$,

$$x_{\sigma(1)}^{a_{\sigma(1)}} \cdots x_{\sigma(n)}^{a_{\sigma(n)}} \equiv x_1^{a_1} \cdots x_n^{a_n} (\text{mod } Id(C,*)).$$

Moreover the set $\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \{1,*\}\}$ is linearly independent modulo $Id(C,*)$. It follows that $c_n(C,*) = 2^n$. Since $c_n(C,*) \leq c_n(A,*)$ we get a contradiction.

Suppose now that $A = B + J$, $c_n(A)$ is polynomially bounded and $*$ is the identity on B. In this case, if $a \in A$, write $a = b + j$, $b \in B, j \in J$. Then $a - a^* = j - j^* \in J$ and $A^- \subseteq J$.

Notice that if $f(x_1, \ldots, x_n) \in V_n \cap Id(A)$, then, for every $r = 0, \ldots, n$,

$$f(s_1, \ldots, s_r, k_{r+1}, \ldots, k_n) \in V_{r,n-r} \cap Id(A,*).$$

Hence $c_{r,n-r}(A,*) \leq c_n(A) \leq \alpha n^t$, for some α, t, for all r. Let $J^q = 0$. Since $A^- \subseteq J$, then, for all $r \leq n - q$, $V_{r,n-r} \cap Id(A,*) = V_{r,n-r}$ and $c_{r,n-r}(A,*) = 0$ follows. By Theorem 5 for all n we obtain

$$c_n(A,*) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A,*) \leq \alpha n^t \sum_{r=n-q+1}^{n} \binom{n}{r}$$

$$= \alpha n^t \sum_{r=0}^{q-1} \binom{n}{r} \leq \alpha n^{t+q}$$

and $c_n(A,*)$ is polynomially bounded. □

Next we want to get an analogue of Theorem 3 above by using the representation theory of the hyperoctahedral group H_n. We write $\chi_{\lambda,\mu}(1) = d_{\lambda,\mu}$ for the degree of the irreducible H_n-character $\chi_{\lambda,\mu}$. Recall that if $\lambda \vdash n, \mu \vdash n - r$, then $d_{\lambda,\mu} = \binom{n}{r} d_\lambda d_\mu$ (see [DG]).

Theorem 7. *Let A be a finite dimensional algebra with involution over a field F. Then the sequence of $*$-codimensions $\{c_n(A,*)\}_{n\geq 1}$ is polynomially bounded if and only if*

$$\chi_n(A,*) = \sum_{\substack{|\lambda|+|\mu|=n \\ n-\lambda_1<q}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

where $J(A)^q = 0$.

Proof. Since the decomposition of $V_n(*)$ into irreducible H_n-modules does not change by extending the scalars, as in the proof of Theorem 3 we may assume that F is algebraically closed.

Suppose that the $*$-codimensions of A are polynomially bounded and let $\lambda = (\lambda_1, \dots, \lambda_t) \vdash r, \mu \vdash n-r$ be such that $n - \lambda_1 \geq q$. Suppose by contradiction that $m_{\lambda,\mu} \neq 0$; then there exist tableaux T_λ, T_μ such that $e_{T_\lambda}e_{T_\mu}$ has a non-trivial action on $V_{r,n-r}(A,*)$. This says that there exists a non trivial polynomial $f \in e_{T_\lambda}e_{T_\mu}V_{r,n-r}$ such that $f = f(s_1, \dots, s_r, k_{r+1}, \dots, k_n)$ is not a $*$-identity of A.

We have that f is a linear combination of polynomials each alternating on t disjoint sets of $\lambda_1', \dots, \lambda_t'$ symmetric variables respectively. Let g be one such polynomial; it is clear that, in order to finish the proof, it is enough to show that $g \equiv 0$ in A.

Since $B_iB_k = B_iJB_k = 0$ for all $i \neq k$, we get $g \equiv 0$ on A unless we substitute for the symmetric variables elements from one simple component, say B_i, and from J. Also, since $\dim B_i = 1$, only one element of B_i can be replaced for a variable in each alternating set. Thus, since $A^- \subseteq J$, in all we substitute at least $|\lambda| - \lambda_1 + |\mu| = n - \lambda_1 \geq q$ elements from J. Since $J^q = 0$ we get that $g \equiv 0$ also in this case and the proof of the first part is complete.

Suppose now that $\chi_n(A,*) = \sum_{|\lambda|+|\mu|=n, n-\lambda_1<q} m_{\lambda,\mu}\chi_{\lambda,\mu}$. By a result of Berele ([B, Theorem 15]) the multiplicities $m_{\lambda,\mu}$ are polynomially bounded. By recalling that if $|\lambda| - \lambda_1$ is bounded by a constant, then d_λ is polynomially bounded, we get

$$c_n(A,*) = \sum_{\substack{|\lambda|+|\mu|=n \\ n-\lambda_1<q}} m_{\lambda,\mu}d_{\lambda,\mu} \leq \alpha n^t \sum_{r=n-q}^{n} \sum_{\substack{\lambda\vdash r,\mu\vdash n-r \\ n-\lambda_1<q}} \binom{n}{r} d_\lambda d_\mu$$

$$\leq \alpha_1 n^{t_1} \sum_{r=0}^{q} \binom{n}{r} \leq \alpha_1 n^{t_1} n^{q+1}.$$

□

The previous theorem says that a finite dimensional algebra with involution A has polynomial growth of the $*$-codimensions if and only if all the irreducible H_n-characters $\chi_{\lambda,\mu}$ appearing with non-zero multiplicity in $\chi_n(A,*)$ are such that the diagram of λ, without the first row, and the diagram of μ contain in all at most q boxes.

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