PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 1, Pages 165–172
S 0002-9939(00)05541-6

UNBOUNDED QUASI-INTEGRALS

ALF BIRGER RUSTAD

(Communicated by Palle E. T. Jorgensen)

Abstract. Let X be a locally compact Hausdorff space. We define a quasi-measure in X, a quasi-integral on $C_0(X)$, and a quasi-integral on $C_c(X)$. We show that all quasi-integrals on $C_0(X)$ are bounded, continuity properties of the quasi-integral on $C_c(X)$, representation of quasi-integrals on $C_c(X)$ in terms of quasi-measures, and unique extension of quasi-integrals on $C_c(X)$ to $C_0(X)$.

1. Introduction

The notion of a quasi-measure was introduced in [1] by J. F Aarnes. In [1], physical states on commutative unital C*-algebras were represented by quasi-measures. The quasi-measure in [1] was defined as a regular, finitely additive set function on open and closed subsets of a compact Hausdorff space X. The quasi-integral (physical state) with respect to a quasi-measure was constructed on the space of continuous functions on X (denoted $C(X)$). The quasi-integrals were shown to be the maps linear on each uniformly closed, singly generated subalgebra of $C(X)$.

Recent results (cf. [2], [4] and [6]) indicate that the quasi-measures are interesting as a generalization of regular Borel measures. The restriction of a quasi-measure to a compact Hausdorff space is therefore unfortunate. Accordingly, the work presented here aims to extend the theory in [1] to X being a locally compact Hausdorff space.

In the sequel we let X denote a locally compact Hausdorff space. A set is called bounded if its closure is compact. $\mathcal{F}$ and $\mathcal{O}$ denote respectively the class of closed and the class of open subsets of X. Similarly, $\mathcal{C}$ and $\mathcal{O}^*$ denote respectively the class of compact and the class of open bounded subsets. Furthermore we put $\mathcal{A} = \mathcal{F} \cup \mathcal{O}$ and $\mathcal{A}^* = \mathcal{C} \cup \mathcal{O}^*$.

Definition 1.1. A quasi-measure in X is a function $\mu : A \to [0, \infty]$ satisfying the following conditions:

1. $\mu(A) < \infty$ if $A \in \mathcal{A}^*$.
2. For any finite, disjoint collection $\{A_i\}_{i=1}^n \subset \mathcal{C} \cup \mathcal{O}$ with $\bigcup_{i=1}^n A_i \in \mathcal{C} \cup \mathcal{O}$, then
$$\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$
3. $\mu(U) = \sup\{\mu(K) : K \subset U, K \in \mathcal{C}\}, U \in \mathcal{O}$.
4. $\mu(F) = \inf\{\mu(U) : F \subset U, U \in \mathcal{O}\}, F \in \mathcal{F}$.

Received by the editors October 14, 1996 and, in revised form, March 22, 1999.
1991 *Mathematics Subject Classification.* Primary 28A25.

Our quasi-measure corresponds to the quasi-measure in [3], and the reader will find numerous properties of the quasi-measure there. The notion of a quasi-measure in a locally compact Hausdorff space can also be found in [7]. The definition in [7] is more restrictive than ours and does not produce the quasi-integrals given below.

Let $C_0(X)$ denote the real-valued continuous functions on X vanishing at infinity and let $C_c(X)$ be the functions in $C_0(X)$ with compact support. The support of a function $a \in C_0(X)$ will be denoted by $\operatorname{supp} a$ and the range of a in $\mathbf{R}$ by $\operatorname{sp} a$. If $a \in C_0(X)$, let $\mathbf{A}_0(a)$ denote the smallest uniformly closed subalgebra of $C_0(X)$ containing a. For subsets $K, O \subset X$ we will let $K \prec a$ and $a \prec O$ denote that $a \in C_c(X), \operatorname{sp} a \subset [0,1]$ and respectively that $x \in K \Rightarrow a(x) = 1$ and $\operatorname{supp} a \subset O$.

Definition 1.2. A real-valued function ρ on $C_0(X)$ is called a *quasi-integral* if the following conditions are satisfied:

1. $b \geq 0 \Rightarrow \rho(b) \geq 0$ whenever $b \in C_0(X)$.
2. ρ is linear on $\mathbf{A}_0(a)$ for each $a \in C_0(X)$.

When $\sup\{\rho(a) : a \prec X\} < \infty$ we say that ρ is bounded. If in addition $\sup\{\rho(a) : a \prec X\} = 1$ we say that ρ is a *quasi-state.*

In the C*-algebra setting this corresponds to the commutative, nonunital case, where ρ is characterized by linearity on closed subalgebras generated by self-adjoint elements.

If $a \in C_c(X)$, then we have $\mathbf{A}_0(a) \subset C_c(X)$. Hence we may define a quasi-integral on $C_c(X)$ similarly as above:

Definition 1.3. A real-valued function ρ on $C_c(X)$ is called a *quasi-integral* if:

1. $f \geq 0 \Rightarrow \rho(f) \geq 0$ whenever $f \in C_c(X)$.
2. ρ is linear on $\mathbf{A}_0(f)$ for each $f \in C_c(X)$.

If in addition $\sup\{\rho(f) : f \prec X\} < \infty$, then ρ is bounded and we put $\|\rho\| = \sup\{\rho(f) : f \prec X\}$.

The only difference between the definition above and Definition 1.2 is that ρ is now restricted to $C_c(X)$. However we will show that if ρ is bounded, these two definitions coincide (Corollary 3.10). The key results in this article are boundedness of quasi-integrals on $C_0(X)$ and a representation theorem between the quasi-measures in X and the quasi-integrals on $C_c(X)$. The representation is a generalization of the Riesz Representation Theorem in [5].

The section below presents some preparatory results on the quasi-measures and quasi-integrals on $C_0(X)$. The section ends with the boundedness theorem for quasi-integrals on $C_0(X)$. The next and last section presents construction of the quasi-integral on $C_c(X)$ with respect to a quasi-measure. Monotonicity and continuity properties of the quasi-integral are given. The section highlights with the representation theorem for quasi-measures and quasi-integrals on $C_c(X)$. Finally, unique extension to $C_0(X)$ of quasi-integrals on $C_c(X)$ is given.

2. Quasi-integrals on $C_0(X)$

Throughout this article we will assume that X is a locally compact Hausdorff space. The results in the following proposition were given in [3]. We will only give a brief outline of the proofs here.

Proposition 2.1. *Let μ be a quasi-measure in X.*

1. $\mu(\emptyset) = 0$.
2. $A \subset B \Rightarrow \mu(A) \leq \mu(B), A, B \in \mathcal{A}$.
3. *If* $K \in \mathcal{C}, F \in \mathcal{F}$ *are disjoint, then* $\mu(F \cup K) = \mu(F) + \mu(K)$.
4. μ *is countably additive on open sets.*
5. *Let* μ *be a quasi-measure in* X. *For any increasing family of open sets* $\{V_\lambda\}$, *if* $V_\lambda \nearrow V$ *(i.e.* $\bigcup V_\lambda = V$*), then* $\mu(V_\lambda) \nearrow \mu(V)$.

Proof. With $A_1 = A_2 = \emptyset$ in item 2 of Definition 1.1 we get $\mu(\emptyset) = 0$. The monotonicity follows from regularity (items 3 and 4 of Definition 1.1). The third statement follows from regularity and a Urysohn's lemma argument. The fifth statement follows from regularity (item 3 of Definition 1.1). The fifth statement and finite additivity (item 2 of Definition 1.1) imply the fourth statement.

Proposition 2.2. *A set function* $\mu : A \to [0, \infty]$ *satisfying items* 1, 3, *and* 4 *of Definition 1.1 is a quasi-measure if and only if the following are satisfied*:

1. *If* $O_1, O_2 \in O$ *are disjoint, then* $\mu(O_1 \cup O_2) = \mu(O_1) + \mu(O_2)$.
2. *If* $K \subset O \in O$ *with* K *compact, then* $\mu(O) = \mu(O \backslash K) + \mu(K)$.

Proof. The proof of the third statement in Proposition 2.1 holds for μ. Hence by induction μ is finitely additive on $\mathcal{C}$. Similarly, μ is finitely additive on $\mathcal{O}$ by assumption 2.2.1. Let $\{A_i\}_{i=1}^n \subset \mathcal{C} \cup \mathcal{O}$ with disjoint union $A = \bigcup_{i=1}^n A_i \in \mathcal{C} \cup \mathcal{O}$. We may split the union to a disjoint union of a compact and an open set by $A = (\bigcup_{A_i \in \mathcal{C}} A_i) \cup (\bigcup_{A_i \notin \mathcal{C}} A_i)$. If A is open, then $\mu(A) = \mu(\bigcup_{A_i \in \mathcal{C}} A_i) + \mu(\bigcup_{A_i \notin \mathcal{C}} A_i)$ by assumption 2.2.2. With μ finitely additive on $\mathcal{C}$ and on $\mathcal{O}$ we obtain $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$. If A is compact we may use a similar argument. Hence it suffices to show that if O is open and $O \subset K \in \mathcal{C}$, then $\mu(K) = \mu(K \backslash O) + \mu(O)$. Let $K' \subset O$ be compact. Then since μ is monotone $\mu(K) \geq \mu(K \backslash O) + \mu(K')$. Taking supremum of all $K' \subset O$, regularity yields $\mu(K) \geq \mu(K \backslash O) + \mu(O)$. Conversely, given $\epsilon > 0$ pick an open set $U \supset K \backslash O$ with $\mu(U) < \mu(K \backslash O) + \epsilon$. Observing that $K \backslash U \subset O$ yields

$$\begin{array}{rcl} \mu(K) & \leq & \mu(U \cup O) = \mu(U) + \mu(K \backslash U) \\ & < & \mu(K \backslash O) + \mu(O) + \epsilon. \end{array}$$

Equality follows. We have shown finite additivity for μ on $\mathcal{C} \cup \mathcal{O}$ which completes the proof.

Lemma 2.3. *A quasi-integral* ρ *on* $C_0(X)$ *is bounded on* $\mathbf{A}_0(a)$ *for each* $a \in C_0(X)$.

Proof. Suppose $\sup\{\rho(f) : 0 \leq f \leq 1, f \in \mathbf{A}_0(a)\} = \infty$ for some $a \in C_0(X)$. Choose ϕ_i with $\phi_i \circ a \in \mathbf{A}_0(a)$, $\rho(\phi_i(a)) > 2^{2i}$ and $0 \leq \phi_i(a) \leq 1$ for $i = 1, 2, \ldots$. Then with $\phi = \sum_{i=1}^\infty 2^{-i} \phi_i$ we have $\phi \circ a \in \mathbf{A}_0(a)$, $0 \leq \phi(a) \leq 1$ and $\rho(\phi(a)) = \infty$, which is a contradiction. Hence we must have ρ bounded on $\mathbf{A}_0(a)$.

Remark 1. Note that ρ is a linear functional on $\mathbf{A}_0(a)$ and thus boundedness implies that ρ is continuous on $\mathbf{A}_0(a)$ for each $a \in C_0(X)$. Hence

$$\sup\{\rho(a) : a \prec X\} = \sup\{\rho(a) : 0 \leq a \leq 1, a \in C_0(X)\}$$

for all quasi-integrals ρ on $C_0(X)$. Moreover, the complexification of $\mathbf{A}_0(a)$ is a C*-algebra so Lemma 2.3 is not a new result. We included it for completeness and the reader's convenience.

Lemma 2.4. *Suppose* $a \in C_c(X)$ *with* $0 \leq a \leq 1$. *Then there is a function* $f \in C_c(X)$ *with* $\operatorname{supp} a \prec f$. *Moreover,* $\operatorname{supp} a \prec f \prec X$ *implies that* $a, f \in \mathbf{A}_0(a+f)$ *and* $\rho(a) \leq \rho(f)$.

Proof. If $a \in C_c(X)$, then $\operatorname{supp} a = K$ is compact. There is an open bounded set V containing K. By Urysohn's lemma there is a function f with $K \prec f \prec V$ which implies that $f \in C_c(X)$. Define ϕ_1 and ϕ_2 by

$$\phi_1(x) = \begin{cases} 1, & x \geq 1, \\ x, & x < 1, \end{cases} \quad \text{and } \phi_2 = \begin{cases} x-1, & x \geq 1, \\ 0, & x < 1. \end{cases}$$

Then $\phi_1(a+f) = f$ and $\phi_2(a+f) = a$; thus $a, f \in \mathbf{A}_0(a+f)$ and we get $\rho(a) \leq \rho(f)$.

Theorem 2.5 (Boundedness of quasi-integrals)**.** *All quasi-integrals on* $C_0(X)$ *are bounded.*

Proof. Let ρ be a quasi-integral on $C_0(X)$ and suppose $\sup\{\rho(a) : a \prec X\} = \infty$. By Lemma 2.4 recursively construct a sequence $\{a_i\}_{i=1}^{\infty}$ where $\rho(a_i) \geq 2^{2i}$ and $\operatorname{supp} a_i \prec a_{i+1} \prec X$ for each i. Let $f = \sum_{i=1}^{\infty} 2^{-i} a_i$. Then $f \in C_0(X)$ since $C_0(X)$ is complete. Define ϕ_i for $i = 1, 2, \ldots$ by

$$\phi_i(x) = \begin{cases} 1, & x \geq 2^{-i+1}, \\ 2^i(x - 2^{-i}), & 2^{-i} \leq x \leq 2^{-i+1}, \\ 0, & x \leq 2^{-i}, \end{cases}$$

we have $\phi_i \in C(\operatorname{sp} f)$, $\phi_i(0) = 0$ and $\phi_i(f) = a_i$ for each i and thus $\{a_i\}_{i=1}^{\infty} \subset \mathbf{A}_0(f)$. Finally $f \geq 2^{-i} a_i$ implies $\rho(f) \geq 2^{-i}\rho(a_i) \geq 2^i$ for $i = 1, 2, \ldots$ which in turn implies that $\rho(f) = \infty$. This is a contradiction since ρ is supposed to be a quasi-integral on $C_0(X)$; we may conclude that $\sup\{\rho(a) : a \prec X\} < \infty$ so ρ is bounded.

Remark 2. Theorem 2.5 shows that the local linearity of the quasi-integrals impose strong restrictions on their global behavior. This suggests that unbounded quasi-integrals on $C_c(X)$ may exhibit nice properties. Indeed, this is what we will devote the next and last section to.

3. Quasi-integrals on $C_c(X)$

Proposition 3.1. *Suppose that* μ *is a quasi-measure in* X *and* $f \in C_c(X)$. *Then there is a unique bounded regular Borel measure* μ_f *on* $\mathbf{R}\backslash\{\mathbf{0}\}$ *with* $\mu_f(O) = \mu(f^{-1}(O))$ *for all open sets* $O \subset \mathbf{R}\backslash\{\mathbf{0}\}$.

Proof. Let $\check{f}(x) = \mu(f^{-1}(-\infty, x)\backslash\{0\})$ which implies that $\check{f}$ is increasing. Since $f \in C_c(X)$ we have that $f(\operatorname{supp} f)$ is compact. Hence $\check{f}$ is constant outside an interval $[a, b]$ for some $a, b \in \mathbf{R}$. By Proposition 2.1 $\check{f}(x^-) = \check{f}(x)$ for each $x \in \mathbf{R}$, so $\check{f}$ is continuous from the left. Thus $\mu_f[x, y) = \check{f}(y) - \check{f}(x)$ uniquely determines a regular Borel measure in $\mathbf{R}$ and by regularity $\mu_f(x, y) = \check{f}(y) - \check{f}(x^+)$. If $x_\lambda \in (x, y)$, then $\check{f}(y) \geq \mu(f^{-1}[(x_\lambda, y)\backslash\{0\}]) + \check{f}(x_\lambda)$ since μ is monotone; hence $\check{f}(y) - \check{f}(x^+) \geq \mu(f^{-1}[(x, y)\backslash\{0\}])$. Conversely finite additivity and monotonicity of μ yields

$$\begin{aligned} \check{f}(y) &= \mu(f^{-1}[(x, y)\backslash\{0\}]) + \mu f^{-1}(\{x\}\backslash\{0\}) + \check{f}(x) \\ &\leq \mu(f^{-1}[(x, y)\backslash\{0\}]) + \check{f}(x_\lambda), \end{aligned}$$

so $\check{f}(y) - \check{f}(x^+) \leq \mu(f^{-1}[(x, y)\backslash\{0\}])$. We have $\mu(f^{-1}[(x, y)\backslash\{0\}]) = \check{f}(y) - \check{f}(x^+) = \mu_f(x, y)$, and since both μ and μ_f are countably additive on open sets, the proof is complete.

Remark 3. We will call μ_f the measure corresponding to μ and f. Notice that Proposition 3.1 is stated only for open sets not containing $\{0\}$, whereas the proof produces a measure on $\mathbf{R}$ with $\mu_f(\{0\}) = 0$. This is convenient when the quasi-measure is an extended real-valued function. In fact, Lemma 3.3 is not valid unless zero is omitted.

Definition 3.2. A map $f \mapsto \mu_f$ from $C_c(X)$ into the regular Borel measures in $\mathbf{R}\backslash\{\mathbf{0}\}$ is consistent if $\mu_{\phi\circ f} = \mu_f \circ \phi^{-1}$ for each $f \in C_c(X)$ and $\phi \in C(\operatorname{sp} f), \phi(0) = 0$.

Lemma 3.3. *Let μ be a quasi-measure in X. Let μ_f denote the measure corresponding to μ and $f \in C_c(X)$. Then the map $f \mapsto \mu_f$ is consistent.*

Proof. Let $f \in C_c(X)$, $\phi\circ f \in \mathbf{A}_0(f)$ and $K \subset \mathbf{R}\backslash\{0\}$ be compact. Now $0 \notin \phi^{-1}(K)$ implies:

$$\begin{aligned} \mu_{\phi\circ f}(K) &= \mu((\phi \circ f)^{-1}(K)) \\ &= \mu(f^{-1}(\phi^{-1}(K))) \\ &= \mu_f(\phi^{-1}(K)). \end{aligned}$$

Note that since K is compact in $\mathbf{R}\backslash\{0\}$, then K is compact in $\mathbf{R}$ by the identity map. So $f^{-1}(\phi^{-1}(K))$ is a closed subset of $\operatorname{supp} f$, and thus compact. The result now follows from the regularity of μ_f.

In the sequel we will assume that the measure corresponding to a quasi-measure μ and a function $f \in C_c(X)$ is extended to $\mathbf{R}$ by $\mu_f\{0\} = 0$.

Proposition 3.4. *Let ρ be a quasi-integral on $C_c(X)$. If $f, g \in C_c(X)$ and $f \leq g$, then $\rho(f) \leq \rho(g)$.*

Proof. Given $\delta > 0$, suppose $f \geq 0$ and $g(x) \geq \delta + f(x)$ when $x \in \operatorname{supp} f$. Pick a natural number n such that $n\delta > \max g$ and define $\phi_i \in C(\operatorname{sp} f)$, $1 \leq i \leq n$ by:

$$\phi_i(x) = \begin{cases} 0, & x \leq (i-1)\delta, \\ x - (i-1), & (i-1)\delta < x < i\delta, \\ \delta, & x \geq i\delta. \end{cases}$$

Then $x \in \operatorname{supp} \phi_i(f) \Rightarrow \phi_i(g(x)) = \delta$; thus $\phi_i(f), \phi_i(g) \in \mathbf{A}_0(\phi_i(f) + \phi_i(g))$ which imply $\rho(\phi_i(f)) \leq \rho(\phi_i(g))$. Now $\sum \phi_i(f) = f, \sum \phi_i(g) = g$ implies $\rho(f) \leq \rho(g)$. Given $\epsilon > 0$, suppose now that $0 \leq f \leq g$, choose h and $\delta > 0$ with $\operatorname{supp} g \prec h \in C_c(X)$ and $\rho(\delta h) < \epsilon$. We have $\rho(f) \leq \rho(g + \delta h) < \rho(g) + \epsilon$. Let $f \leq g \in C_c(X)$ be arbitrary. Then $f^+, f^- \in A(f)$ and $f^+ \leq g^+, f^- \geq g^-$. We have $\rho(f) = \rho(f^+) - \rho(f^-) \leq \rho(g^+) - \rho(g^-) = \rho(g)$ by the previous argument. The proof is complete.

Corollary 3.5. *Let ρ be a quasi-integral on $C_c(X)$ and let K be an arbitrary compact subset of X. Then there is a $k \in \mathbf{R}$ such that whenever* $\operatorname{supp} f_i \subset K, f_i \in C_c(X)$ *for $i = 1, 2$, we have:*

$$|\rho(f_1) - \rho(f_2)| \leq k \left\| f_1 - f_2 \right\|.$$

Proof. Pick a $g \succ K$ and let $\rho(g) = k$. Then $f_1 \leq f_2 + g \left\| f_1 - f_2 \right\|$ which implies that $\rho(f_1) - \rho(f_2) \leq \rho(g) \left\| f_1 - f_2 \right\|$ and conversely $\rho(f_2) - \rho(f_1) \leq \rho(g) \left\| f_1 - f_2 \right\|$. But then we must have $|\rho(f_1) - \rho(f_2)| \leq k \left\| f_1 - f_2 \right\|$.

Remark 4. In general ρ is not uniformly continuous (since it is a generalization of regular Borel measures). However, ρ is continuous with respect to the topology of uniform convergence on compacta. Hence this is a sharp result; we cannot expect stronger continuity properties.

Corollary 3.6. *Let ρ be a bounded quasi-integral on $C_c(X)$. Then for each pair $f_1, f_2 \in C_c(X)$ we have*

$$|\rho(f_1) - \rho(f_2)| \leq \|\rho\| \, \|f_1 - f_2\| .$$

Proof. Pick a function $g \succ \operatorname{supp} f_1 \cup \operatorname{supp} f_2$. Then $\rho(g) \leq \|\rho\|$ and the result follows from Corollary 3.5.

Proposition 3.7. *Let μ be a quasi-measure in X. Define*

$$\rho(f) = \int i \; d\mu_f \text{ for each } f \in C_c(X),$$

where μ_f is the measure corresponding to μ and f and i is the identity map on $\boldsymbol{R}$. Then ρ is a quasi-integral on $C_c(X)$.

Proof. By the transformation theorem for integrals and Lemma 3.3, the result follows.

Lemma 3.8. *Let μ be a quasi-measure in X and let ρ be the corresponding quasi-integral. Then for each open set $O \subset X$ we have:*

$$\mu(O) = \sup\{\rho(f) : f \prec O\}.$$

Moreover, if $\mu(X) < \infty$, then ρ is bounded and $\|\rho\| = \mu(X)$.

Proof. First suppose $\mu(O) < \infty$. Choose a compact set $K \subset O$ with $\mu(K) > \mu(O) - \epsilon$, and a function f with $K \prec f \prec O$. We have:

$$\begin{aligned} \rho(f) &= \textstyle\int_{\operatorname{sp} f} i \; d\mu_f = \int_{\{1\}} d\mu_f + \int_{(0,1)} i \; d\mu_f \\ &\geq \textstyle\int_{\{1\}} d\mu_f = \mu_f(\{1\}) = \mu(f^{-1}\{1\}) \\ &\geq \mu(K) \text{ since } K \subset f^{-1}\{1\}. \end{aligned}$$

On the other hand we have:

$$\begin{aligned} \rho(f) &\leq \textstyle\int_{\operatorname{sp} f} d\mu_f && (\operatorname{sp} f \subset [0,1]) \\ &= \mu_f(\operatorname{sp} f \backslash \{0\}) \\ &= \mu(f^{-1}(0,\infty)) && (f^{-1}(0,\infty) = f^{-1}(\operatorname{sp} f \backslash \{0\})) \\ &\leq \mu(O) && (f \prec O \Rightarrow f^{-1}(0,\infty) \subset O). \end{aligned}$$

These together imply $\mu(O) = \sup\{\rho(f) : f \prec O\}$. If $\mu(O) = \infty$, then there is, for every natural number n, a compact set $K \subset O$ with $\mu(K) > n$. By the previous argument there is then a function f with $K \prec f \prec O$ and $\rho(f) > n$. Hence $\mu(O) = \sup\{\rho(f) : f \prec O\} = \infty$. If $\mu(X) < \infty$, put $O = X$ in the previous argument. Then $\mu(X) = \sup\{\rho(f) : f \prec X\} = \|\rho\| < \infty$. The proof is complete.

Theorem 3.9 (The representation theorem)**.** *Let X be a locally compact Hausdorff space.*

1. *To each quasi-measure μ in X there is a unique quasi-integral ρ on $C_c(X)$ such that for any $f \in C_c(X)$ we have*

$$\rho(\phi(f)) = \int \phi(i) \; d\mu_f$$

for all $\phi \in \{\phi \in C(\operatorname{sp} f) : \phi(0) = 0\}$*. Here* μ_f *is the regular Borel measure in* $\boldsymbol{R}$ *corresponding to* μ *and* f*.*

2. *Conversely, for any quasi-integral* ρ *on* $C_c(X)$ *there is a unique quasi-measure* μ *in* X *such that* ρ *is the quasi-integral corresponding to* μ*. Specifically we have, for any open set* $O \subset X$*:*

$$\mu(O) = \sup\{\rho(f) : f \prec O\}. \tag{3.1}$$

Proof. The first part of the theorem follows from Proposition 3.7. Suppose ρ is a quasi-integral on $C_c(X)$. Define a set function $\mu : \mathcal{O} \to \mathbf{R} \cup \{\infty\}$ by (3.1). Extend μ to the closed subsets F of X by $\mu(F) = \inf\{\mu(O) : F \subset O, O \text{ is open}\}$. Notice that this implies $\mu(K) = \inf\{\rho(f) : f \succ K\}$ when K is compact by Urysohn's lemma and the monotonicity of ρ. We will show that μ is a quasi-measure in X. Note that $\mu(A) < \infty$ when $A \in \mathcal{A}^*$ by Urysohn's lemma and Corollary 3.5. Suppose that O_1 and O_2 are open disjoint subsets of X. Pick f_i with $f_i \prec O_i$ and $\rho(f_i) > \mu(O_i) - \epsilon$ for $i = 1, 2$. We have $f_1 f_2 = 0$ which implies $f_1, f_2 \in \mathbf{A}(f_1 - f_2)$ and thus

$$\begin{aligned} \mu(O_1 \cup O_2) &\geq \rho(f_1 + f_2) = \rho(f_1) + \rho(f_2) \\ &\geq \mu(O_1) + \mu(O_2) + 2\epsilon. \end{aligned}$$

Conversely if $f \prec O_1 \cup O_2$, the opposite equality follows from observing that $f = f_1 + f_2$ where $f_i(x) = f(x)$ if $x \in O_i$ and elsewhere zero. Let $K \subset O \subset X$ where K is compact and O is open. By Urysohn's lemma there is an open bounded set U and functions f_K, f_U such that $K \subset U \subset \bar{U} \subset O$, $K \prec f_K \prec O$ and $\bar{U} \prec f_U \prec O$ with $\rho(f_U) > \mu(O) - \epsilon$. Then $f_K, f_U \in \mathbf{A}(f_K + f_U)$ and $f_U - f_K \prec O \backslash K$; thus

$$\begin{aligned} \mu(O \backslash K) &\geq \rho(f_U - f_K) = \rho(f_U) - \rho(f_K) \\ &> \mu(O) - \mu(K) - \epsilon, \end{aligned}$$

which yields $\mu(O) \leq \mu(O \backslash K) + \mu(K)$ when $\mu(O) < \infty$ and equality when $\mu(O) = \infty$. Conversely, if $f \prec O \backslash K$ with $\rho(f) > \mu(O \backslash K) - \epsilon$, then $K' = \operatorname{supp} f \subset O \backslash K$, so $(X \backslash K') \cap U$ is an open set containing K. Pick f_K such that $K \prec f_K \prec (X \backslash K') \cap U$; then $f f_K = 0$. We have

$$\begin{aligned} \mu(O) &\geq \rho(f_K + f) = \rho(f_K) + \rho(f) \\ &> \mu(K) + \mu(O \backslash K) - \epsilon. \end{aligned}$$

We have shown that μ is a quasi-measure in X. The uniqueness of μ follows from Lemma 3.8. Let ρ_μ denote the quasi-integral corresponding to μ; it remains to prove that ρ_μ is equal to ρ. Let $f \in C_c(X)$ be arbitrary. Then $\rho_f : \phi \to \rho(\phi(f))$ is a functional on $\{\phi : \phi \in C(\operatorname{sp} f), \phi(0) = 0\}$. Extend ρ_f to a functional F on $C(\operatorname{sp} f)$ by $F(\phi) = \rho_f(\phi - \phi(0)) + \phi(0)$, $\phi \in C(\operatorname{sp} f)$. Now $\operatorname{sp} f$ is compact and thus F determines a unique regular Borel measure ν_f on $\operatorname{sp} f$ such that

$$F(\phi) = \rho(\phi(f)) = \int \phi(i)\, d\nu_f \text{ when } \phi \in C(\operatorname{sp} f) \text{ and } \phi(0) = 0.$$

So by regularity it suffices to show that $\mu_f = \nu_f$ on the open subsets of $E = \operatorname{sp} f \backslash \{0\}$. Suppose $\epsilon > 0$ and $U \subset E$ open are arbitrary. Pick a compact set $K \subset U$ such that $\nu_f(K) > \nu_f(U) - \epsilon$. Choose an Urysohn function $K \prec \phi \prec U$; then $\nu_f(U) - \epsilon < \nu_f(K) \leq \rho(\phi(f)) \leq \mu(f^{-1}(U)) = \mu_f(U)$ since $\phi \circ f \prec f^{-1}(U)$. Conversely, pick a compact set $K \subset f^{-1}(U)$ such that $\mu(K) > \mu(f^{-1}(U)) - \epsilon$ and a function ϕ with $f(K) \prec \phi \prec U$. Then since $\phi \circ f \succ f^{-1}(K)$ we have

$$\mu_f(U) - \epsilon = \mu(f^{-1}(U)) - \epsilon < \mu(K) \leq \rho(\phi \circ f) = \rho_f(\phi) \leq \nu_f(U).$$

The proof is complete.

Corollary 3.10. *Let ρ be a quasi-integral on $C_c(X)$. If ρ is bounded, then ρ has a unique extension to a quasi-integral on $C_0(X)$.*

Proof. By Corollary 3.6 ρ is uniformly continuous. Extend ρ by continuity to a function $\rho_0 : C_0(X) \to \mathbf{R}$. For example by the functions ϕ_ϵ defined by

$$\phi_\epsilon(x) = \begin{cases} 0, & x < \epsilon, \\ 2x - 2\epsilon, & \epsilon \le x \le 2\epsilon, \\ x, & x > 2\epsilon. \end{cases}$$

Obviously $\rho_0(\alpha f) = \alpha \rho_0(f)$ for all $\alpha \in \mathbf{R}, f \in C_0(X)$. Suppose $f \in C_0(X)$ and $\phi_1(f), \phi_2(f) \in \mathbf{A}_0(f)$. Then $\phi_i(\phi_\epsilon(f)) \in \mathbf{A}_0(\phi_\epsilon(f))$ for all $\epsilon > 0$ and $i = 1, 2$. Note that $\phi_i(\phi_\epsilon(f))$ converges uniformly to $\phi_i(f)$ when ϵ tends to zero. Hence by continuity

$$\begin{aligned} \rho(\phi_1(f) + \phi_2(f)) &= \lim_{\epsilon \to 0} \rho(\phi_1(\phi_\epsilon(f)) + \phi_2(\phi_\epsilon(f))) \\ &= \lim_{\epsilon \to 0} [\rho(\phi_1(\phi_\epsilon(f))) + \rho(\phi_2(\phi_\epsilon(f)))] \\ &= \rho(\phi_1(f)) + \rho(\phi_2(f)). \end{aligned}$$

We have shown that ρ_0 is a quasi-integral on $C_0(X)$. The uniqueness of the extension is immediate from the continuity of ρ_0. The proof is complete.

References

[1] J. F. Aarnes: "Quasi-states and Quasi-measures," Advances in Mathematics, vol. 86 (1991). MR **92d:**46152

[2] J. F. Aarnes: "Image transformations and attractors," Dept. of Math., University of Trondheim, Preprint no. 2 (1994).

[3] J. F. Aarnes: "Quasi-measures in locally compact spaces," Norwegian University of Science and Technology, Preprint no. 1 (1996).

[4] J. F. Aarnes and A. B. Rustad: "Probability and Quasi-measures -a new interpretation," To appear in Math. Scand.

[5] W. Rudin: "Real and complex analysis," 3d ed., McGraw-Hill Book Company, Singapore (1987). MR **88k:**00002

[6] A. B. Rustad: "The multidimensional median as a quasi-measure," Norwegian University of Science and Technology, Preprint no. 5 (1998).

[7] G. Taraldsen: "Image-transformations and quasi-measures in locally compact Hausdorff spaces," University of Trondheim, Preprint (1995).

Faculty of Mathematics, Norwegian University of Science and Technology, Sem Saelandsv 9, Gloshaugen, 7055 Dragvoll, Norway

E-mail address: `alfr@math.ntnu.no`

Current address: Department of Mathematical Sciences, Lade Norwegian University of Science and Technology, 7491 Trondheim, Norway