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A MATRIX-VALUED CHOQUET–DENY THEOREM

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Abstract. Let σ be a positive matrix-valued measure on a locally compact abelian group G such that $\sigma(G)$ is the identity matrix. We give a necessary and sufficient condition on σ for the absence of a bounded non-constant matrix-valued function f on G satisfying the convolution equation $f * \sigma = f$. This extends Choquet and Deny's theorem for real-valued functions on G.

1. Introduction

Let σ be a probability measure on a locally compact abelian group G. A real Borel function f on G is called *σ-harmonic* if it satisfies the integral equation

$$f(x) = (f * \sigma)(x) = \int_G f(x-y)d\sigma(y) \qquad (\text{for all } x \in G).$$

A celebrated theorem of Choquet and Deny [2] asserts that every bounded σ-harmonic function on G is constant if (and only if) σ is *adapted*; that is, the support of σ generates a dense subgroup of G. Choquet and Deny's theorem plays an important role in probability theory and has been extended to various non-abelian groups (see, for example, [4, 5, 7, 8, 10]). Recently in [11], a vector-valued version of the Choquet–Deny theorem has been proved and used to obtain a vector-valued renewal theorem for the study of the L^p dimension of some vector-valued self-similar measures. In a related paper [3], the equation $f * \sigma = f$ has been studied under the assumption that both σ and f are operator-valued, but with *commuting ranges*. In this paper, we remove the restriction of commuting ranges and prove a Choquet–Deny type theorem for matrix-valued functions defined on G. This theorem uses positive definite matrices and differs from that of [11] where matrices with non-negative entries are considered instead, and consequently different techniques are used.

Let M_n be the C^*-algebra of $n \times n$ complex matrices. The pure states of M_n are exactly the *vector states* $\rho(\cdot) = \langle \cdot\, \xi, \xi \rangle$ where ξ is a unit vector in $\mathbb{C}^n$. Let M_n^+ be the positive cone of M_n, consisting of all self-adjoint matrices with non-negative eigenvalues. An M_n^+-valued measure σ on G will be called a *positive* M_n-valued measure and its *support* is defined to be

$$\operatorname{supp} \sigma = \{x \in G : \sigma(V) \neq 0 \text{ for all open sets } V \text{ containing } x\}.$$

We say that σ is *adapted* if $\rho \circ \sigma$ is adapted on G for every pure state ρ of M_n. We note that $\operatorname{supp}(\rho \circ \sigma) \subseteq \operatorname{supp} \sigma$. We can write $\sigma = (\sigma_{ij})$ where each σ_{ij} is a

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complex-valued measure on G. A function $f\colon G \longrightarrow M_n$ can also be denoted by $f = (f_{ij})$ where each f_{ij} is a complex-valued function on G. The convolution $f * \sigma$ can be defined naturally in terms of $f_{kl} * \sigma_{ij}$ via matrix multiplication. Details are given later. An M_n-valued function f on G is called *σ-harmonic* if it satisfies the convolution equation $f * \sigma = f$. We can now state our main result.

Theorem 1. *Let σ be a positive M_n-valued measure on G such that $\sigma(G)$ is the identity matrix. The following conditions are equivalent:*

i) every bounded σ-harmonic M_n-valued function on G is constant;
ii) σ is adapted.

2. Choquet–Deny type theorem

We need some vector measure preliminaries. Let $\mathcal{B}$ be the algebra of Borel sets in G. By an M_n-valued *measure* on G, we mean a (norm) countably additive function $\sigma\colon \mathcal{B} \longrightarrow M_n$. If we use the matrix notation $\sigma = (\sigma_{ij})$, then each σ_{ij} is a complex-valued measure on G. We adopt the definition of a complex measure in [13] and note that complex-valued measures are not only bounded, but also of bounded total variation [13, Theorem 6.4]. As in [6, p. 2], we define the *semivariation* of σ to be a non-negative function $\|\sigma\|$ whose value on a set $S \in \mathcal{B}$ is given by

$$\|\sigma\|(S) = \sup\{|\rho \circ \sigma|(S) : \rho \in M_n^*, \|\rho\| \le 1\}$$

where $|\rho \circ \sigma|$ is the total variation of the complex measure $\rho \circ \sigma$. We will write $\rho\sigma$ for $\rho \circ \sigma$. We say that σ is of *bounded semivariation* if $\|\sigma\|(G) < \infty$ which is equivalent to the condition that $\sigma(\mathcal{B})$ is norm-bounded in M_n by [6, Proposition 11]. Given any $A = (a_{ij}) \in M_n$, we have

$$\|A\| \le \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2} \le n\|A\|.$$

The above inequality together with the boundedness of complex measures implies that every M_n-valued measure on G is of bounded semivariation.

Given a Borel function $\lambda\colon G \longrightarrow \mathbb{C}$, the vector integral $\int_S \lambda d\sigma$ is defined in the usual way (see [6, Definition 12]). If λ is bounded, then we have

$$\left\|\int_G \lambda d\sigma\right\| \le \|\lambda\|_\infty \|\sigma\|(G).$$

In general, we note that the bound of the above inequality cannot be sharpened to $\|\lambda\|_\infty\|\sigma(G)\|$, as the following example shows, although it can be if σ is positive with commuting range.

Example 1. Let $\alpha = -\frac{\pi}{4}$ and $\beta = \frac{3\pi}{4}$. Let σ be the following M_2-valued measure on $\mathbb{R}$,

$$\sigma = A\delta_\alpha + B\delta_\beta$$

where δ_x is the point mass at $x \in \mathbb{R}$,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and $\sigma(\mathbb{R})$ is the identity matrix. Let

$$D = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}, \qquad |D| = \sqrt{D^*D} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

and define $\rho\colon M_2 \longrightarrow \mathbb{C}$ by

$$\rho(X) = \frac{1}{2}\text{trace}(DX)$$

for $X \in M_2$. Then $\|\rho\| = \frac{1}{2}\text{trace}(|D|) = 1$. Therefore

$$\|\sigma\|(\mathbb{R}) \geq |\rho\sigma|(\mathbb{R}) \geq |\rho(A)| + |\rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\sigma(\mathbb{R})\|.$$

Given $\lambda(x) = e^{ix}$, we have

$$\left\| \int_{\mathbb{R}} \lambda(x) d\sigma(x) \right\| \geq |e^{\alpha i}\rho(A) + e^{\beta i}\rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\lambda\|_\infty \|\sigma(\mathbb{R})\|.$$

As in [1, 3], using the natural bilinear map

$$(A, B) \in M_n \times M_n \mapsto AB \in M_n,$$

we can define the σ-*integrable functions* $f\colon G \longrightarrow M_n$ and the *bilinear vector integrals* $\int_S f d\sigma$ for $S \in \mathcal{B}$. For our purpose, we can simplify the construction in the following way. Given $\sigma = (\sigma_{ij})$ and $f = (f_{ij})$, we say that f is σ-*integrable* on S if the integral $\int_S f_{ik} d\sigma_{kj}$ exists for all i, j, k in which case we define

$$\int_S f d\sigma = \left(\sum_{k=1}^{n} \int_S f_{ik} d\sigma_{kj} \right) \in M_n.$$

For example, if $f_{ij} = \delta_{ij}\lambda$ where δ_{ij} is the Kronecker delta, then we have the (i, j)-th entry $\left(\int_S f d\sigma\right)_{ij} = \int_S \lambda d\sigma_{ij}$ and $\int_S f d\sigma = \int_S \lambda d\sigma$. We can also define the following convolution of f and σ if it exists:

$$f * \sigma(x) = \int_G f(x-y) d\sigma(y).$$

Given two matrix-valued measures $\sigma = (\sigma_{ij})$ and $\gamma = (\gamma_{ij})$, their convolution can be defined as

$$\sigma * \gamma = \left(\sum_k \sigma_{ik} * \gamma_{kj} \right).$$

Given a complex-valued Borel measure μ on G, the integral $\int_S f d\mu \in M_n$ denotes the Bochner integral of $f = (f_{ij})$ whenever it is well defined (see [6, p. 44]).

In the sequel, we will always assume that all M_n-valued measures σ are *regular* which means that each $\rho\sigma$ is a regular Borel measure on G, for every $\rho \in M_n^*$.

Let $\sigma = (\sigma_{ij})$ be an M_n-valued measure on G. We define its *Fourier transform* $\widehat{\sigma}$ on the dual group $\widehat{G}$ by

$$\widehat{\sigma}(\lambda) = \int_G \lambda(-x) d\sigma(x) \in M_n$$

for $\lambda \in \widehat{G}$. We also define the *determinant* $\mathbf{det}\,\sigma$ by convolution

$$\mathbf{det}\,\sigma = \sum_\pi \text{sgn}(\pi)\sigma_{1\pi(1)} * \cdots * \sigma_{n\pi(n)}$$

where π is a permutation. But for an M_n-valued function $f = (f_{ij})$ on G, we define the *determinant* $\det f$ by pointwise multiplication

$$\det f = \sum_{\pi} \operatorname{sgn}(\pi) f_{1\pi(1)} \cdots f_{n\pi(n)}.$$

Using the above notation, we have

$$\det \widehat{\sigma}(\lambda) = \widehat{\mathbf{det}\,\sigma}(\lambda).$$

We also have

$$\rho(\widehat{\sigma}(\lambda)) = \int_G \lambda(-x) d\rho\sigma(x) = \widehat{\rho\sigma}(\lambda)$$

for $\rho \in M_n^*$.

The proof of Theorem 1 is achieved by writing the equation $f = f * \sigma$ in the form $f * \mu = 0$ and convolving it with a judiciously chosen M_n-valued measure which reduces the equation to simultaneous scalar convolution equations, but convolved by the *same* scalar measure, as in the proof of Lemma 3 *ii)* $\Rightarrow$ *i)*, to which one can apply the $L^1(G)$-Tauberian theorem [9, 39.27] and get the result readily. The setting where the Tauberian theorem applies is the following lemma which can be proved as in [12, Theorem 2].

Lemma 2. *Let μ be a complex-valued measure on a locally compact abelian group G. The following conditions are equivalent:*

i) for every $f \in L^\infty(G)$, $f * \mu = 0$ implies that f is constant;
ii) $\widehat{\mu}(\lambda) \neq 0$ for $\lambda \in \widehat{G} \setminus \{\iota\}$ where ι is the identity in $\widehat{G}$.

We need to extend the above lemma to the following matrix-valued setting.

Lemma 3. *Let μ be an M_n-valued measure on a locally compact abelian group G. The following conditions are equivalent:*

i) for every bounded M_n-valued function f on G, $f * \mu = 0$ implies that f is constant;
ii) $\det \widehat{\mu}(\lambda) \neq 0$ for $\lambda \in \widehat{G} \setminus \{\iota\}$.

Proof. i) $\Longrightarrow$ *ii)*. Suppose that $\det \widehat{\mu}(\lambda) = 0$ for some $\lambda \in \widehat{G} \setminus \{\iota\}$. Then there exists $\xi \in \mathbb{C}^n \setminus \{0\}$ such that $\widehat{\mu}(\lambda)^{\mathrm{T}} \xi = 0$, where T denotes transpose. Define $f \colon G \longrightarrow M_n$ by $f(x) = \lambda(x)(\zeta_{ij})$ where

$$(\zeta_{i1}, \zeta_{i2}, \cdots, \zeta_{in}) = (\xi_1, \xi_2, \cdots, \xi_n)$$

for all i. Then $f * \mu = 0$, but f is not constant.

ii) $\Longrightarrow$ *i)*. Let $f = (f_{ij})$ be a bounded M_n-valued function such that $f * \mu = 0$. Let $\gamma = (\gamma_{ij})$ be the M_n-valued measure defined as the adjoint matrix of $\mu = (\mu_{ij})$, using convolution, so that

$$\mu * \gamma = \begin{pmatrix} \mathbf{det}\,\mu & & 0 \\ & \ddots & \\ 0 & & \mathbf{det}\,\mu \end{pmatrix}.$$

Then we have

$$f * \begin{pmatrix} \mathbf{det}\,\mu & & 0 \\ & \ddots & \\ 0 & & \mathbf{det}\,\mu \end{pmatrix} = f * \mu * \gamma = 0$$

which gives

$$f_{ij} * \mathbf{det}\,\mu = 0$$

for all i, j. Since $\widehat{\mathbf{det}\,\mu}(\lambda) = \det \widehat{\mu}(\lambda) \neq 0$ for $\lambda \in \widehat{G} \setminus \{\iota\}$, Lemma 2 implies that f_{ij} is constant. □

Lemma 4. *Let $A \in M_n^+$ be such that $\langle A\xi, \xi\rangle = 0$ for some $\xi \in \mathbb{C}^n$. Then $A\xi = 0$.*

Proof. We have $A = B^2$ for some $B \in M_n^+$. Hence $\langle B^2\xi, \xi\rangle = 0$ which gives $B\xi = 0$ and $A\xi = 0$. □

We also need the following well-known result for Theorem 1.

Lemma 5. *Let ν be a probability measure on a locally compact abelian group G. Then ν is adapted if, and only if, $\widehat{\nu}(\lambda) \neq 1$ for every $\lambda \in \widehat{G} \setminus \{\iota\}$.*

Let f be an M_n-valued function on G and let Δ_e be the diagonal matrix in which each diagonal entry is the point mass δ_e at the identity e of G. Then we have $f * \sigma = f$ if, and only if, $f * (\sigma - \Delta_e) = 0$. Given $\lambda \in \widehat{G}$, we have

$$\Big(\mathbf{det}\,(\sigma - \Delta_e)\Big)\widehat{\ }(\lambda) = \det(\widehat{\sigma - \Delta_e}(\lambda)) = \det(\widehat{\sigma}(\lambda) - I_n).$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. i) $\Longrightarrow$ ii). Let $\rho(\cdot) = \langle \cdot\, \xi, \xi\rangle$ be a pure state of M_n. We need to show that $\rho\sigma$ is adapted. Suppose otherwise. By Lemma 5, there exists $\lambda \in \widehat{G} \setminus \{\iota\}$ such that $\widehat{\rho\sigma}(\lambda) = 1$; that is $\langle \widehat{\sigma}(\lambda)\xi, \xi\rangle = 1$. As we do not know if $\|\widehat{\sigma}(\lambda)\|$ is at most 1 (see Example 1), we cannot conclude immediately that $\widehat{\sigma}(\lambda)\xi = \xi$ although it is true but requires the following arguments. Since $\rho\sigma$ is a probability measure, we have

$$\rho\sigma\{x \in G : \lambda(-x) = 1\} = 1.$$

Write $V = \{x \in G : \lambda(-x) \neq 1\}$; then

$$\langle \sigma(V)\xi, \xi\rangle = \rho\sigma(V) = 0.$$

It follows from Lemma 4 that $\sigma(V)\xi = 0$ and hence also

$$\sigma(G \setminus V)\xi = \xi.$$

Thus

$$\begin{aligned} \widehat{\sigma}(\lambda)\xi &= \left(\int_G \lambda(-x)\, d\sigma(x)\right)\xi \\ &= \left(\int_{G\setminus V} 1\, d\sigma(x)\right)\xi + \left(\int_V \lambda(x)\, d\sigma(x)\right)\xi \\ &= \sigma(G \setminus V)\xi \;=\; \xi. \end{aligned}$$

Therefore $\det(\widehat{\sigma}(\lambda) - I_n) = 0$. By Lemma 3, there is a non-constant bounded M_n-valued function f such that $f * (\sigma - \Delta_e) = 0$ which contradicts condition *i)*.

ii) $\Longrightarrow$ i). By Lemma 3, it suffices to show that $\det(\widehat{\sigma - \Delta_e})(\lambda) \neq 0$ for all $\lambda \in \widehat{G} \setminus \{\iota\}$. Suppose otherwise, so that $\det(\widehat{\sigma}(\lambda) - I_n) = 0$ for some $\lambda \neq \iota$. Then there is a unit vector $\xi \in \mathbb{C}^n$ such that $(\widehat{\sigma}(\lambda) - I_n)\xi = 0$; that is, $\widehat{\sigma}(\lambda)\xi = \xi$. Let

$\rho(\cdot) = \langle \cdot\, \xi, \xi \rangle$. Then we have

$$\widehat{\rho\sigma}(\lambda) = \rho(\widehat{\sigma}(\lambda)) = \langle \widehat{\sigma}(\lambda)\xi, \xi \rangle = 1.$$

Therefore, by Lemma 5, $\rho\sigma$ is not adapted, contradicting condition *ii*). □

We end with an example which shows that condition *ii*) in Theorem 1 cannot be replaced by the condition that $\operatorname{supp}\sigma$ generates a dense subgroup of G.

Example 2. Let ν be any adapted probability measure on $\mathbb{R}$ with $\nu\{0\} \geq \frac{1}{2}$ and let

$$\sigma = \begin{pmatrix} \nu & \delta_0 - \nu \\ \delta_0 - \nu & \nu \end{pmatrix}.$$

Then σ is a positive M_2-valued measure on $\mathbb{R}$ such that $\sigma(\mathbb{R})$ is the identity matrix and $\operatorname{supp}\sigma = \operatorname{supp}\nu$ generates a dense subgroup of $\mathbb{R}$. A direct calculation reveals that every M_2-valued function $f = (f_{ij})$, with $f_{11} = f_{12}$ and $f_{21} = f_{22}$, is σ-harmonic and need not be constant.

In fact under the change of coordinates $u = x + y$ and $v = x - y$, the measure σ is transformed to

$$\begin{pmatrix} \delta_0 & 0 \\ 0 & 2\nu - \delta_0 \end{pmatrix}.$$

The bounded solutions of the convolution equation with this measure are then of the form

$$\begin{pmatrix} g & \alpha \\ h & \beta \end{pmatrix},$$

with α and β constants, g and h any bounded functions. Undoing the change of coordinates we see that all bounded solutions of $f = f * \sigma$ are the bounded functions $f = (f_{ij})$, with $f_{11} - f_{12}$ and $f_{21} - f_{22}$ both constant.

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