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SPACES OF $\mathcal{D}_{L^p}$−TYPE AND THE HANKEL CONVOLUTION

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Abstract. In this paper we introduce new function spaces that are denoted by $\mathcal{H}_{\mu,p}$, $\mu > -1/2$ and $1 \leq p \leq \infty$, and that are spaces of $\mathcal{D}_{L^p}$−type where the Hankel convolution and the Hankel transformation are defined. The spaces $\mathcal{H}_{\mu,p}$ will play the same role in the Hankel setting that the spaces $\mathcal{D}_{L^p}$ play in the theory of Fourier transformation.

1. Introduction

The spaces $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, have been studied by many authors (see [2], [3], [4], and [21], among others).

In this paper we introduce, for every $\mu > -1/2$ and $1 \leq p \leq \infty$, function spaces, represented by $\mathcal{H}_{\mu,p}$, similar to $\mathcal{D}_{L^p}$ but replacing the usual derivative by the Bessel operator $\Delta_\mu = x^{-2\mu-1}Dx^{2\mu+1}D$. Throughout this note μ will always denote a real number greater than $-1/2$. We characterize the elements and the bounded sets of the dual space $\mathcal{H}'_{\mu,p}$ of $\mathcal{H}_{\mu,p}$ through the Hankel convolution. Also, we define the Hankel convolution on $\mathcal{H}'_{\mu,p} \times \mathcal{H}'_{\mu,q}$. One of the forms that Hankel transformation takes is the following (see G. Altenburg [1], I. I. Hirschman [14] and A. L. Schwartz [20]):

$$h_\mu(\phi)(y) = \int_0^\infty x^{2\mu+1}(xy)^{-\mu} J_\mu(xy)\phi(x)dx, \qquad y \in (0,\infty),$$

where, as usual, J_μ denotes the Bessel function of the first kind and order μ. Here we study h_μ on $\mathcal{H}_{\mu,p}$ and $\mathcal{H}'_{\mu,p}$. The spaces $\mathcal{H}_{\mu,p}$ will play the same role in the Hankel setting that the spaces $\mathcal{D}_{L^p}$ ([21, p. 199]) play in the theory of the usual convolution and the Fourier transformation. Here we establish the first properties of the spaces $\mathcal{H}_{\mu,p}$. However, there exist other interesting questions (convolution equations, boundary values of holomorphic functions,...) that will be investigated related to our new spaces. In Section 3 we present some of these open questions.

Before stating our results we recall some known definitions and results about Hankel transformation and Hankel convolution.

G. Altenburg [1] considered the space H constituted by all those complex valued and smooth functions ϕ on $(0,\infty)$ such that

$$\gamma_{m,n}(\phi) = \sup_{x\in(0,\infty)} \left(1+x^2\right)^m \left|\left(\frac{1}{x}D\right)^n \phi(x)\right| < \infty, \qquad \text{for every } m,n \in \mathbb{N}.$$

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When H is endowed with the topology associated to the family $\{\gamma_{m,n}\}_{m,n\in\mathbb{N}}$ of seminorms, H is a Fréchet space and the Hankel transformation h_μ is an automorphism of H [1, Satz 5]. The Hankel transform is defined on H', the dual space of H, as the transpose of h_μ on H and it is denoted by h'_μ. That is, if $T \in H'$, then the Hankel transform $h'_\mu T$ of T is the element of H' defined by

$$\left\langle h'_\mu T, \phi \right\rangle = \left\langle T, h_\mu \phi \right\rangle, \qquad \phi \in H.$$

For every $1 \leq p < \infty$, the space $L_{\mu,p}$ consists of all those measurable functions ϕ on $(0,\infty)$ such that

$$\|\phi\|_{\mu,p} = \left\{ \int_0^\infty |\phi(x)|^p \, x^{2\mu+1} dx \right\}^{1/p} < \infty.$$

By $L_{\mu,\infty}$ we represent the space of essentially (with respect to the measure $x^{2\mu+1}dx$ or, equivalently, with respect to the Lebesgue measure) bounded functions on $(0,\infty)$. The usual norm in $L_{\mu,\infty}$ is denoted by $\| \ \|_{\mu,\infty}$.

C. Herz [13] proved that the Hankel transformation h_μ can be extended to $L_{\mu,p}$ as a bounded operator form $L_{\mu,p}$ into $L_{\mu,p'}$, provided that $1 \leq p \leq 2$. Here and in the sequel by p' we denote the conjugate of p, that is: $p' = \dfrac{p}{p-1}$, when $1 < p \leq \infty$; $p' = 1$ when $p = \infty$, and $p' = \infty$ when $p = 1$.

It is not hard to see that if $f \in L_{\mu,p}$, for some $1 \leq p \leq \infty$, then f defines an element of H' through

$$\langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)x^{2\mu+1}dx, \qquad \phi \in H.$$

Thus, $L_{\mu,p}$ can be seen as a subspace of H'. Moreover, $h'_\mu f = h_\mu f$ when $f \in L_{\mu,p}$ and $1 \leq p \leq 2$.

The Hankel convolution was introduced and investigated on $L_{\mu,p}$ by I. I. Hirschman [14] and D. T. Haimo [12]. They defined the Hankel convolution $f\#g$ of f and g by

$$(f\#g)(x) = \int_0^\infty f(y)\,(\tau_x g)(y)\,\frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}dy \tag{1.1}$$

where the Hankel translation operator τ_x, $x \in (0,\infty)$, is defined through

$$(\tau_x g)(y) = \int_0^\infty g(z)\,D(x,y,z)\,\frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}dz,$$

provided the above integrals exist. Here, for every $x,y,z \in (0,\infty)$, $D(x,y,z)$ represents the area of the triangle having sides with length x,y and z, when that triangle exists, and $D(x,y,z) = 0$, otherwise.

In a series of papers J. J. Betancor and I. Marrero ([6], [7], [8] and [16]) studied the Hankel convolution on Zemanian spaces of distributions. The convolution operation considered by the last author is different although closely connected to the $\#-$convolution defined by (1.1). We need to rewrite for $\#$ some of the results of J. J. Betancor and I. Marrero. According to Proposition 2.1 of [16], for each $x \in (0,\infty)$ the Hankel translation τ_x defines a continuous linear mapping from H into itself. We define the Hankel convolution $T\#\phi$ of $T \in H'$ and $\phi \in H$ by $(T\#\phi)(x) = \langle T, \tau_x\phi \rangle$, $x \in (0,\infty)$. By Proposition 3.5 of [16] $T\#\phi \in \mathcal{O}$ where $\mathcal{O}$ is the space of multipliers of H (see [24, p. 134] and [5]), for every $T \in H'$ and $\phi \in H$. Also, if $T \in H'$ and $\phi, \psi \in H$, then

$$\langle T\#\phi, \psi\rangle = \langle T, \phi\#\psi\rangle . \tag{1.2}$$

Inspired in [7] we introduce for every $a \in (0,\infty)$ the space W_a^m constituted by all those complex valued functions $\psi \in \mathcal{C}^{2m}(0,\infty)$ such that $\psi(x) = 0$, for each $x > a$, and the limit $\lim_{x\to 0^+} \left(\frac{1}{x}D\right)^k \psi(x)$ exists for every $k \in \mathbb{N}$, $0 \le k \le 2m$.

By proceeding as in Lemma 2.1 of [7] we can also prove the following

Proposition 1.1. *Let $a \in (0,\infty)$ and $m \in \mathbb{N}$. Then there exists $r_0 > 0$ such that for each $r \in \mathbb{N}$, $r \ge r_0$, we can find $\varphi_r \in W_a^m$ and $\psi_r \in H$, with $\psi_r(x) = 0$, $x > a$, for which*

$$\delta = (1-\Delta_\mu)^r \varphi_r + \psi_r,$$

in the sense of equality in H'. Here δ is the element of H' defined by

$$\langle \delta, \phi\rangle = \lim_{x\to 0^+} \phi(x), \qquad \phi \in H. \qquad \square$$

2. The spaces $\mathcal{H}_{\mu,p}$ and $\mathcal{H}'_{\mu,p}$ and the Hankel convolution

Let $1 \le p \le \infty$. We say that a measurable function f on $(0,\infty)$ is in $\mathcal{H}_{\mu,p}$ if for every $k \in \mathbb{N}$, $\Delta_\mu^k f \in L_{\mu,p}$, that is, there exists $h_k \in L_{\mu,p}$ such that

$$\begin{aligned}\left\langle \Delta_\mu^k f, \phi\right\rangle &= \int_0^\infty \left(\Delta_\mu^k \phi\right)(x)\, f(x) x^{2\mu+1} dx \\ &= \int_0^\infty \phi(x)\, h_k(x)\, x^{2\mu+1} dx, \qquad \phi \in H.\end{aligned}$$

The space $\mathcal{H}_{\mu,p}$ is endowed with the topology generated by the system $\left\{\gamma_{\mu,p}^k\right\}_{k\in\mathbb{N}}$ of seminorms, where

$$\gamma_{\mu,p}^k(f) = \left\|\Delta_\mu^k f\right\|_{\mu,p}, \qquad f \in \mathcal{H}_{\mu,p} \text{ and } k \in \mathbb{N}.$$

It is not hard to see that H is continuously contained in $\mathcal{H}_{\mu,p}$.

Remark 1. It is well known that if f is a measurable function on $\mathbb{R}^n$ and f admits distributional derivatives $\frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{a_1}\ldots\partial x_n^{\alpha_n}} f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, for every $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, then $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ ([11, Corollaire IX.13]). We do not know if a similar result holds for $\mathcal{H}_{\mu,p}$, $p > 2$, that is, if $\mathcal{H}_{\mu,p}$ is contained in $\mathcal{C}^\infty(0,\infty)$, when $2 < p \le \infty$. However, we can prove that $\mathcal{H}_{\mu,p}$ is contained in $\mathcal{C}^\infty(0,\infty)$, provided that $1 \le p \le 2$. Indeed, let $1 \le p \le 2$ and $f \in \mathcal{H}_{\mu,p}$. According to [13, Theorem 3], a distributional argument allows us to conclude that for every $k \in \mathbb{N}$, $y^{2k} h_\mu(f) \in L_{\mu,p'}$.

Hence, Hölder's inequality implies that $y^{2k} h_\mu(f) \in L_{\mu,r}$, for every $k \in \mathbb{N}$, provided that $1 \le r \le p'$. In particular, by [13, Theorem 3], $h_\mu(h_\mu(f)) \in L_{\mu,2}$ and

$$f(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\, h_\mu(f)(y)\, y^{2\mu+1} dy, \qquad \text{a.e.} \quad x \in (0,\infty).$$

Moreover, by invoking well-known properties of the Bessel function J_μ, we can see that the function in the right-hand side of the last equality is in $\mathcal{C}^\infty(0,\infty)$. Note that we can also obtain from [13, Theorem 3] that $\Delta_\mu^k f \in L_{\mu,r}$, for every $k \in \mathbb{N}$ and $2 \le r \le \infty$. Thus, we prove that $\mathcal{H}_{\mu,p}$ is continuously contained in $\mathcal{H}_{\mu,r}$, for each $2 \leq r < \infty$, provided that $1 \leq p \leq 2$. This can be seen as an embedding theorem in the setting of our spaces.

Proposition 2.1. *For every* $1 \leq p \leq \infty$, $\mathcal{H}_{\mu,p}$ *is a Fréchet space.*

Proof. Let $1 \leq p \leq \infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}_{\mu,p}$. Since $L_{\mu,p}$ is a Banach space, there exists $h_k \in L_{\mu,p}$ such that $\Delta_\mu^k f_n \longrightarrow h_k$, as $n \to \infty$, in $L_{\mu,p}$, for each $k \in \mathbb{N}$. Moreover, $\Delta_\mu^k h_0 = h_k$, $k \in \mathbb{N}$. Indeed, let $k \in \mathbb{N}$. We can write

$$\left\langle \Delta_\mu^k h_0, \phi \right\rangle = \left\langle h_0, \Delta_\mu^k \phi \right\rangle = \int_0^\infty h_0(x) \left(\Delta_\mu^k \phi\right)(x)\, x^{2\mu+1} dx$$
$$= \lim_{n\to\infty} \int_0^\infty f_n(x) \left(\Delta_\mu^k \phi\right)(x)\, x^{2\mu+1} dx = \lim_{n\to\infty} \left\langle f_n, \Delta_\mu^k \phi \right\rangle$$
$$= \lim_{n\to\infty} \left\langle \Delta_\mu^k f_n, \phi \right\rangle = \left\langle h_k, \phi \right\rangle, \qquad \phi \in H.$$

Thus the proof is finished. □

It is clear that the mapping $f \longmapsto \Delta_\mu f$ is continuous from $\mathcal{H}_{\mu,p}$ into itself, for each $1 \leq p \leq \infty$.

Also, for every $x \in (0, \infty)$ and $1 \leq p \leq \infty$, the Hankel translation τ_x defines a continuous mapping from $\mathcal{H}_{\mu,p}$ into itself. Indeed, let $x \in (0, \infty)$, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. From Proposition 2.1 (ii) of [16] we can deduce that

$$\left\langle \Delta_\mu^k \tau_x f, \phi \right\rangle = \left\langle \tau_x \Delta_\mu^k f, \phi \right\rangle, \qquad f \in \mathcal{H}_{\mu,p} \text{ and } \phi \in H.$$

Since τ_x is a contractive operator in $L_{\mu,p}$ [23, p. 16], $\Delta_\mu^k \tau_x f \in L_{\mu,p}$, and

$$\left\| \Delta_\mu^k \tau_x f \right\|_{\mu,p} \leq \left\| \Delta_\mu^k f \right\|_p$$

for every $f \in \mathcal{H}_{\mu,p}$. Thus we conclude that τ_x is bounded from $\mathcal{H}_{\mu,p}$ into itself, for all $x \in (0, \infty)$.

As usual, we denote by $\mathcal{H}'_{\mu,p}$ the dual space of $\mathcal{H}_{\mu,p}$. In the following we characterize the elements of $\mathcal{H}'_{\mu,p}$.

Proposition 2.2. *Let T be a functional on $\mathcal{H}_{\mu,p}$ where $1 \leq p < \infty$. Then, T is in $\mathcal{H}'_{\mu,p}$ if, and only if, there exist $r \in \mathbb{N}$ and $f_k \in L_{\mu,p'}$, $k = 0, 1, \ldots, r$, for which*

$$T = \sum_{k=0}^{r} \Delta_\mu^k f_k, \qquad \text{on } \mathcal{H}_{\mu,p}. \tag{2.1}$$

Proof. First, assume that T is in $\mathcal{H}_{\mu,p}$. Then there exists $C > 0$ and $r \in \mathbb{N}$ such that

$$|\langle T, f \rangle| \leq C \max_{0\leq k\leq r} \left\| \Delta_\mu^k f \right\|_p \qquad \text{for every } f \in \mathcal{H}_{\mu,p}. \tag{2.2}$$

We define the mappings

$$J : \mathcal{H}_{\mu,p} \longrightarrow \Pi_p^{r+1}$$
$$f \longrightarrow \left(\Delta_\mu^k f\right)_{k=0}^r$$

where $\Pi_p^{r+1} = L_{\mu,p} \times \overset{r+1}{\cdots} \times L_{\mu,p}$, and

$$L : J\mathcal{H}_{\mu,p} \longrightarrow \mathbb{C}$$
$$\left(\Delta_\mu^k f\right)_{k=0}^r \longrightarrow \langle T, f \rangle .$$

Note that, since J is one to one, the mapping L is well defined. Moreover, according to (2.2), L is a continuous linear mapping when in $J\mathcal{H}_{\mu,p}$ we consider the topology induced by Π_p^{r+1}. Hence, according to the Hahn-Banach Theorem, L can be extended to Π_p^{r+1} as an element of $\left(\Pi_p^{r+1}\right)'$. Then, there exists $f_k \in L_{\mu,p'}$, $k = 0, 1, \ldots, r$, such that (2.1) holds.

Conversely, if T takes the form (2.1), for some $f_k \in L_{\mu,p'}$, $k = 0, 1, \ldots, r$, with $r \in \mathbb{N}$, Hölder's inequality implies that $T \in \mathcal{H}'_{\mu,p}$. □

Now we analyze the behaviour of the Hankel transformation h_μ on the spaces $\mathcal{H}_{\mu,p}$ and $\mathcal{H}'_{\mu,p}$.

Proposition 2.3. (a) *If $f \in \mathcal{H}_{\mu,p}$ with $1 \leq p \leq 2$, then $P(y)\, h_\mu(f) \in L_{\mu,p'}$, for every polynomial P.*

(b) *If $T \in \mathcal{H}'_{\mu,p}$, with $2 \leq p < \infty$, then $h'_\mu T = P\left(y^2\right) F$, where P is a polynomial and $F \in L_{\mu,p}$.*

Proof. (a) Let f be in $\mathcal{H}_{\mu,p}$, with $1 \leq p \leq 2$. Then, by invoking Theorem 3 of [13] it follows that

$$h'_\mu\left(\Delta_\mu^k f\right) = h_\mu\left(\Delta_\mu^k f\right) \in L_{\mu,p'}, \qquad k \in \mathbb{N}.$$

Moreover, according to [24, Lemma 5.4-1], we have

$$h'_\mu\left(\Delta_\mu^k f\right) = \left(-y^2\right)^k h'_\mu(f) = \left(-y^2\right)^k h_\mu(f), \qquad k \in \mathbb{N}.$$

Hence, $h_\mu\left(\Delta_\mu^k f\right) = \left(-y^2\right)^k h_\mu(f) \in L_{\mu,p'}$, $k \in \mathbb{N}$.

(b) Let $T \in \mathcal{H}'_{\mu,p}$, with $2 \leq p < \infty$. By virtue of Proposition 2.2 we can find $n \in \mathbb{N}$ and $f_k \in L_{\mu,p'}$, $k = 0, \ldots, n$, such that

$$T = \sum_{k=0}^{n} \Delta_\mu^k f_k.$$

Then, by [24, Lemma 5.4-1] we obtain

$$h'_\mu T = \sum_{k=0}^{n} h'_\mu \Delta_\mu^k f_k = \sum_{k=0}^{n} \left(-y^2\right)^k h_\mu f_k.$$

Therefore, the function F defined by

$$F(y) = \sum_{k=0}^{n} \frac{\left(-y^2\right)^k h_\mu f_k(y)}{\left(1+y^2\right)^n}, \qquad y \in (0, \infty),$$

is in $L_{\mu,p}$ [13, Theorem 3], and $h'_\mu(T) = \left(1+y^2\right)^n F$. □

The result in Proposition 2.3 (b) can be improved when $p = 2$.

Proposition 2.4. *Let $T \in H'$. Then $T \in \mathcal{H}'_{\mu,2}$ if, and only if, there exist a polynomial P and a function F in $L_{\mu,2}$ such that $h'_\mu T = P\left(y^2\right) F$.*

Proof. Assume that $h'_\mu T = P\left(y^2\right) F$, where P is a polynomial and $F \in L_{\mu,2}$. Then, according to [24, Lemma 5.4-1],

$$T = h'_\mu\left(P\left(y^2\right) F\right) = P\left(-\Delta_\mu\right) h'_\mu F = P\left(-\Delta_\mu\right) h_\mu F.$$

Since $h_\mu F \in L_{\mu,2}$ [13, Theorem 3], from Proposition 2.2 we deduce that $T \in \mathcal{H}'_{\mu,2}$.

The proof of this proposition can be completed by invoking Proposition 2.3 (b). □

Next we obtain a characterization of $\mathcal{H}_{\mu,p}$ involving the Hankel convolution. We will consider the space β of functions defined as follows. Let $a \in (0,\infty)$. A function $\phi \in H$ is in β_a if, and only if, $\phi(x) = 0$, for every $x \geq a$. β_a is endowed with the topology induced on it by H. The space $\beta = \bigcup_{a\in(0,\infty)} \beta_a$ is equipped with the inductive topology.

Theorem 2.1. *Let $T \in H'$ and $1 \leq p < \infty$. Then T is in $\mathcal{H}'_{\mu,p}$ if and only if $T\#\phi \in L_{\mu,p'}$, for every $\phi \in \beta$.*

Proof. First, suppose that $T \in \mathcal{H}'_{\mu,p}$. According to Proposition 2.2 we can assume without loss of generality that $T = \Delta_\mu^k f$, where $k \in \mathbb{N}$ and $f \in L_{\mu,p'}$. Then, for every $\phi \in \beta$

$$(T\#\phi)(x) = \langle T, \tau_x\phi\rangle = \left\langle f, \tau_x \Delta_\mu^k \phi\right\rangle = \left(f\#\Delta_\mu^k\phi\right)(x), \qquad x \in (0,\infty).$$

Since $\Delta_\mu^k\phi \in \beta$, for each $\phi \in \beta$, and since β is contained in $L_{\mu,1}$, from Theorem 2.b of [14] we deduce that $T\#\phi \in L_{\mu,p'}$.

Assume now that $T\#\phi \in L_{\mu,p'}$, for every $\phi \in \beta$. We will see that T takes the form (2.1) for certain $r \in \mathbb{N}$ and $f_k \in L_{\mu,p'}$, $k = 0, 1, \dots, r$.

According to (1.2) for every $\phi, \varphi \in \beta$ we have

$$\langle T\#\phi, \varphi\rangle = \langle T, \phi\#\varphi\rangle = \langle T\#\varphi, \phi\rangle .$$

Then, Hölder's inequality leads to

$$|\langle T\#\varphi, \phi\rangle| \leq \|T\#\phi\|_{\mu,p'} \|\varphi\|_{\mu,p}, \qquad \phi, \varphi \in \beta. \tag{2.3}$$

Denote by $\mathbb{B}$ the intersection between β and the unit ball of $L_{\mu,p}$. From (2.3) it is deduced that the set $\{T\#\varphi\}_{\varphi\in\mathbb{B}}$ is bounded in β' when we consider in β' the weak* topology (also, when β' is endowed with the strong topology [6, p. 281]). Hence, by invoking [6, Proposition 2.5], for every $a > 0$ there exists $m \in \mathbb{N}$ such that for every $\varphi \in \mathbb{B}$ there exists an extension of $T\#\varphi$ to W_a^m, denoted again by $T\#\varphi$, with $\{T\#\varphi\}_{\varphi\in\mathbb{B}}$ pointwise bounded on W_a^m. As we mentioned in Section 1, for every $m \in \mathbb{N}$ and $a > 0$, the space W_a^m is constituted by all those complex valued functions $\psi \in \mathcal{C}^{2m}(0,\infty)$ such that $\psi(x) = 0$, $x > a$, and the limit $\lim_{x\to 0^+} \left(\frac{1}{x}D\right)^k \psi(x)$ exists for every $k \in \mathbb{N}$, $0 \leq k \leq 2m$.

Moreover, if $\phi \in W_a^m$, then $h_\mu(\phi)$ is a multiplier of H. Therefore, according to the interchange formula [14, Theorem 2.b], every $\phi \in W_a^m$ defines a convolution operator on H; that is, the mapping $\varphi \longrightarrow \phi\#\varphi$ is continuous from H into itself. Then by proceeding as in [8, Proposition 3] we obtain

$$\langle T\#\phi, \varphi\rangle = \langle T\#\varphi, \phi\rangle, \qquad \varphi \in \beta \text{ and } \phi \in W_a^m.$$

Hence, $T\#\phi \in L_{\mu,p'}$, for every $\phi \in W_a^m$. Now, by invoking Proposition 1.1 we conclude that

$$T = T\#\delta = (1-\Delta_\mu)^r (T\#\phi) + T\#\Psi$$

where $r \in \mathbb{N}$, $\phi \in W_a^m$ and $\Psi \in \beta_a$.

Thus the proof is finished. □

In the following we characterize the bounded sets in $\mathcal{H}'_{\mu,p}$.

Proposition 2.5. *Let $1 \leq p < \infty$ and let K be a subset of $\mathcal{H}'_{\mu,p}$. The following three assertions are equivalent:*

(i) *K is a bounded set in $\mathcal{H}'_{\mu,p}$ when in $\mathcal{H}'_{\mu,p}$ we consider the weak* topology.*

(ii) *There exist $C > 0$ and $r \in \mathbb{N}$ and for every $T \in K$, $r+1$ functions $f_{0,T}$, $f_{1,T}, \dots, f_{r,T} \in L_{\mu,p'}$ such that*

$$T = \sum_{k=0}^{r} \Delta_\mu^k f_{k,T}$$

and $\sum_{k=0}^{r} \|f_{k,T}\|_{\mu,p'} \leq C$.

(iii) *For every $\phi \in \beta$, the set $\{T\#\phi\}_{T\in K}$ is bounded in $L_{\mu,p'}$.*

Proof. To prove this proposition it is sufficient to repeat the arguments of Propositions 2.2 and Theorem 2.1. □

We now introduce for $1 \leq p < \infty$ the space $H_{\mu,p}$ as the closure of H in $\mathcal{H}_{\mu,p}$. It is obvious that $H_{\mu,p}$ is a Fréchet space when we consider on $H_{\mu,p}$ the topology induced on it by $\mathcal{H}_{\mu,p}$. Also, we can establish a property analogous to Proposition 2.2 when the space $\mathcal{H}_{\mu,p}$ is replaced by $H_{\mu,p}$.

Remark 2. It is an open question if the spaces $\mathcal{H}_{\mu,p}$ and $H_{\mu,p}$ coincide for some μ and p values.

We now analyze the Hankel convolution on $\mathcal{H}'_{\mu,p} \times H_{\mu,p}$. As was mentioned above, for every $x \in (0,\infty)$, the Hankel translation τ_x is a continuous mapping from $\mathcal{H}_{\mu,p}$ into itself.

For every $T \in \mathcal{H}'_{\mu,p}$ and $\phi \in \mathcal{H}_{\mu,p}$, where $1 \leq p < \infty$, we define the convolution $T\#\phi$ of T and ϕ by

$$(T\#\phi)(x) = \langle T, \tau_x \phi \rangle, \qquad x \in (0,\infty).$$

By virtue of Proposition 2.2 there exist $n \in \mathbb{N}$ and $f_k \in L_{\mu,p'}$, $k = 0,1,\dots,n$, for which

$$\begin{aligned}(T\#\phi)(x) &= \sum_{k=0}^{n} \int_0^\infty f_k(y)\, \tau_x\left(\Delta_\mu^k \phi\right)(y)\, y^{2\mu+1} dy \\ &= 2^\mu \Gamma(\mu+1) \sum_{k=0}^{n} \left(f_k \# \Delta_\mu^k \phi\right)(x). \end{aligned} \tag{2.4}$$

Hence, according to [14, Theorem 2.b], $T\#\phi \in L_{\mu,\infty}$. Moreover, if we also assume that $\phi \in H_{\mu,p}$, for every $k \in \mathbb{N}$, $\Delta_\mu^k (T\#\phi) = T\#\Delta_\mu^k\phi$. Indeed, let $f \in L_{\mu,p'}$, $\phi \in H_{\mu,p}$ and $\psi \in H$. We have by using [14, Theorem 2.b] again that

$$\begin{aligned}\langle \Delta_\mu (f\#\phi), \psi\rangle &= \langle f\#\phi, \Delta_\mu \psi\rangle = \int_0^\infty (f\#\phi)(y)\, \Delta_\mu\psi(y)\, y^{2\mu+1} dy \\ &= \int_0^\infty \int_0^\infty f(z)\, (\tau_y \phi)(z)\, \Delta_\mu \psi(y) \frac{y^{2\mu+1}}{\Gamma(\mu+1)} dy \quad z^{2\mu+1} dz \\ &= \int_0^\infty f(z)\, (\phi \# \Delta_\mu \psi)(z)\, z^{2\mu+1} dz = \langle f\#\Delta_\mu \phi, \psi\rangle . \end{aligned}$$

Hence, by (2.4) it follows that

$$\Delta_\mu^k (T\#\phi) = T\#\Delta_\mu^k \phi \in L_{\mu,\infty}, \qquad k \in \mathbb{N}.$$

Thus, we establish that $T\#\phi \in \mathcal{H}_{\mu,\infty}$ for each $T \in \mathcal{H}'_{\mu,p}$ and $\phi \in H_{\mu,p}$ with $1 \leq p < \infty$.

Again let $T \in \mathcal{H}'_{\mu,p}$. Suppose that

$$T = \sum_{k=0}^{m} \Delta_\mu^k f_k \tag{2.5}$$

where $m \in \mathbb{N}$ and $f_k \in L_{\mu,p'}$, $k = 0, 1, \ldots, m$. We define the Hankel convolution $T\#\phi$ of T and $\phi \in H_{\mu,r}$, $\quad 1 \leq r < p$, by

$$T\#\phi = \sum_{k=0}^{m} f_k \# \Delta_\mu^k \phi.$$

According to [14, Theorem 2.b], if $T \in \mathcal{H}'_{\mu,p}$, then the mapping defined by $\phi \longrightarrow T\#\phi$ is continuous from $H_{\mu,r}$ into $L_{\mu,q}$ provided that $\dfrac{1}{q} = \dfrac{1}{r} - \dfrac{1}{p}$.

Moreover, if $\phi \in H$ we can write

$$(T\#\phi)(x) = \sum_{k=0}^{m} \left(f_k \# \Delta_\mu^k \phi\right)(x) = \langle T, \tau_x \phi \rangle, \qquad x \in (0,\infty).$$

Then, since H is a dense subspace of $H_{\mu,r}$, the definition of $T\#\phi$, for every $T \in \mathcal{H}'_{\mu,p}$ and $\phi \in H_{\mu,r}$, is not dependent of the representation (2.5) of T.

Since $\Delta_\mu^k (T\#\phi) = T\#\Delta_\mu^k\phi$, $T \in \mathcal{H}'_{\mu,p}$, $\phi \in H_{\mu,r}$ and $k \in \mathbb{N}$ [14, Theorem 2.b], we can conclude that, for every $T \in \mathcal{H}'_{\mu,p}$, the mapping $\phi \longrightarrow T\#\phi$ is continuous from $H_{\mu,r}$ into $\mathcal{H}_{\mu,q}$ provided that $\dfrac{1}{q} = \dfrac{1}{r} - \dfrac{1}{p}$.

Let $1 \leq p, q < \infty$. If $T \in \mathcal{H}'_{\mu,p}$ and $S \in \mathcal{H}'_{\mu,q}$, we define the $\#-$convolution $S\#T$ of S and T by

$$\langle S\#T, \phi \rangle = \langle S, T\#\phi \rangle, \qquad \phi \in H_{\mu,r},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$. Then, $S\#T \in H'_{\mu,r}$.

3. Open questions

In this section we present some interesting questions related to the spaces $\mathcal{H}_{\mu,p}$ introduced in this paper that can be studied and that are our next objectives.

3.1. Asymptotic of elements in $\mathcal{H}_{\mu,p}$ and $\mathcal{H}'_{\mu,p}$. B. Stankovic [22] investigated the asymptotic of elements of $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^p}$. The concept of asymptotic that is suitable to the Hankel setting must involve the Hankel translation.

We say that a distribution $T \in \mathcal{H}'_{\mu,p}$ has asymptotic behavior in infinity, with respect to a positive function c defined on $(0,\infty)$, to $S \in \mathcal{H}'_{\mu,p}$, if and only if

$$\lim_{x\to\infty} \left\langle \frac{(\tau_x T)(y)}{c(x)}, \phi(y) \right\rangle = \langle S, \phi \rangle, \qquad \phi \in \mathcal{H}_{\mu,p}.$$

Several questions can be asked about the function c appearing in the above definition (see [22]).

3.2. Convolution operators and multipliers in $\mathcal{H}_{\mu,p}$ and $\mathcal{H}'_{\mu,p}$. In [2] S. Abdullah characterized the spaces of convolution operators and multipliers of distributions of L^p-growth. It is an open question to characterize the Hankel convolution operators in $\mathcal{H}_{\mu,p}$ and $\mathcal{H}'_{\mu,p}$.

3.3. **Hankel convolution equations in $\mathcal{H}'_{\mu,p}$.** Convolution equations in $\mathcal{D}'_{L^p}$ were considered by D. H. Pahk [17]. He found a condition for convolution operators to be hypoelliptic in $\mathcal{D}'_{L^\infty}$ in terms of their Fourier transforms and he showed that the same conditions work for the solvability of convolution operators in the tempered distribution spaces $\mathcal{S}'$ and $\mathcal{D}'_{L^p}$.

Hankel convolution equation in spaces of distributions of slow growth and of exponential growth were studied in [9]. We think that the analysis of the hypoellipticity and the solubility of Hankel convolution equations in $\mathcal{H}'_{\mu,p}$ is an interesting question.

3.4. **Other spaces of type $\mathcal{H}_{\mu,p}$.** S. Pilipovic ([18] and [19]) and D. Kovacevic [15], among others, have investigated spaces of Beurling and Roumieu ultradistributions that are generalizations of the spaces $\mathcal{D}'_{L^p}$. We can define, in a similar way, new spaces of type $\mathcal{H}_{\mu,p}$.

Let $(M_r)_{r\in\mathbb{N}}$ be a strictly increasing sequence of positive numbers and let $h>0$. For every $p\geq 1$, we define the space $\mathcal{H}_{\mu,p}^{h,(M_r)_{r\in\mathbb{N}}}$ as follows:

$$\mathcal{H}_{\mu,p}^{h,(M_r)_{r\in\mathbb{N}}}=\left\{\phi\in\mathcal{C}^\infty(0,\infty):\quad \sup_{r\in\mathbb{N}}\left\{\frac{h^r}{M_r}\left\|\Delta_{\mu,p}^r\phi\right\|\right\}<\infty\right\}.$$

Also we introduce the spaces

$$\mathcal{H}_{\mu,p}^{(M_r)_{r\in\mathbb{N}}}=\underset{h\to\infty}{\text{projlim}}\mathcal{H}_{\mu,p}^{h,(M_r)_{r\in\mathbb{N}}} \text{ and } \mathcal{H}_{\mu,p}^{\{M_r\}_{r\in\mathbb{N}}}=\underset{h\to 0}{\text{indlim}}\mathcal{H}_{\mu,p}^{h,(M_r)_{r\in\mathbb{N}}}.$$

Inspired in [15], [18] and [19] we think that it is an interesting problem to obtain representations of elements belonging to the dual spaces of $\mathcal{H}_{\mu,p}^{(M_r)_{r\in\mathbb{N}}}$ and $\mathcal{H}_{\mu,p}^{\{M_r\}_{r\in\mathbb{N}}}$, $p\in(1,\infty)$, where $(M_r)_{r\in\mathbb{N}}$ is a nonquasianalytic sequence, as boundary values of holomorphic functions which satisfy appropriated estimates on the boundary of their domains. Also, the hypoellipticity of the Hankel convolution equations in the above spaces can be investigated.

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