

REALIZING ALTERNATING GROUPS AS MONODROMY GROUPS OF GENUS ONE COVERS

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ABSTRACT. We prove that if $n \geq 4$, a generic Riemann surface of genus 1 admits a meromorphic function (i.e., an analytic branched cover of \mathbb{P}^1) of degree n such that every branch point has multiplicity 3 and the monodromy group is the alternating group A_n . To prove this theorem, we construct a Hurwitz space and show that it maps (generically) onto the genus one moduli space.

1. INTRODUCTION

Associated to any n -sheeted branched cover of \mathbb{P}^1 with branch set $B \subset \mathbb{P}^1$ is a homomorphism $\pi_1(\mathbb{P}^1 - B) \rightarrow S_n$ (the symmetric group) called the monodromy representation of the branched cover. The image of this homomorphism in S_n is simply called the monodromy group of the cover (this group is well-defined up to conjugacy in S_n). If Σ is a compact Riemann surface and ϕ is a nonconstant meromorphic function on Σ , then $\phi : \Sigma \rightarrow \mathbb{P}^1$ is a branched cover and so we may speak of the monodromy group of ϕ . In [GN], it is stated that “Thompson (private correspondence) has verified that A_4 is the monodromy group of the generic Riemann surface of genus 1 (as far as we are aware, this is the only known example of a cover of a generic genus $g > 0$ surface with monodromy group different from a symmetric group)”. Our main result in this paper (Theorem 1, stated formally and proved in Section 4) states that this is true for *all* A_n , where $n \geq 4$. More precisely, Theorem 1 asserts that if $n \geq 4$, then a generic Riemann surface of genus one admits a meromorphic function of degree n whose monodromy group is the alternating group A_n and all of whose branch points have multiplicity 3. By *generic*, we mean that for a given n , all but a finite number of genus 1 Riemann surfaces admit such functions.

It also says, the set of Riemann surfaces of genus $g \geq 1$, with functions admitting branch points of multiplicity only 3, defines an algebraic set of dimension ≥ 1 in the module space of curves of genus g (Section 4, Comment 0). There is only one Riemann surface of genus one which admits a meromorphic function with monodromy A_3 : it is the Fermat curve $x^3 + y^3 + z^3 = 0$, and the meromorphic function is projection onto any one of the three coordinate axes in \mathbb{P}^2 . To see that there is only one such curve, note that, first, the location of the three branch points in \mathbb{P}^1

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is irrelevant to the moduli and, second, the combinatorics is completely determined by the monodromy requirements (since the only way to select three 3-cycles in A_3 whose product is 1 is to select the same 3-cycle three times).

We now give a brief summary of our proof. Given a topological branched cover $\phi : \Sigma \rightarrow \mathbb{P}^1$, one may form the corresponding *Hurwitz space* \mathcal{H} , a moduli space whose points represent those branched covers $\Sigma \rightarrow \mathbb{P}^1$ which may be obtained from ϕ by moving around the images of the branch points in \mathbb{P}^1 while holding constant the combinatorial branch structure over these points as they move. Each of these branched covers gives rise to a complex structure on Σ by pulling back the one on \mathbb{P}^1 . This defines a map $\Psi : \mathcal{H} \rightarrow \mathcal{M}_\Sigma$, where \mathcal{M}_Σ is the moduli space of complex structures on Σ . Under the assumption that $\phi : \Sigma \rightarrow \mathbb{P}^1$ is *completely non-Galois* (i.e., it has no non-trivial deck transformations), one may show that Ψ lifts to a map $\tilde{\Psi} : Q \rightarrow \mathcal{T}_\Sigma$, where Q is a regular covering space of \mathcal{H} and \mathcal{T}_Σ is the Teichmüller space of Σ , and that $\tilde{\Psi}$ is equivariant with respect to a natural group homomorphism $R : \text{Deck}(Q \rightarrow \mathcal{H}) \rightarrow \Gamma_\Sigma$, where Γ_Σ denotes the mapping class group of Σ acting on \mathcal{T}_Σ . Our proof then proceeds as follows: We first prove an algebraic lemma enabling us to construct a topological branched cover $\phi : \Sigma \rightarrow \mathbb{P}^1$, which has the branch structure and monodromy specified in the theorem. We then show that for the corresponding Hurwitz space, R has infinite image. Because every point of \mathcal{T}_Σ has finite stabilizer in Γ_Σ , it follows that $\tilde{\Psi}$ and hence Ψ are nonconstant. Because Ψ is an algebraic map between algebraic varieties, when \mathcal{M}_Σ has dimension 1, Ψ maps \mathcal{H} onto a Zariski open subset of \mathcal{M}_Σ , i.e., onto a set with finite complement. This proves the theorem; given any genus one Riemann surface, elements of the inverse image of the corresponding point in \mathcal{M}_Σ are branched covers with the desired property (since combinatorially, they are identical to ϕ). Whenever the genus of \mathcal{M}_Σ is positive, Ψ is non-constant.

2. A TOPOLOGICAL CONSTRUCTION OF THE BRANCHED COVERING

We begin by reminding the reader how any given n -sheeted branched covering $\phi : \Sigma \rightarrow \mathbb{P}^1$ may be described combinatorially. Let $\{x_1, \dots, x_r\} \subset \mathbb{P}^1$ denote the points over which branching occurs, and choose a basepoint $x_0 \in \mathbb{P}^1$ disjoint from the other x_i 's. Let w_1, \dots, w_r denote simple closed curves in $\mathbb{P}_0^1 := \mathbb{P}^1 - \{x_1, \dots, x_r\}$, all based at x_0 , which satisfy (see Figure 1):

- (1) Each w_i bounds a disc $D_i \subset \mathbb{P}^1$ such that $D_i \cap \{x_1, \dots, x_r\} = \{x_i\}$.
- (2) If $i \neq j$, then $D_i \cap D_j = \{x_0\}$.
- (3) Each w_i is oriented counterclockwise as the boundary of D_i .
- (4) $\prod_{i=1}^r w_i = 1$ in $\pi_1(\mathbb{P}_0^1, x_0)$.

Label the points in $\phi^{-1}(x_0)$ by the numbers $\{1, \dots, n\}$. Then each loop w_i gives rise to a permutation $\rho_i \in S_n$, the symmetric group. Think of ρ_i as acting on $\{1, \dots, n\}$ from the right. Define a group homomorphism $\rho : \pi_1(\mathbb{P}_0^1, x_0) \rightarrow S_n$ by $\rho(w_i) = \rho_i$ (this ρ is the *monodromy representation* of ϕ). Define the *signature* of the branched cover $\phi : \Sigma \rightarrow \mathbb{P}^1$ to be the n -tuple of permutations (ρ_1, \dots, ρ_r) . Conversely, suppose we are just given the points $\{x_1, \dots, x_r\} \subset \mathbb{P}^1$, the loops w_1, \dots, w_r (as above), and the permutations $\rho_1, \dots, \rho_r \in S_n$ satisfying $\prod_{i=1}^r w_i = 1$. Reconstruct the surface Σ and the branched covering $\phi : \Sigma \rightarrow \mathbb{P}^1$ as follows. First construct the (unbranched) cover $\phi_0 : \Sigma_0 \rightarrow \mathbb{P}_0^1$ corresponding to ρ using covering space theory. Then fill in one point for each end of Σ_0 to obtain Σ , and extend ϕ_0 continuously to ϕ on Σ in the only possible way.

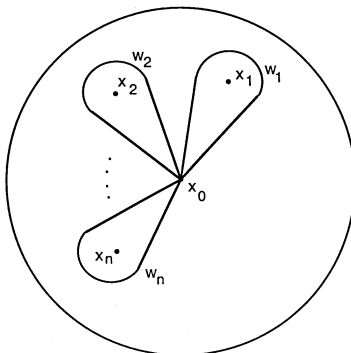


FIGURE 1.

Thus, to create a branched covering with certain properties, one needs to produce permutations with corresponding properties. Hence the following lemma:

Lemma 1. *Let $n \geq 4$, and consider $\rho_1 = (123)$ and $\rho_2 = (132)$ in S_n . Then it is possible to choose $\rho_3, \dots, \rho_n \in S_n$ such that:*

- (1) ρ_i is a 3-cycle for each i .
- (2) $\prod_{i=1}^n \rho_i = 1$.
- (3) The number “1” doesn’t occur in any of the 3-cycles ρ_3, \dots, ρ_n : all fix 1.
- (4) The subgroup of S_n generated by $\{\rho_3, \dots, \rho_n\}$ acts transitively on $\{2, \dots, n\}$.
- (5) $\{\rho_1, \dots, \rho_n\}$ generate A_n .

Proof. We will denote by $\vec{\rho}_n$ the n -tuple (ρ_1, \dots, ρ_n) . Let

$$\vec{\rho}_4 = ((123), (132), (234), (243))$$

and

$$\vec{\rho}_5 = ((123), (132), (234), (245), (253)).$$

It is easily verified that these signatures satisfy the five conditions specified in the lemma. Inductively, if $n > 5$ define $\vec{\rho}_n$ by adjoining the permutations $\rho_{n-1} = (2 \ n \ n-1)$ and $\rho_n = (2 \ n \ n-1)$ to the $(n-2)$ -tuple $\vec{\rho}_{n-2}$. It is an elementary exercise (which we omit) to show that $\vec{\rho}_n$ satisfies the conditions of the theorem for all n . This completes the proof of Lemma 1.

Fix an $n \geq 4$, choose n distinct points $x_1, \dots, x_n \in \mathbb{P}^1$, a basepoint $x_0 \in \mathbb{P}_0^1$, and n based loops w_i related to the x_i ’s as described above. Use the signature $\vec{\rho}_n$ produced in Lemma 1 to construct a branched cover $\phi : \Sigma \rightarrow \mathbb{P}^1$, branched over the x_i ’s. By construction, this n -sheeted cover will be connected and have monodromy group A_n . By the Riemann-Hurwitz formula, $\text{genus}(\Sigma) = 1$.

3. HURWITZ SPACES

In this section we give a construction of the Hurwitz space corresponding to a branched cover. (Note: The definition of a *Hurwitz space* given in this paper corresponds to a single connected component of a Hurwitz space as defined in [F2].) We will start with a general finite-sheeted branched cover, and then specialize to the ones constructed in the last section. So, begin by letting $\phi : \Sigma \rightarrow \mathbb{P}^1$ be any n -sheeted branched cover, branched over $\{x_1, \dots, x_r\}$. Let $\text{Homeo}(\mathbb{P}^1)$ denote the

topological group of orientation preserving self-homeomorphisms of \mathbb{P}^1 . Define the *Hurwitz space* \mathcal{H} corresponding to the branched cover ϕ by

$$\mathcal{H} = \{g \circ \phi : \Sigma \rightarrow \mathbb{P}^1 \text{ such that } g \in \text{Homeo}(\mathbb{P}^1)\} / \sim$$

where $g_1 \circ \phi \sim g_2 \circ \phi$ if and only if there exists a homeomorphism $h : \Sigma \rightarrow \Sigma$ such that the diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{h} & \Sigma \\ & \searrow g_2 \circ \phi & \swarrow g_1 \circ \phi \\ & \mathbb{P}^1 & \end{array}$$

commutes. Note that $g_1 \circ \phi \sim g_2 \circ \phi$ if and only if there exists an $h \in \text{Homeo}(\Sigma)$ such that $(g_1^{-1}g_2)\phi = \phi h$. Thus we may write $\mathcal{H} \cong \text{Homeo}(\mathbb{P}^1)/G$, where $G \subset \text{Homeo}(\mathbb{P}^1)$ is the subgroup consisting of those homeomorphisms g of \mathbb{P}^1 which lift to a homeomorphism h_g of Σ making the diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{h_g} & \Sigma \\ \phi \downarrow & & \phi \downarrow \\ \mathbb{P}^1 & \xrightarrow{g} & \mathbb{P}^1 \end{array}$$

commute. Let S_r act on $(\mathbb{P}^1)^r$ by permuting the coordinates, and define $\Delta \subset (\mathbb{P}^1)^r$ by $\Delta = \{(y_1, \dots, y_r) \in (\mathbb{P}^1)^r : y_i = y_j \text{ for some } i \neq j\}$. Define

$$\Pi = ((\mathbb{P}^1)^r - \Delta) / S_r.$$

Define a map $P : \text{Homeo}(\mathbb{P}^1) \rightarrow \Pi$ by $P(f) = [f(x_1), \dots, f(x_r)]$. Define the following two subgroups of $\text{Homeo}(\mathbb{P}^1)$:

$$\begin{aligned} \mathcal{G} &= P^{-1}[x_1, \dots, x_r], \\ \mathcal{G}_0 &= \text{the identity component of } \mathcal{G}. \end{aligned}$$

We now observe that

$$\mathcal{G}_0 \subseteq G \subseteq \mathcal{G}.$$

The second of these inclusions is completely elementary; since $\phi \circ h$ and $g \circ \phi$ are two ways of writing the same branched cover, they must have the same branch locus in \mathbb{P}^1 . Hence, $g[x_1, \dots, x_r] = [x_1, \dots, x_r]$.

To prove the first inclusion, $\mathcal{G}_0 \subseteq G$, we quote two lemmas from [KlKo, Lemmas 2 and 3]:

Lemma 2. *If $r \geq 3$, then $\pi_i(\mathcal{G}_0) = 0$ for all i .*

We omit the proof of Lemma 2; the reader is referred to [KlKo].

Lemma 3. *Given $g \in \mathcal{G}_0$, there is a homeomorphism $h_g : \Sigma \rightarrow \Sigma$ such that $g \circ \phi = \phi \circ h_g$. If $r \geq 3$, then h_g is uniquely determined by g and, in fact, $g \mapsto h_g$ defines a continuous group homomorphism $\mathcal{G}_0 \rightarrow \text{Homeo}(\Sigma)$ such that ϕ is equivariant with respect to the resulting action of \mathcal{G}_0 on Σ .*

Proof. Let $g \in \mathcal{G}_0$. Choose a path g_t in \mathcal{G}_0 from the identity to g . Let $y \in \phi^{-1}(\mathbb{P}_0^1)$. Let $\alpha : I \rightarrow \phi^{-1}(\mathbb{P}_0^1)$ be the lift of the path $g_t(\phi(y))$ which starts at y , and define $h_g(y) = \alpha(1)$. Define h_g to be the identity on $\phi^{-1}(\{x_1, \dots, x_n\})$. Then $h_g : \Sigma \rightarrow \Sigma$ is a homeomorphism and $g \circ \phi = \phi \circ h_g$. Furthermore, if $r \geq 3$, then, since $\pi_1(\mathcal{G}_0) = 0$, any two such paths g_t would lead to homotopic paths in \mathbb{P}_0^1 . Hence, for $r \geq 3$, $g \mapsto h_g$ is a well-defined homomorphism $\mathcal{G}_0 \rightarrow \text{Homeo}(\Sigma)$ making ϕ equivariant. This completes the proof of Lemma 3.

For the rest of this section assume that the branched cover $\phi : \Sigma \rightarrow \mathbb{P}^1$ is *completely non-Galois*: It has no non-trivial deck transformations. This is equivalent to the algebraic assumption that the monodromy group of ϕ has trivial centralizer in S_n . From this, we have a well-defined group homomorphism $G \rightarrow \text{Homeo}(\Sigma)$ given by $g \mapsto h_g$, where h_g is defined as in the definition of G .

Next, we construct some useful covering maps. Let $\Pi = ((\mathbb{P}^1)^r - \Delta)/S_r$, which is homeomorphic to $\text{Homeo}(\mathbb{P}^1)/\mathcal{G}$ and let $Q = \text{Homeo}(\mathbb{P}^1)/\mathcal{G}_0$. Because \mathcal{G}_0 is the identity component of G and of \mathcal{G} , it follows that the natural quotient maps $Q \rightarrow \mathcal{H} \rightarrow \Pi$ are both covering maps. Furthermore, since \mathcal{G}_0 is a normal subgroup of G , the covering map $Q \rightarrow \mathcal{H}$ is regular (Galois), with deck group equal to G/\mathcal{G}_0 . This deck group acts on Q from the right in the obvious manner, with quotient \mathcal{H} . Note that Q is almost, but not quite, the universal cover of \mathcal{H} ; $\pi_1(Q) = \pi_1(\text{Homeo}(\mathbb{P}^1)) = Z_2$, since \mathcal{G}_0 is contractible and $SO(3) \rightarrow \text{Homeo}(\mathbb{P}^1)$ is a homotopy equivalence (a fact dating back to Kneser [K] in 1926).

We now remind the reader of some basic Teichmüller theory. Given the closed oriented (topological) surface Σ , define the Teichmüller space \mathcal{T}_Σ by

$$\mathcal{T}_\Sigma = \{(\Sigma_0, [q_0]) : \Sigma_0 \text{ is a Riemann surface and}$$

$$[q_0] \text{ is an isotopy class of homeomorphisms } \Sigma \rightarrow \Sigma_0\} / \sim$$

where we define $(\Sigma_0, q_0) \sim (\Sigma_1, q_1)$ if there is an analytic isomorphism $h : \Sigma_0 \rightarrow \Sigma_1$ such that $q_1 \circ h$ is isotopic to q_0 .

The mapping class group of Σ , defined by $\Gamma_\Sigma = \text{Homeo}(\Sigma)/\text{isotopy}$, acts on \mathcal{T}_Σ from the right by

$$(\Sigma_0, [q_0]) \cdot [h] = (\Sigma_0, [q_0 \circ h]).$$

The quotient of \mathcal{T}_Σ under this action is the *moduli space* of Σ , defined by

$$\mathcal{M}_\Sigma = \{\text{Riemann surfaces } \Sigma_0 \text{ homeomorphic to } \Sigma\} / \text{analytic isomorphism.}$$

Let $p : \Sigma \rightarrow \mathbb{P}^1$ be any branched cover; define Σ_p to be the Riemann surface with underlying space Σ and with the unique complex structure making p analytic. We now define maps $\Psi : \mathcal{H} \rightarrow \mathcal{M}_\Sigma$ and $\tilde{\Psi} : Q \rightarrow \mathcal{T}_\Sigma$ by $\Psi(fG) = \Sigma_{f\phi}$ and $\tilde{\Psi}(f\mathcal{G}_0) = (\Sigma_{f\phi}, \text{id}_\Sigma)$. It is immediately clear that the following diagram commutes:

$$\begin{array}{ccccc} \text{Homeo}(\mathbb{P}^1)/\mathcal{G}_0 & = & Q & \xrightarrow{\tilde{\Psi}} & \mathcal{T}_\Sigma \\ \downarrow & & \downarrow & & \downarrow \\ \text{Homeo}(\mathbb{P}^1)/G & = & \mathcal{H} & \xrightarrow{\Psi} & \mathcal{M}_\Sigma \end{array}$$

The vertical arrows in this diagram are simply quotient maps involving the right action of G/\mathcal{G}_0 on Q and the right action of Γ_Σ on \mathcal{T}_Σ . Define a group homomorphism $R : G/\mathcal{G}_0 \rightarrow \Gamma_\Sigma$ by $g\mathcal{G}_0 \mapsto [h_g]$. The fact that R is well-defined follows from the proof of Lemma 3, which actually shows that if $g \in \mathcal{G}_0$, then h_g is homotopic (hence isotopic) to the identity. In [KIKo] we give a general algorithm for computing the composition of R with the natural homomorphism $\Gamma_\Sigma \rightarrow SL(2g, \mathbb{Z})$ (defined by action on $H_1(\Sigma)$). In the genus one case, this gives R precisely, since $\Gamma_\Sigma \rightarrow SL(2, \mathbb{Z})$ is an isomorphism. In the current paper, instead of using this general method, we get the information we need from a specific geometric observation in the next section.

Lemma 4. $\tilde{\Psi}$ is equivariant with respect to the homomorphism $R : G/\mathcal{G}_0 \rightarrow \Gamma_\Sigma$.

Proof. We need to show that if $f \in \text{Homeo}(\mathbb{P}^1)$ and $g \in G$, then $\tilde{\Psi}(f\mathcal{G}_0 \cdot g) = (\tilde{\Psi}(f\mathcal{G}_0)) \cdot [h_g]$. Restating using the definitions, we need to show that $(\Sigma_{fg\phi}, [id]) \sim (\Sigma_{f\phi}, [h_g])$. In other words, we need to show that the diagram

$$\begin{array}{ccc} & \Sigma & \\ id \swarrow & & \searrow h_g \\ \Sigma_{fg\phi} & \xrightarrow{h_g} & \Sigma_{f\phi} \end{array}$$

commutes up to homotopy (which is obvious!), and that $h_g : \Sigma_{fg\phi} \rightarrow \Sigma_{f\phi}$ is analytic. To prove this second fact, consider the diagram

$$\begin{array}{ccc} \Sigma_{fg\phi} & \xrightarrow{h_g} & \Sigma_{f\phi} \\ \searrow fg\phi & & \swarrow f\phi \\ & \mathbb{P}^1 & \end{array}$$

which commutes by definition of h_g . The two vertical branched cover maps are analytic by definition of the complex structures on the Σ 's. Conclude the homeomorphism h_g is analytic as well. This completes the proof of Lemma 4.

4. STATEMENT AND PROOF OF THEOREM 1

Theorem 1. *Let $n \geq 4$ be an integer. There exists a finite subset $Y \subset \mathcal{M}_1$ (where \mathcal{M}_1 is the moduli space of genus one Riemann surfaces) with the following property. If Σ_0 is a Riemann surface of genus one, and $[\Sigma_0] \notin Y$, then there exists a holomorphic function $f : \Sigma_0 \rightarrow \mathbb{P}^1$ of degree n such that all branch points of f have multiplicity 3, no two branch points of f map to the same point in \mathbb{P}^1 , and the monodromy group of f is the full alternating group A_n .*

Proof. Fix n . Let $\phi : \Sigma \rightarrow \mathbb{P}^1$ be the topological branched cover with monodromy group A_n constructed in Section 2 using Lemma 1. In building this cover, we may choose our branch points x_1, \dots, x_n and our basepoint x_0 arbitrarily in \mathbb{P}^1 . Since A_n has trivial centralizer in S_n , the branched cover $\phi : \Sigma \rightarrow \mathbb{P}^1$ is completely non-Galois, and hence we can use ϕ to make all the constructions of Section 3 involving Hurwitz spaces, Teichmuller theory, etc. Express \mathbb{P}^1 as the union of two discs B_1 and B_2 whose intersection and common boundary is a smooth circle C . Choose these discs so that B_1 contains $D_1 \cup D_2$, B_2 contains $D_3 \cup \dots \cup D_n$ and, for $i = 1, \dots, n$, $C \cap D_i = x_0$. See Figure 2.

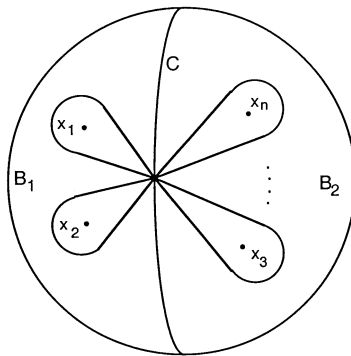


FIGURE 2.

We now wish to visualize the topology of $\phi^{-1}(B_1)$ and $\phi^{-1}(B_2)$. The monodromy along the curve C is trivial: $\rho_1\rho_2$ is the identity. Conclude that $\phi^{-1}(C)$ consists of n disjoint circles, each mapped homeomorphically to C by ϕ . Since we numbered the points of $\phi^{-1}(x_0)$ using $\{1, \dots, n\}$, this allows labelling components of $\phi^{-1}(C)$ as C_1, \dots, C_n according to which point of $\phi^{-1}(x_0)$ they contain. Using the algebraic properties of ρ_1, \dots, ρ_n in Lemma 1, conclude the following: $\phi^{-1}(B_1)$ consists of one component with boundary $C_1 \cup C_2 \cup C_3$ and $n - 3$ other components; each of these other components has as its boundary one of the remaining C_i 's (for $i > 3$), and is mapped homeomorphically onto B_1 . On the other hand, $\phi^{-1}(B_2)$ consists of only two components: the first maps homeomorphically to B_2 and has as its boundary C_1 and the second has as its boundary $C_2 \cup \dots \cup C_n$. We illustrate this situation in Figure 3, with \mathbb{P}^1 and Σ shown split in two along C and $\phi^{-1}(C)$.

Let $A \subset B_1$ be a thin collar of $C = \partial B_1$, i.e., an annulus in B_1 one of whose boundary components is C . Define $g \in \text{Homeo}(\mathbb{P}^1)$ to be a single Dehn twist along A . More precisely, the Dehn twist g is defined as follows. Identify A with $S^1 \times [0, 1]$ and define $g : A \rightarrow A$ by $g(z, t) = (e^{2\pi it} z, t)$. Clearly, g is a homeomorphism of A which is the identity on ∂A . Extend g to all of \mathbb{P}^1 by defining it to be the identity outside of A . If we define $h_g \in \text{Homeo}(\Sigma)$ to consist of simultaneous Dehn twists along all n components of $\phi^{-1}(A)$, then $\phi \circ h_g = g \circ \phi$. We conclude that $g \in G$ and $R(g\mathcal{G}_0) = [h_g]$. Referring to Figure 3, note that all the C_i 's except C_2 and C_3 bound discs in Σ (C_1 bounds a disc in $\phi^{-1}(B_2)$ while C_4, \dots, C_n bound discs

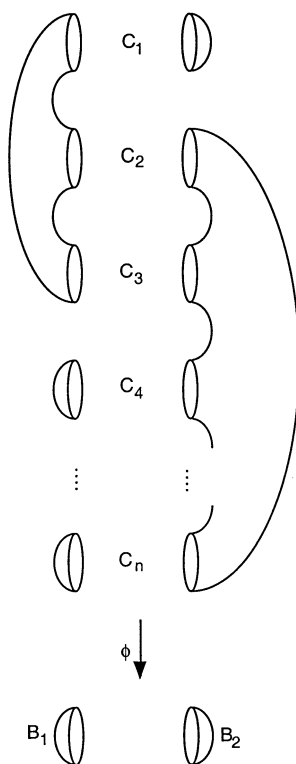


FIGURE 3.

in $\phi^{-1}(B_1)$); hence the corresponding Dehn twists are trivial in the mapping class group Γ_Σ . The curves C_2 and C_3 are isotopic to each other in Σ (by inspection of Figure 3); hence their Dehn twists are equal to each other in Γ_Σ . We conclude that $[h_g]$ is a double Dehn twist along the essential curve C_2 in the torus Σ . Hence $[h_g]$ is of infinite order in Γ_Σ : That a Dehn twist along an essential curve in a closed orientable surface has infinite order in the mapping class group follows easily by considering its action on the fundamental group. Since each point in \mathcal{T}_Σ has finite stabilizer in Γ_Σ , it follows that $\tilde{\Psi} : Q \rightarrow \mathcal{T}_\Sigma$ and, hence, $\Psi : \mathcal{H} \rightarrow \mathcal{M}_\Sigma$ are non-constant functions. Since \mathcal{H} and \mathcal{M}_Σ both have the structure of quasiprojective varieties (see [M], p. 25, for \mathcal{M}_g and [F1], p. 53, for \mathcal{H}), Ψ is an algebraic map which extends to the compactification of \mathcal{H} (see [Gr], p. 247). When \mathcal{M}_Σ has dimension 1 (Σ has genus one), conclude that the image of \mathcal{H} in \mathcal{M}_Σ is a quasiprojective subvariety of dimension one. Hence $\mathcal{M}_\Sigma - \Psi(\mathcal{H})$ consists of at most a finite number of points. This finishes the proof of Theorem 1.

Comment 0. [F3] shows, for each $r \geq n$, \mathcal{H} has exactly two components. Further, the Riemann surfaces parametrized by each have branch cycles satisfying the conditions of Lemma 1. In particular, Ψ restricted to each component is non-constant.

Comment 1. We originally conceived of Theorem 1 as an application of Fried's Theorem 3.6 in [F2], which states that if a certain representation of $\pi_1(\mathcal{H})$ on $H_1(\Sigma; \mathbb{Z})$ has infinite image, then Ψ is non-constant. However, one gets a similar result by considering our homomorphism R instead, which is a natural lift of Fried's representation. In addition, R has infinite image by the pictorial argument involving Dehn twists given here, rather than by the more algebraic computations involving $H_1(\Sigma; \mathbb{Z})$ (see for example [F2] and [KlKo]). We present this view for the sake of variety, and because we think it may appeal to the more geometrically-minded reader.

Comment 2. Having proved that, for each $n \geq 4$, the map $\Psi : \mathcal{H} \rightarrow \mathcal{M}_1$ misses at most a finite number of points of \mathcal{M}_1 , it is natural to ask, for each such n , whether the map does in fact miss some points or whether it might actually be surjective. Mark van Hoeij, using very nice computations involving J -invariants, has shown that in the case $n = 5$ the map Ψ is actually surjective. For higher values of n , it seems likely that it remains surjective but someone needs to prove it! For $n = 4$, we don't have a conjecture.

Comment 3. The preprint [F3] makes further applications of Dehn twists in order to compute explicitly the monodromy action of $\pi_1 Q$ on the cohomology of a Riemann surface corresponding to a point on a Hurwitz space. (For other examples of this, see [F2] and [KlKo].) As a result, the map Ψ is shown in [F3] to have higher dimensional image on other components of Hurwitz spaces constructed from r -tuples of 3-cycles corresponding to higher genus covers of \mathbb{P}^1 . We don't, however, know this dimension as a precise function of r and n .

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