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POSITIVE AND NEGATIVE 3-K-CONTACT STRUCTURES

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Abstract. The aim of this paper is to give a characterization of 3-K-contact and quasi 3-K-contact manifolds.

1. Introduction

The aim of this paper is to describe the class of Riemannian manifolds that is closely related to quaternionic-Kähler manifolds. It is well known that there is a close connection between the quaternionic-Kähler manifolds with positive scalar curvature and the class of 3-Sasakian manifolds (see [K], [Ku], [I1], [BGM]). Ishihara [I1] has shown that every fibered Riemannian 3-Sasakian bundle has a base which is a quaternionic-Kähler manifold with positive scalar curvature. The 3-Sasakian manifolds are essentially the $SO(3)$-principal bundles (or principal orbifold bundles) over quaternionic-Kähler manifolds (see [BGM]). We have proved in [J2] that for any quaternionic-Kähler manifold M with negative scalar curvature there exists an $SO(3)$-principal fibre bundle P such that P is an $\mathcal{A}$-manifold, the projection $p : P \to M$ is a Riemannian submersion and P admits three K-contact structures satisfying the relations very similar to those characterizing the 3-K-contact structures. We have called such structures the quasi 3-K-contact structures. Recently similar structures were introduced by S. Tanno ([T]). S. Tanno introduced nS-structures (an nS-structure is a quasi 3-K-contact structure satisfying an additional condition analogous to that characterizing the 3-Sasakian structure in the positive case). S. Tanno proved that every quaternionic-Kähler manifold with negative scalar curvature admits an $SO(3)$-principal fibre bundle P with canonical nS-structure and that any 3-K-contact structure on a 7-dimensional manifold has to be 3-Sasakian (respectively any quasi 3-K-contact structure on a 7-dimensional manifold has to be an nS-structure). In the present paper we extend these results to any dimension $4n + 3 > 11$. The work is devoted to the study of general quasi 3-K-contact structures. We shall also call them negative 3-K-contact structures and the usual 3-K-contact structure we shall call positive 3-K-contact structure. The manifold P with 3-K-contact (positive or negative) structure has dimension $4n+3$. We show that if $n \neq 2$, then every (positive) 3-K-contact structure is 3-Sasakian and every negative 3-K-contact manifold is an $\mathcal{A}$-manifold whose Ricci tensor has two constant eigenvalues.

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2. Preliminaries

We start by recalling some basic facts concerning the quaternion-Kähler geometry (see [S], [Sw], [B], [GL]) and K-contact structures. By $\mathfrak{X}(M)$ we denote the Lie algebra of all local vector fields on M. If D is a vector bundle over M, then by $\Gamma(D)$ we denote the set of all local sections of D. We also write $\mathcal{A}^k(M) = \Gamma(\wedge^k TM^*)$. Let (M,g) be an oriented Riemannian manifold, let $\dim M = 4n$ and let $SO(M)$ be the $SO(4n)$-principal fibre bundle of oriented orthonormal frames $u : \mathbb{R}^{4n} \to TM$. By ∇ we denote the Levi-Civita connection of (M,g). Let $\mathcal{G} \subset \operatorname{End}(TM)$ be the 3-dimensional subbundle locally generated by three almost complex structures $\{I,J,K\}$ compatible with the metric g and satisfying the additional condition $I \circ J = -J \circ I = K$. If $n > 1$, then a manifold (M,g) with the bundle $\mathcal{G}$ satisfying the above conditions is called the almost quaternion Hermitian manifold. The subbundle $\mathcal{G}$ is called parallel (with respect to ∇) if for every section $A \in \Gamma(\mathcal{G})$ and for every $X \in \mathfrak{X}(M)$ we have $\nabla_X A \in \mathcal{G}$. An almost quaternion Hermitian manifold (M,g) with a parallel bundle $\mathcal{G}$ is called a quaternionic-Kähler manifold. If $\dim M = 4$, then we shall call (M,g) a quaternionic-Kähler manifold if it is anti-self-dual and Einstein (we shall always choose an orientation of M in such a way that $\mathcal{G} = \bigwedge^+ M$). Every quaternionic-Kähler manifold is an Einstein manifold. Let (M,g) be a Riemannian manifold and ξ be a unit Killing vector field on M. Let us define a tensor field ϕ by $\phi(X) = \nabla_X \xi$ and a 1-form $\eta(X) := g(\xi, X)$. Then we call (M,g,ξ,ϕ,η) a K-contact structure if the following relation is satisfied:

$$\phi^2 = -id + \eta \otimes \xi. \tag{K}$$

Let us assume that ξ_0 is a Killing vector field of constant length on M. We shall find the conditions under which the Killing vector field $\xi = c\xi_0$ where $c = \frac{1}{\|\xi_0\|}$ defines the K-contact metric structure. Let us denote by

$$H = ker\, \eta = \{X : g(\xi, X) = 0\}$$

the distribution of horizontal vectors on M. The following Lemma is well known (see [J1]).

Lemma. *Under the above assumptions the Killing vector field ξ gives the K-contact structure on M if and only if the tensor $J = \phi_{|H}$ is the almost complex structure on the bundle H, i.e. $J^2 = -id_{|H}$.*

The mapping $p : P \to M$ is a Riemannian submersion (see [ON]) if for every $y \in P$ the mapping $d_y p : H_y \to T_x M$ is an isometry, where $x = p(y)$ and H_y is an orthogonal complement of the vertical space $V_y = T_y F_x$ where $F_x = p^{-1}(x)$. In the sequel we shall use the O'Neill's tensors T, A. They are defined as follows:

$$A_X Y = \mathcal{V}(\nabla_{\mathcal{H}X}\mathcal{H}Y) + \mathcal{H}(\nabla_{\mathcal{H}X}\mathcal{V}Y),$$
$$T_X Y = \mathcal{H}(\nabla_{\mathcal{V}X}\mathcal{V}Y) + \mathcal{V}(\nabla_{\mathcal{V}X}\mathcal{H}Y),$$

where $\mathcal{H}, \mathcal{V}$ respectively denote the projections on the horizontal and vertical subbundles H, V of $TP = H \oplus V$. Finally, let us recall that a Riemannian manifold (M,g) is called an $\mathcal{A}$-manifold (see [G]) (we shall write $M \in \mathcal{A}$ in such a case) if the Ricci tensor of (M,g) satisfies the condition $\nabla_X \rho(X,X) = 0$ for all local vector fields $X \in \mathfrak{X}(M)$.

3. Quasi 3-K-contact structures

We start with a definition of the 3-K-contact and quasi 3-K-contact structure.

Definition. Let (P, g) be a Riemannian manifold that admits three distinct K-contact structures (ϕ_i, ξ_i, η_i) such that

$$(a)\ g(\xi_i, \xi_j) = \delta_{ij}, \quad (b)\ [\xi_i, \xi_j] = 2\epsilon_{ijk}\xi_k, \quad (c)\ \phi_i\xi_j = -\epsilon_{ijk}\xi_k$$

where $\phi_i = \nabla\xi_i,\ \eta_i(X) = g(\xi_i, X)$. Let us denote by H the horizontal distribution $H = \ker\eta_1 \cap \ker\eta_2 \cap \ker\eta_3 = \bigcap \ker\eta_i$ and let us define the almost complex structures J_i on H by the formulas $J_i = -\phi_{i|H}$. We shall call (P, ξ_1, ξ_2, ξ_3) the 3-K-contact structure (or positive 3-K-contact structure) if (for $i \neq j$)

$$J_i \circ J_j = \epsilon_{ijk} J_k$$

and the quasi 3-K-contact structure (or negative 3-K-contact structure) if $(i \neq j)$

$$J_i \circ J_j = -\epsilon_{ijk} J_k.$$

The Riemannian manifold (P, g) with positive (negative) 3-K-contact structure we shall call positive (negative) 3-K-contact manifold.

By $V = span_{\mathbb{R}}\{\xi_1, \xi_2, \xi_3\}$ we shall denote the vertical bundle of P. It is clear that $TP = H \oplus V$ and $H \perp V$. The distribution V is integrable and the leaves are totally geodesic submanifolds of P. If P is complete, then the leaves are 3-dimensional spherical space forms.

Remark. Our definition of positive 3-K-contact structure is equivalent to the usual one (see [Ku], [J1]). S. Tanno defined in [T] an nS-structure. The nS-structure is a negative 3-K-contact structure (condition (6.3) in [T] is equivalent to $J_i \circ J_j = -\epsilon_{ijk} J_k$) satisfying an additional condition ((6.4) in [T]). We shall show in the sequel that if $\dim P = 4n + 3 \neq 11$, then every negative 3-K-contact structure must be an nS-structure.

Note that if P is complete, then it admits an action of the group $SU(2)$ or $SO(3)$ of isometries of P. Let us assume that (P, g) is a fibre bundle $p : P \to M$ and the group G ($G = SU(2)$ or $G = SO(3)$) acts on P on the right by isometries such that the orbits of the action coincides with the fibers of p, i.e. $p^{-1}(p(x)) = orb_G(x)$ and $M = P/G$. Let us assume that Killing tensors ξ_1, ξ_2, ξ_3 corresponding to the basis of the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ of G define on P the (positive or negative) 3-K-contact structure. Thus the fibers are totally geodesic submanifolds of P isometric to G/Γ where Γ is a discrete subgroup of G. If $\Gamma = \{e\}$, i.e. $p : P \to M$ is a G-principal fibre bundle over M, then we shall call P the 3-K-contact principal fibre bundle. If $X \in \mathfrak{X}(M)$, then by $X^* \in \Gamma(H)$ we mean a horizontal lift of X, i.e. the horizontal vector field $X^* \in \mathfrak{X}(P)$ which is p related with X $(dp(X^*) = X \circ p)$.

Theorem 1. *Let (P, g) be an $SO(3)$ principal 3-K-contact bundle (positive or negative) over a manifold M. Then the metric g induces a metric g_* on M such that $p : (P, g) \to (M, g_*)$ is a Riemannian submersion and (M, g_*) is an almost quaternion Hermitian manifold.*

Proof (Compare [I1]). We start by constructing a bundle $\mathcal{G} \subset \mathrm{End}(TM)$ locally spanned by three almost complex structures giving the quaternionic structure on

M. Let $\sigma \in \Gamma(P)$ be a local section of the bundle P. We shall define on $U =$domσ three almost complex structures

$$J_i^\sigma = \epsilon dp \circ \phi_i \circ \sigma^* \tag{3.1}$$

where $\sigma^*(X)(x) = X^*_{\sigma(x)}, \epsilon = -1$ if (P, g) is a positive 3-K-contact manifold, $\epsilon = 1$ if (P, g) is a negative manifold and X^* denotes the horizontal lift of the field $X \in \mathfrak{X}(M)$. We have $dp\phi_i d\sigma(X) = dp \circ \phi_i(d\sigma(X) - \sigma^*(X) + \sigma^*(X)) = dp \circ \phi_i \sigma^*(X)$ since $d\sigma(X) - \sigma^*(X) \in \Gamma(V)$ and $\phi_i(V) \subset V$. Thus in fact we have

$$J_i^\sigma = \epsilon dp \circ \phi_i \circ d\sigma. \tag{3.2}$$

It is obvious from the definition of positive (negative) 3-K-contact structure that the almost complex structures $\{J_1^\sigma, J_2^\sigma, J_3^\sigma\}$ define on U the almost quaternionic structure, i.e. $J_i^\sigma \circ J_j^\sigma = \epsilon_{ijk} J_k^\sigma$. Thus dim$M = 4n$. The structures $\{J_1^\sigma, J_2^\sigma, J_3^\sigma\}$ are the sections spanning the 3-dimensional bundle $\mathcal{G}_U \subset End(TP)$. We shall show that $\mathcal{G}_U$ does not depend on σ and there exists a global bundle $\mathcal{G}$ such that bundles $\mathcal{G}_U$ are the restrictions of $\mathcal{G}$, i.e. $\mathcal{G}|_U = \mathcal{G}_U$.

The group $SO(3)$ has an adjoint representation ad in the vector space $\mathfrak{so}(3)$ defined by $ad_g X = gXg^{-1}$ for $X \in \mathfrak{so}(3)$. Let us denote

$$ad_g(E_i) = \sum_{j=1}^{3} A_j^i(g) E_j$$

where $\{E_1, E_2, E_3\}$ is the standard basis of $\mathfrak{so}(3)$ corresponding to the Killing fields ξ_1, ξ_2, ξ_3. Let V be the 3-dimensional vector space and $\mathcal{C} = (e_1, e_2, e_3)$ be a basis of V. Then by $ad^{\mathcal{C}}$ we shall mean the linear representation of $SO(3)$ in V defined on $\mathcal{C}$ by $ad_g^{\mathcal{C}} e_i = A_i^j(g) e_j$. The group $G = SO(3)$ acts on (P, g) from the right by the isometries R_g. We shall also write pg instead of $R_g p$. Note that $\nabla_{(R_g)_* X}(R_g)_* Y = (R_g)_*(\nabla_X Y)$ where $((R_g)_* X)_p = d_{pg^{-1}} R_g(X_{pg^{-1}})$. Let $X = \xi^+$ be the fundamental Killing vector field corresponding via the action of G to the vector $\xi \in \mathfrak{so}(3)$. Let us write $a_t = \exp(t\xi)$. Note that $((R_g)_*(\xi)^+)_p = \frac{d}{dt}(pg^{-1}a_t g) = \frac{d}{dt}(p(ad(g^{-1})(a_t))) = (ad(g^{-1})\xi)_p^+$. It is also clear that

$$p(\nabla_X (ad(g^{-1})\xi_i)^+) = p(\sum_{j=1}^{3} A_i^j(g^{-1}) \nabla_X \xi_j^+).$$

Thus

$$\begin{aligned} p\phi_i(R_g \sigma^* X) &= p(\sum_{j=1}^{3} A_i^j(g^{-1}) \nabla_{\sigma^* X} \xi_j^+) \\ &= p(\sum_{j=1}^{3} A_i^j(g^{-1}) \phi_j \sigma^* X) = \sum_{j=1}^{3} A_i^j(g^{-1}) p\phi_j \sigma^* X) \end{aligned} \tag{3.3}$$

for any section $\sigma \in \Gamma(P)$. Let σ_1, σ_2 be two sections of the bundle P, such that $U_{12} =$dom$\,\sigma_1 \cap$dom$\,\sigma_2 \neq \emptyset$. Then $\sigma_1 = \sigma_2 g_{12}$ where $g_{12} : U_{12} \to G$ is a transition function. From (3.3) it follows that

$$(J_1^{\sigma_1}, J_2^{\sigma_1}, J_3^{\sigma_1}) = ad^{\mathcal{C}}(g_{12}^{-1})(J_1^{\sigma_2}, J_2^{\sigma_2}, J_3^{\sigma_2}) \tag{3.4}$$

where $\mathcal{C} = \{J_1^{\sigma_2}, J_2^{\sigma_2}, J_3^{\sigma_2}\}$.

Hence there exists a global bundle $\mathcal{G} \subset End(TP)$ which is locally spanned by the bases $\{J_1^\sigma, J_2^\sigma, J_3^\sigma\}$. From (3.4) it follows that the bundle P is isomorphic to the G-principal fibre bundle associated with vector bundle $\mathcal{G}$. We also have

$$\mathcal{G} = P \times_{SO(3)} \mathfrak{so}(3). \tag{3.5}$$

Hence M is an almost quaternion manifold. In particular $dim M = 4n$. Let us define the metric g_* on M by the formula $g_*(X,Y)_x = g(X^*,Y^*)_y$ where $p(y) = x$ and X^*, Y^* are horizontal lifts of the fields $X, Y \in \mathfrak{X}(M)$. Since G acts by isometries it is clear that $g(X^*,Y^*)$ is constant on the fibers and the metric g is well defined. From the definition of g it is obvious that $p : (P,g) \to (M,g_*)$ is a Riemannian submersion. Note that each almost complex structure J_i^σ is compatible with the metric g_*, i.e.

$$g_*(X, J_i^\sigma Y) = -g_*(J_i^\sigma X, Y)$$

which is a straightforward consequence of (3.1). Thus $(M, g_*, \mathcal{G})$ is an almost quaternion Hermitian manifold. □

The 3-K-contact bundle P admits a natural connection form (see [BGM], p. 192)

$$\omega = \eta_1 E_1 + \eta_2 E_2 + \eta_3 E_3 \in \mathcal{A}(P) \otimes \mathfrak{so}(3).$$

Note that the metric on P can be written as

$$g = B(\omega,\omega) + p^* g_*$$

where B is the standard metric on $\mathfrak{so}(3)$ inducing on $SO(3)$ the metric of constant sectional curvature equal to 1. The horizontal and vertical subbundles H_ω, V_ω with respect to the connection on P defined by the form ω coincide with the horizontal and vertical bundles defined by us earlier. The vector bundle $\mathcal{G}$ defines a reduction of the $SO(4n)$ bundle $SO(M)$ to the $Sp(n)Sp(1)$ subbundle

$$Q = \{u \in SO(M) : uI_0u^{-1} \in \mathcal{G}, uJ_0u^{-1} \in \mathcal{G}, uK_0u^{-1} \in \mathcal{G}\}$$

and we have the homomorphism of principal fibre bundles $F : Q \to P$ where we identify P with the bundle of orthonormal bases of $\mathcal{G}$ defined by

$$F(u) = (uI_0u^{-1}, uJ_0u^{-1}, uK_0u^{-1}). \tag{3.6}$$

It is clear that the Levi-Civita connection of M reduces to Q if and only if the bundle $\mathcal{G}$ is parallel, i.e. if for any section $\sigma \in \Gamma(\mathcal{G})$ and any vector field $X \in \mathfrak{X}(M)$ we have $\nabla_X \sigma \in \Gamma(\mathcal{G})$. We shall find the conditions under which $\nabla\mathcal{G} = 0$. Note that for $X, Y \in \mathfrak{X}(M)$

$$\epsilon g_*(X, J_i^\sigma Y)_x = g(X^*, \phi_i Y^*)_{\sigma(x)}. \tag{3.7}$$

Let us assume that $X, Y, Z \in \mathfrak{X}(M)$,$\nabla_Z X_x = \nabla_Z Y_x = 0$. Thus we have from (3.7)

$$\begin{aligned}\epsilon g_*(X, \nabla J_i^\sigma(Z,Y) &= g(\nabla_{\sigma_* Z} X^*, \phi_i Y^*) \\ &+ g(X^*, \nabla\phi_i(\sigma_* Z, Y^*)) + g(X^*, \phi_i(\nabla_{\sigma_* Z} Y^*)).\end{aligned} \tag{3.8}$$

Note that $(\sigma_* Z)_x = Z^*_{\sigma(x)} + V_{\sigma(x)}$ where $V \in T_{\sigma(x)}P$ is a vertical vector. We can extend V to a (vertical) Killing vector field $V = \omega(\sigma_* Z)^+ \in \Gamma(V)$. Note that $(\nabla_V X^*)_{\sigma(x)} = (\nabla^*_X V)_{\sigma(x)}$. We also have $\nabla_{Z^*} X^* \in \Gamma(V)$, $\nabla_{Z^*} Y^* \in \Gamma(V)$. Let us write $V = \omega(\sigma_*(Z)) = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ where $\alpha_i \in \mathbb{R}$. Note that

$$\begin{aligned}\epsilon g_*(X, \nabla J_i^\sigma(Z,Y)) &= g(\nabla_V X^*, \phi_i Y^*) + g(X^*, \nabla\phi_i(\sigma_* Z, Y^*) \\ &+ g(X^*, \phi_i(\nabla_{\sigma_* Z} Y^*)).\end{aligned} \tag{3.9}$$

We also have

$$\begin{aligned} &g(\nabla_V X^*, \phi_i Y^*) + g(X^*, \phi_i(\nabla_{\sigma_* Z} Y^*) \\ &= \sum_{j=1}^{3} \alpha_j (g(\phi_j(X^*), \phi_i(Y^*)) - g(\phi_i(X^*), \phi_j(Y^*))) \\ &= \sum_{j \neq i} 2\alpha_j g(X^*, \phi_i \circ \phi_j(Y^*)) = 2 \sum_{j \neq i} \alpha_j \epsilon \epsilon_{ijk} g(X^*, \phi_k(Y^*)). \end{aligned}$$

It follows that

$$\epsilon g_*(X, \nabla J_i^\sigma(Z, Y)) = 2 \sum_{j \neq i} \alpha_j \epsilon \epsilon_{ijk} g(X^*, \phi_k(Y^*)) + g(X^*, \nabla \phi_i(\sigma_* Z, Y^*)).$$

Consequently

$$g_*(X, \epsilon \nabla J_i^\sigma(Z, Y) - 2 \sum_{j \neq i} \alpha_j \epsilon_{ijk} J_k^\sigma(Y)) = g(X^*, \nabla \phi_i(\sigma_* Z, Y^*)). \tag{3.10}$$

Hence we have proved:

Proposition 1. *Let P be a 3-K-contact principal fibre bundle. Then the following conditions are equivalent:*
(a) $\nabla \phi_i(X, Y) \in \Gamma(V)$ *for* $i \in \{1, 2, 3\}$ *and any* $X \in \mathfrak{X}(P)$, $Y \in \Gamma(H)$,
(b) $R(X, \xi_i)Y \in \Gamma(V)$ *for* $i \in \{1, 2, 3\}$ *and any* $X \in \mathfrak{X}(P)$, $Y \in \Gamma(H)$,
(c) $R(X, Y)\xi_i = 0$ *for* $i \in \{1, 2, 3\}$ *and any* $X, Y \in \Gamma(H)$,
(d) $R(X, Y)Z \in \Gamma(H)$ *for any* $Z \in \mathfrak{X}(P)$ *and* $X, Y \in \Gamma(H)$.
Each of these conditions implies the following condition:
(e) *the bundle $\mathcal{G}$ is parallel.*

Proof. It follows from (3.10), the equality $R(X, \xi_i)Y = \nabla \phi_i(X, Y)$ and the properties of the Riemannian curvature tensor R. □

Remark. If the 3-K-contact structure is Sasakian, then for $X \in \mathfrak{X}(P)$ and $Y \in \mathfrak{X}(M)$

$$\nabla \phi_i(X, Y^*) = \eta_i(Y^*)X - g(X, Y^*)\xi_i = -g(X, Y^*)\xi_i \in \Gamma(V)$$

and the condition (a) is satisfied. Hence (M, g_*) is a quaternionic-Kähler manifold if $dim\, M = 4n > 4$. We obtain in this way the result of Ishihara ([I1]).

Next we shall prove the following theorem (for $n = 1$ this is a result of S. Tanno [T]; we include a proof of this case for the completeness):

Theorem 2. *Let us assume that dim $M = 4n \neq 8$. Let (P, g) be an $SO(3)$ principal 3-K-contact bundle (positive or negative) over a manifold M. Then the metric g induces a metric g_* on M such that $p : (P, g) \to (M, g_*)$ is a Riemannian submersion and (M, g_*) is a (positive or negative respectively) quaternionic-Kähler manifold. The Riemannian manifold (P, g) is a 3-Sasakian (hence Einstein) manifold if (P, g, ξ_i) is a positive 3-K-contact structure and an $\mathcal{A}$-manifold whose Ricci tensor has two constant eigenvalues $\lambda = 4n + 2$ and $\mu = -4n - 14$ if (P, g, ξ_i) is a negative 3-K-contact structure.*

Proof. Let $\omega = \eta_1 E_1 + \eta_2 E_2 + \eta_3 E_3$ be as above the connection form on the bundle P. We have

$$D\omega = \Omega = \Omega_1 E_1 + \Omega_2 E_2 + \Omega_3 E_3$$

where $\Omega_i = d\eta_i \circ h$ and $h : TP \to H$ is the horizontal projection. Hence from the structural equation we get

$$\Omega_i = d\eta_i - \epsilon_{jki}\eta_j \wedge \eta_k.$$

Let us define the 4-form $\tilde{\Omega} \in \mathcal{A}^4(P)$ by the formula $\tilde{\Omega} = \Omega_1 \wedge \Omega_1 + \Omega_2 \wedge \Omega_2 + \Omega_3 \wedge \Omega_3$. Note that $L_{\xi_j}\Omega_i = 2(\epsilon_{ijk}d\eta_k - \eta_j \wedge \eta_i) = 2\epsilon_{jik}\Omega_k$. Hence $L_{\xi_j}\tilde{\Omega} = 0$ for $j = 1, 2, 3$ and the 4-form $\tilde{\Omega} \in \mathcal{A}^4(P)$ is the horizontal 4-form invariant with respect to the action of G. Let us define for a section $\sigma \in \Gamma(P)$ the three 2-forms $\Omega^i_\sigma = g_*(J^\sigma_i X, Y)$. Then $2(p^*\Omega^i_\sigma)_{\sigma(x)} = \epsilon d\eta_i \circ h_{\sigma(x)} = \epsilon(\Omega_i)_{\sigma(x)}$ where h denotes the horizontal projection and $\epsilon \in \{-1, 1\}$ was defined above. Let us define the 4-form $\Omega_\sigma = \Omega^1_\sigma \wedge \Omega^1_\sigma + \Omega^2_\sigma \wedge \Omega^2_\sigma + \Omega^3_\sigma \wedge \Omega^3_\sigma$. Then $\Omega_\sigma \in \mathcal{A}^4(dom\sigma)$. Note that in fact Ω_σ does not depend on σ and there exists a global 4-form $\bar{\Omega}$ such that $\bar{\Omega}_{|dom\sigma} = \Omega_\sigma$. We also have $4p^*\bar{\Omega} = \tilde{\Omega}$. Thus

$$4p^*d\bar{\Omega} = d\tilde{\Omega}. \tag{3.11}$$

Since the form $p^*d\bar{\Omega}$ is horizontal it follows that the form $d\tilde{\Omega} = D\tilde{\Omega}$ is horizontal. Thus

$$\begin{aligned} d\tilde{\Omega} &= 2(d\Omega_1 \wedge \Omega_1 + d\Omega_2 \wedge \Omega_2 + d\Omega_3 \wedge \Omega_3) \\ &= 2(D\Omega_1 \wedge \Omega_1 + D\Omega_2 \wedge \Omega_2 + D\Omega_3 \wedge \Omega_3). \end{aligned}$$

From the Bianchi identity we have $D\Omega_i = 0$. Consequently $d\bar{\Omega} = 0$, $d\tilde{\Omega} = 0$. From the result of Swann [Sw] it follows that if $dim\, M = 4n \geq 12$, then $\nabla\bar{\Omega} = 0$ and thus $(M, g_*, \mathcal{G})$ is a quaternionic-Kähler manifold. In [J1], [J2] we have proved that the principal $SO(3)$-bundle P associated with $\mathcal{G}$ admits canonical negative 3-K-contact structure (P_0, g_0) if (M, g_*) has negative scalar curvature and that (P_0, g_0) is then an $\mathcal{A}$-manifold whose Ricci tensor has two eigenvalues $\lambda = 4n+2$ and $\mu = -4n-14$ of multiplicity 3 and $4n$ respectively. In the case of positive scalar curvature (P_0, g_0) is a 3-Sasakian manifold (see [J1], [J2], [BGM]). We shall show that if $dim\, M = 4$, then (M, g_*) is anti-self-dual Einstein and if $n \neq 2$, then (P, g) is isometric to the canonical 3-K-contact bundle (P_0, g_0). We can treat P as the bundle of orthonormal frames of $\mathcal{G}$; hence $P = P_0$. We shall show that $\omega = \omega_0$ where ω_0 is the canonical metric connection form on P_0. To this end it is enough to prove that

$$\nabla J^\sigma_i(Z, Y) = -2\sum \epsilon_{ijk}\sigma^*\omega^j(Z)J^\sigma_k(Y) \tag{3.12}$$

for every local section $\sigma \in \Gamma(P)$. We first show that $\nabla\phi_i(Z^*, Y^*)_{p_0} \in \Gamma(V)$ for every $Z, Y \in \mathfrak{X}(M)$ and a point $p_0 \in P$. We can find a section $\sigma \in \Gamma(P)$ such that $\sigma(x_0) = p_0$ and $d_{x_0}\sigma(U) = U^*_{p_0} \in H_{p_0}$ for every $U \in T_{x_0}M$. Since either $(M, \mathcal{G})$ is quaternionic-Kähler or $dim\, M = 4$ and $\mathcal{G} = \wedge^+ M$ we have $\nabla J^\sigma_i(Z, Y) = \sum \theta^j_i(Z)J^\sigma_j(Y)$ where $\theta^i_j = -\theta^j_i$. Hence from (3.10) it follows that

$$g(X^*, \nabla\phi_i(Z^*, Y^*))_{p_0} = \epsilon \sum g_*(X, \theta^j_i(Z)J^\sigma_j(Y))_{x_0}.$$

Consequently

$$\epsilon\nabla\Omega_i(Z^*, X^*, Y^*)_{p_0} = \sum \theta^j_i(Z)_{x_0}\Omega^\sigma_j(X, Y)_{x_0}. \tag{3.13}$$

Let $Z \in T_{x_0}M$ be any vector and let $X = I_1 Z, Y = I_2 Z$. Note that $\Omega^\sigma_j(Z, X) = \delta^1_j g_*(Z, Z), \Omega^\sigma_j(Z, Y) = \delta^2_j g_*(Z, Z), \Omega^\sigma_j(X, Y) = \delta^3_j g_*(Z, Z)$. Since $D\Omega_i = 0$ we have

$\mathfrak{C}_{Z^*,X^*,Y^*}\nabla\Omega_i(Z^*,X^*,Y^*) = 0$ where $\mathfrak{C}$ denotes the cyclic sum. Consequently

$$\sum_j \mathfrak{C}_{Z,X,Y}\theta_i^j(Z)\Omega_j^\sigma(X,Y) = 0.$$

Thus $(\theta_i^3(Z)+\theta_i^1(Y)-\theta_i^2(X))g_*(Z,Z) = 0$. Consequently we get $\theta_1^3(Z)-\theta_1^2(I_1Z) = 0, \theta_2^3(Z)+\theta_2^1(I_2Z) = 0, \theta_3^1(I_2Z)-\theta_3^2(I_1Z) = 0$. Since Z was arbitrary we have $\theta_1^3(Z) = \theta_3^2(I_1I_2Z) = -\theta_2^3(I_3Z) = -\theta_1^2(I_2I_3Z) = -\theta_1^2(I_1Z)$. Thus $(\theta_1^3)_{x_0} = 0$ and $(\theta_j^i)_{x_0} = 0$. Since p_0 was an arbitrary point we have $\nabla\Omega_i\circ h = 0$, which means that $\nabla\phi_i(Z^*,Y^*)\in\Gamma(V)$. Note that from the first Bianchi equation we obtain (R being the curvature tensor of (P,g)) $R(\xi_j,\xi_i)Y^*+R(Y^*,\xi_j)\xi_i-R(Y^*,\xi_i)\xi_j = 0$. Consequently

$$\nabla\phi_i(\xi_j,Y^*) = R(\xi_j,\xi_i)Y^* = -R(Y^*,\xi_j)\xi_i+R(Y^*,\xi_i)\xi_j. \tag{3.14}$$

If (P,ξ_i) is a positive 3-K-contact structure, then $R(Y^*,\xi_i)\xi_j = \eta_j(Y^*)\xi_i = 0$, condition (a) of Proposition 1 is satisfied and consequently (3.12) holds. In the case of negative structure we have $R(Y^*,\xi_i)\xi_j = -2\epsilon_{ijk}\phi_k(Y^*)$ (see [J1]). Thus from (3.14) it follows that

$$\nabla\phi_i(\xi_j,Y^*) = -4\epsilon_{ijk}\phi_k(Y^*).$$

Consequently we obtain for an arbitrary section $\sigma\in\Gamma(P)$ (where $\alpha_j = \sigma^*\omega^j(Z)$)

$$g(X^*,\nabla\phi_i(\sigma_*Z,Y^*)) = \sum\alpha_j g(X^*,\nabla\phi_i(\xi_j,Y^*)) = -4\sum\epsilon_{ijk}\alpha_j g(X,J_k^\sigma Y)$$

and from (3.10) it follows that $\nabla J_i^\sigma(Z,Y) = -2\sum_{j\neq i}\alpha_j\epsilon_{ijk}J_k^\sigma(Y)$. Since $\alpha_j = \sigma^*\omega^j(Z)$ again formula (3.12) is satisfied. If $n = 1$, then from the Ricci identity (see [I2]) we have $\mathcal{R}\Omega_i^\sigma = 2\sum_{j<k}\epsilon_{ijk}(d\theta_k^j-\theta_j^i\wedge\theta_k^i)$ where $\mathcal{R}:\wedge M\to\wedge M$ is the curvature operator. Let the section σ be chosen as above. Since $\theta_k^j = -2\epsilon_{ijk}\sigma^*\omega_i$ we infer that at the point x_0 $\mathcal{R}\Omega_i^\sigma = 2\sigma^*d\omega_i = -4\epsilon\Omega_i^\sigma$. Since x_0 was arbitrary it follows that (see [GL] Definition 1.14) (M,g_*) is anti-self-dual Einstein with non-zero scalar curvature. Thus if $n\neq 2$ the positive (negative) 3-K-contact bundles are in one-to-one correspondence with positive (negative) quaternionic-Kähler manifolds. □

Remark. S. Tanno proved (Theorem B in [T]) that for the negative quaternionic-Kähler manifold every canonical 3-K-contact structure is an nS-structure. Consequently from our Theorem 2 it follows that if $n\neq 2$, then every negative 3-K-contact structure is in fact an nS-structure. Let us note that the O'Neill tensor A for the Riemannian submersion $p:P\to M$ is given by the formulas (see [BGM])

$$(a)\ A_XY = \sum_{i=1}^3 g(\phi_iX,Y)\xi_i, \qquad (b)\ A_X\xi_i = \phi_i(X)$$

for any horizontal vector fields X,Y. Since the fibers are totally geodesic the tensor $T = 0$. Hence we could also directly compute the scalar curvature and the eigenvalues of the Ricci S tensor using the O'Neill formulas (see [ON]) and the fact that the connection form ω is Yang-Mills. Since ξ_i define K-contact structures it is obvious that $S_{|V} = (4n+2)id_V$ (each Killing vector field defining K-contact structure is an eigenfield of the Ricci tensor with constant eigenvalue $\dim P-1$). Using the formulas in [B] (9.62) and [BGM] (p.192) we can show that $S_{|H} = \mu id_H$ where $\mu = \frac{1}{4n}(-24n+\tau_*)$ and τ_* is the scalar curvature of (M,g_*). In the positive case since P is Einstein it is clear that $\tau_* = 16n(n+2)$ (see [BGM]). However it needs some work to show that in the negative case $\tau_* = -16n(n+2)$ (see [J2]).

The general 3-K-contact structure is bundle like with respect to the vertical foliation V (see [BGM]). From the general results concerning bundle like manifolds it follows that on an open and dense subset $U \subset P$ the manifold $P_{|U}$ is the bundle associated with the G-principal bundle $\tilde{P}$ with the fibre $F = G/\Gamma$ where Γ is the discrete (hence finite) subgroup of G and $G = Sp(1)$ or $G = SO(3)$. Note also that if $G = Sp(1)$, then $Q = \tilde{P}/\mathbb{Z}_2$ is an $SO(3)$-principal fibre bundle. The metric g induces on Q the metric $\bar{g}$ such that the natural projection $\pi : \tilde{P} \to Q$ is a local isometry. Thus the above results concerning the 3-K-contact $SO(3)$-bundle remain valid for the $Sp(1)$-principal 3-K-contact bundle. If dim$P > 11$, then it follows from Theorem 2 and [J2] that the positive 3-K-contact structure is then the 3-Sasakian structure. From [BGM] it is clear that in this case P is an orbifold bundle over quaternionic-Kähler orbifold of positive scalar curvature $16n(n+2)$. Analogously as in [BGM] (p.192) we have taken account of Theorem 2 and the results from [J1], [J2]:

Theorem 3. *Let (P, g, ξ_i) be a (positive or negative) 3-K-contact manifold. Then dim$P = 4n+3$. Let us assume that P is complete and $n \neq 2$. If P is a positive 3-K-contact structure, then (P, g, ξ_i) is a 3-Sasakian structure. If P is a negative 3-K-contact structure, then*

(a) *(P, g, ξ_i) is an $\mathcal{A}$-manifold of negative scalar curvature which is equal to $-4n(4n+11)+6$,*

(b) *the Ricci tensor S of P has two constant eigenvalues $\lambda = 4n+2$ and $\mu = -4n-14$ of multiplicity 3 and $4n$ respectively,*

(c) *the eigendistributions of the Ricci tensor S are $D_\lambda = ker(S - \lambda Id) = V$ and $D_\mu = ker(S - \mu Id) = H$,*

(d) *the metric g is bundle like with respect to the foliation V defined by the fields ξ_i,*

(e) *each leaf of the foliation V is a 3-dimensional homogeneous spherical space form,*

(f) *the space of leaves P/V is a quaternionic-Kähler orbifold of dimension $4n$ with negative scalar curvature equal to $-16n(n+2)$.*

Remark. Note that we do not know whether Theorems 2 and 3 are true for $n = 2$. Any counterexample to Theorem 2 with $n = 2$ will also be an example of an almost quaternion-Hermitian manifold with closed and non-parallel fundamental 4-form Ω. (See [Sw] for a discussion of the problem of whether $d\Omega = 0$ implies $\nabla\Omega = 0$ also in the case $n = 2$.)

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References

[B] A. Besse, Einstein Manifolds, *Springer-Verlag* Berlin, Heidelberg (1987). MR **88f:**53087

[BGM] C.P. Boyer, K. Galicki, B.M. Mann, *The geometry and topology of 3-Sasakian manifolds*, J. reine. angew. Math **445** (1994), 183-220. MR **96e:**53057

[GL] K. Galicki and H.B. Lawson, *Quaternionic reduction and Quaternionic Orbifolds*, Math. Ann **282** (1988), 1-21. MR **89m:**53075

[G] A. Gray, *Einstein like manifolds which are not Einstein*, Geom. Dedicata. **7** (1978), 259-280. MR **80b:**53034

[I1] S. Ishihara, *Quaternion Kählerian manifolds and fibered Riemannian spaces with Sasakian 3-structure*, Kodai Math. Sem. Rep. **25** (1973), 321-329. MR **48:**2943

[I2] S. Ishihara, *Quaternion Kählerian manifolds,* J. Differential Geom. **9** (1974), 483-500. MR **50:**1184

[J1] W. Jelonek, *Almost Kähler $\mathcal{A}$-structures on twistor bundles,* Ann. of Global Analysis and Geometry **17** (1999), 329–339. CMP 99:17

[J2] W. Jelonek, *Quaternion Kähler geometry and almost Kähler $\mathcal{A}$-manifolds,* (to appear in Ann. Polon. Math.).

[K] M. Konishi, *On manifolds with Sasakian 3 - structures over quaternionic Kähler manifolds,* Kodai Math. Sem. Rep. **26** (1975), 194-200. MR **51:**13951

[Ku] Y-Y. Kuo, *On almost contact 3-structures,* Tohoku Math. J. **22** (1970), 325-332. MR **43:**3956

[ON] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. **13** (1966), 459-469. MR **34:**751

[S] S. Salamon, *Quaternionic Kähler Manifolds,* Inv. Math. **67** (1982), 143-171. MR **83k:**53054

[Sw] A. Swann, *Hyperkähler and Quaternionic Kähler geometry*, Math. Ann. **289** (1991), 421-450. MR **92c:**53030

[T] S. Tanno, *Remarks on a triple of K-contact structures*, Tôhoku Math J. **48** (1996), 519-531. MR **97i:**53055

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