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WEYL SPECTRA OF OPERATOR MATRICES

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ABSTRACT. In this paper it is shown that if $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is a 2×2 upper triangular operator matrix acting on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ and if $\omega(\cdot)$ denotes the "Weyl spectrum", then the passage from $\omega(A) \cup \omega(B)$ to $\omega(M_C)$ is accomplished by removing certain open subsets of $\omega(A) \cap \omega(B)$ from the former, that is, there is equality

$$\omega(A) \cup \omega(B) = \omega(M_C) \cup \mathfrak{S},$$

where $\mathfrak{S}$ is the union of certain of the holes in $\omega(M_C)$ which happen to be subsets of $\omega(A) \cap \omega(B)$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and abbreviate $\mathcal{L}(\mathcal{H}, \mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. When $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are given we denote by M_C an operator acting on $\mathcal{H} \oplus \mathcal{K}$ of the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. The invertibility and spectra of M_C were considered by Du and Jin [5]. In this paper we give some conditions for operators A and B to exist an operator C such that M_C is Weyl, and describe the Weyl spectra of M_C.

Recall ([7], [8]) that an operator $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ for Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is called *regular* if there is an operator $A' \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ for which $A = AA'A$; then A' is called a *generalized inverse* for A. In this case, $\mathcal{X}$ and $\mathcal{Y}$ can be decomposed as follows (cf. [8, Theorem 3.8.2]):

$$A^{-1}(0) \oplus A'A(\mathcal{X}) = \mathcal{X} \quad \text{and} \quad A(\mathcal{X}) \oplus (AA')^{-1}(0) = \mathcal{Y}.$$

It is familiar ([6], [8]) that $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is regular if and only if A has closed range. An operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called *relatively Weyl* if there is an invertible operator $A' \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ for which $A = AA'A$. It is known ([8, Theorem 3.8.6]) that A is relatively Weyl if and only if A is regular and $A^{-1}(0) \cong A(H)^{\perp}$, where "$\cong$" means a topological isomorphism between spaces. An operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called *left-Fredholm* if it is regular with finite dimensional null space and *right-Fredholm* if it is regular with its range of finite co-dimension. If A is both left- and right-Fredholm, we call it *Fredholm*. The *index*, $\operatorname{ind} A$, of a left- or right-Fredholm

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operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is defined by $\operatorname{ind} A = \dim A^{-1}(0) - \dim A(\mathcal{H})^{\perp}$. An operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is called *Weyl* if it is Fredholm of index zero. Thus a relatively Weyl operator with finite dimensional null space or its range of finite co-dimension is Weyl. If $A \in \mathcal{L}(\mathcal{H})$, then the left essential spectrum $\sigma_e^+(A)$, the right essential spectrum $\sigma_e^-(A)$, the essential spectrum $\sigma_e(A)$, and the Weyl spectrum $\omega(A)$ are defined by

$$\begin{aligned} \sigma_e^+(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not left-Fredholm}\};\\ \sigma_e^-(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not right-Fredholm}\};\\ \sigma_e(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\};\\ \omega(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\}. \end{aligned}$$

Evidently

$$\sigma_e^+(A) \cup \sigma_e^-(A) = \sigma_e(A) \subseteq \omega(A).$$

If we write $\operatorname{iso} \mathfrak{C}$ for the isolated points of $\mathfrak{C} \subseteq \mathbb{C}$ and $\sigma(A)$ for the ordinary spectrum of A, then we define

$$\pi_{00}(A) := \{\lambda \in \operatorname{iso} \sigma(A) : 0 < \dim (A - \lambda I)^{-1}(0) < \infty\}$$

for the isolated eigenvalues of finite multiplicity.

Recall that a sequence of module-homomorphisms

$$\mathcal{A}_0 \xrightarrow{f_1} \mathcal{A}_1 \xrightarrow{f_2} \mathcal{A}_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} \mathcal{A}_{n-1} \xrightarrow{f_n} \mathcal{A}_n$$

is said to be *exact* if $\operatorname{ran} f_i = \ker f_{i+1}$ $(i = 1, \cdots, n-1)$.

We begin with:

Lemma 1. *Suppose*

$$0 \longrightarrow \mathcal{A}_0 \xrightarrow{T_1} \mathcal{A}_1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} \mathcal{A}_n \longrightarrow 0 \tag{1.1}$$

is an exact sequence of Banach spaces. If each T_j $(1 \le j \le n)$ is regular, then

$$\begin{cases} \bigoplus_{i=0}^{\frac{n}{2}} \mathcal{A}_{2i} \cong \bigoplus_{i=0}^{\frac{n}{2}-1} \mathcal{A}_{2i+1} & (n \text{ even}),\\ \bigoplus_{i=0}^{\frac{n-1}{2}} \mathcal{A}_{2i} \cong \bigoplus_{i=0}^{\frac{n-1}{2}} \mathcal{A}_{2i+1} & (n \text{ odd}). \end{cases} \tag{1.2}$$

Hence, in particular, if the sequence (1.1) *is an exact sequence of Hilbert spaces, then* (1.2) *holds.*

Proof. If $T_j = T_j T_j' T_j$ with $T_j' \in \mathcal{L}(\mathcal{A}_j, \mathcal{A}_{j-1})$ $(1 \le j \le n)$, then each space $\mathcal{A}_j$ can be decomposed as follows:

$$\mathcal{A}_{j-1} = T_j^{-1}(0) \oplus T_j' T_j(\mathcal{A}_{j-1}) \quad (1 \le j \le n+1), \tag{1.3}$$

where $T_{n+1} : \mathcal{A}_n \to 0$ is the zero operator. Since the given sequence is exact, we have that $T_j(\mathcal{A}_{j-1}) = T_{j+1}^{-1}(0)$ $(1 \le j \le n)$. Since the restriction $T_j^{\#}$ of T_j to $T_j' T_j(\mathcal{A}_{j-1})$ is one-one and $T_j^{\#}(T_j' T_j(\mathcal{A}_{j-1})) = T_j(\mathcal{A}_{j-1})$, it follows that $T_j^{\#} : T_j' T_j(\mathcal{A}_{j-1}) \to T_{j+1}^{-1}(0)$ is an isomorphism, i.e.,

$$T_j' T_j(\mathcal{A}_{j-1}) \cong T_{j+1}^{-1}(0) \quad (1 \le j \le n). \tag{1.4}$$

Now (1.2) follows from (1.3) and (1.4). The second assertion follows from the first together with the observation that the exactness of the sequence gives that T_j has closed range, so that T_j is regular for $1 \le j \le n$. □

From Lemma 1 we can see that if $0 \to \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_n \to 0$ is an exact sequence of finite dimensional spaces, then $\sum_{i=0}^{n}(-1)^i \dim(\mathcal{A}_i) = 0$ (cf. [17, Theorem A.6]).

Corollary 2. *Suppose $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Hilbert spaces. If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $S \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $ST \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ have closed ranges, then there is isomorphism*

$$T^{-1}(0) \oplus S^{-1}(0) \oplus (ST\mathcal{X})^{\perp} \;\cong\; (ST)^{-1}(0) \oplus (T\mathcal{X})^{\perp} \oplus (S\mathcal{Y})^{\perp}.$$

Proof. From the "one-diagram" proof of the index theorem due to Yang [16], we can see that the sequence

$$0 \to T^{-1}(0) \to (ST)^{-1}(0) \to S^{-1}(0) \to (T\mathcal{X})^{\perp} \to (ST\mathcal{X})^{\perp} \to (S\mathcal{Y})^{\perp} \to 0$$

is exact. Thus the result follows at once from Lemma 1. □

Lemma 3. *For a given pair (A, B) of operators, if $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is Weyl, then M_C is Weyl for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. Hence, in particular, we have*

$$\omega(M_C) \subseteq \omega\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \subseteq \omega(A) \cup \omega(B). \tag{3.1}$$

Proof. If $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is Weyl, then A and B are both Fredholm, and $\operatorname{ind} A + \operatorname{ind} B = 0$. Write

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{3.2}$$

Since $\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ is invertible for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, and since $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ are both Fredholm, it follows that M_C is Fredholm. Furthermore we have that $\operatorname{ind} M_C = \operatorname{ind}\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right) + \operatorname{ind}\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right) = 0$ and therefore M_C is Weyl for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. The inclusions in (3.1) are evident from the first assertion. □

The following lemma gives a necessary condition for M_C to be Weyl:

Lemma 4. *If M_C is Weyl for some $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, then $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ satisfy the following conditions:*

(i) *A is left-Fredholm,*
(ii) *B is right-Fredholm,*
(iii) *$A^{-1}(0) \oplus B^{-1}(0) \cong A(\mathcal{H})^{\perp} \oplus B(\mathcal{K})^{\perp}$,*

which in turn implies that $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is relatively Weyl.

Proof. From (3.2) we can see that if M_C is Fredholm, then $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ is left-Fredholm and $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)$ is right-Fredholm, so that A is left-Fredholm and B is right-Fredholm. On the other hand since, evidently, $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ have closed ranges, it follows from Corollary 2 that

$$\begin{aligned}&\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} I & C \\ 0 & B \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \left(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}\right)^{\perp} \\ &\cong \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}\right)^{\perp} \oplus \left(\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}\begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}\right)^{\perp}.\end{aligned} \tag{4.1}$$

Thus if M_C is Weyl, then (4.1) reduces to (iii). For the second assertion, noting that if the pair (A, B) satisfies the conditions (i) and (ii), then (A, B) has a pair of generalized inverses (A', B'), we have that $\left(\begin{smallmatrix} A' & 0 \\ 0 & B' \end{smallmatrix}\right)$ is a generalized inverse of $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ and the condition (iii) is just the equivalence $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \cong \left(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathcal{H} \\ \mathcal{K} \end{smallmatrix}\right)\right)^{\perp}$, which implies that $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is relatively Weyl. □

By the argument of Lemma 4, we can see that if any two of A, B, and $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ are Fredholm, then so is the other and, in that case, $\operatorname{ind}\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \operatorname{ind} A + \operatorname{ind} B$ (cf. [3, Lemma 5.2], [10]).

The first inclusion in (3.1) may be proper. However, we have a large class of operators for which the first inclusion in (3.1) is reversible. To see this recall ([3], [13]) that the *spectral picture* of an operator $A \in \mathcal{L}(\mathcal{H})$, denoted $\mathcal{SP}(A)$, is the structure consisting of the set $\sigma_e(A)$, the collection of holes and pseudoholes in $\sigma_e(A)$, and the indices associated with these holes and pseudoholes, where a *hole* in $\sigma_e(A)$ is a nonempty bounded component of $\mathbb{C} \setminus \sigma_e(A)$ and a *pseudohole* in $\sigma_e(A)$ is a nonempty component of $\sigma_e(A) \setminus \sigma_e^+(A)$ or of $\sigma_e(A) \setminus \sigma_e^-(A)$. We then have:

Corollary 5. *If either $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ has no pseudoholes, then, for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$,*

$$\omega(M_C) = \omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{5.1}$$

Hence, in particular, if either $A \in \mathcal{L}(\mathcal{H})$ or $B \in \mathcal{L}(\mathcal{K})$ is essentially normal (i.e., the self-commutator is a compact operator), then (5.1) *holds.*

Proof. From Lemma 3, we have that $\omega(M_C) \subseteq \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. For the reverse, observe that if $\mathcal{SP}(A)$ has no pseudoholes, then, for every $\lambda \in \mathbb{C}$,

$$A - \lambda I \text{ is left-Fredholm} \Longrightarrow A - \lambda I \text{ is Fredholm}. \tag{5.2}$$

Thus if $\lambda \notin \omega(M_C)$, then by the remark after Lemma 4 and (5.2), $A - \lambda I$ and $B - \lambda I$ are both Fredholm. Further since $\left(\begin{smallmatrix} A-\lambda I & 0 \\ 0 & B-\lambda I \end{smallmatrix}\right)$ is relatively Weyl we must have that $\lambda \notin \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. If instead $\mathcal{SP}(B)$ has no pseudoholes, then the same argument gives the result. ☐

The condition "either $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ has no pseudoholes" is essential in Corollary 5. For example consider the following operators on $\ell_2 \otimes \ell_2$:

$$A = U \otimes 1, \quad B = U^* \otimes 1 \quad \text{and} \quad C = (1 - UU^*) \otimes 1, \tag{5.3}$$

where U is the unilateral shift on ℓ_2. Then $\omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \mathbb{D}$ and $\omega\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \mathbb{T}$.

The following is our main theorem. It says that the passage from $\omega(A) \cup \omega(B)$ to $\omega(M_C)$ is accomplished by removing certain open subsets of $\omega(A) \cap \omega(B)$ from the former.

Theorem 6. *For a given pair (A, B) of operators there is equality, for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$,*

$$\omega(A) \cup \omega(B) = \omega(M_C) \cup \mathfrak{S},$$

where $\mathfrak{S}$ is the union of certain of the holes in $\omega(M_C)$ which happen to be subsets of $\omega(A) \cap \omega(B)$.

Proof. We first claim that, for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$,

$$\big(\omega(A) \cup \omega(B)\big) \setminus \big(\omega(A) \cap \omega(B)\big) \subseteq \omega(M_C) \subseteq \omega(A) \cup \omega(B). \tag{6.1}$$

Indeed the second inclusion in (6.1) follows from Lemma 3. For the first inclusion suppose that $\lambda \notin \omega(M_C)$. Then by Lemma 4, $\left(\begin{smallmatrix} A-\lambda I & 0 \\ 0 & B-\lambda I \end{smallmatrix}\right)$ is relatively Weyl, so that by the remark after Lemma 4, $A - \lambda I$ is Weyl if and only if $B - \lambda I$ is Weyl.

Therefore if $\lambda \in \big(\omega(A) \cup \omega(B)\big) \setminus \omega(M_C)$, then $\lambda \in \omega(A) \cap \omega(B)$, which proves (6.1). We next claim that, for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$,

$$\eta\,\big(\omega(M_C)\big) = \eta\,\big(\omega(A) \cup \omega(B)\big), \tag{6.2}$$

where $\eta\mathfrak{C}$ denotes the "polynomially convex hull" of the compact set $\mathfrak{C} \subseteq \mathbb{C}$. Since by (6.1), $\omega(M_C) \subseteq \omega(A) \cup \omega(B)$ for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, we need to show that $\partial\big(\omega(A) \cup \omega(B)\big) \subseteq \partial\,\omega(M_C)$, where $\partial\,\mathfrak{C}$ denotes the topological boundary of the compact set $\mathfrak{C} \subseteq \mathbb{C}$. But since $\operatorname{int}\omega(M_C) \subseteq \operatorname{int}\big(\omega(A) \cup \omega(B)\big)$, it suffices to show that $\partial\,\big(\omega(A) \cup \omega(B)\big) \subseteq \omega(M_C)$. Indeed there are inclusions

$$\partial\big(\omega(A) \cup \omega(B)\big) \subseteq \partial\omega(A) \cup \partial\omega(B) \subseteq \sigma_e^+(A) \cup \sigma_e^-(B) \subseteq \omega(M_C),$$

where the last inclusion follows from Lemma 4 and the second inclusion follows from the punctured neighborhood theorem ([8, Theorem 9.8.9], [9]): for every operator T,

$$\partial\omega(T) \subseteq \partial\sigma_e(T) \subseteq \sigma_e^+(T) \cap \sigma_e^-(T).$$

This proves (6.2). Consequently, (6.2) says that the passage from $\omega(M_C)$ to $\omega(A) \cup \omega(B)$ is the filling in certain of the holes in $\omega(M_C)$. But since, by (6.1), $\big(\omega(A) \cup \omega(B)\big) \setminus \omega(M_C)$ is contained in $\omega(A) \cap \omega(B)$, it follows that the filling in certain of the holes in $\omega(M_C)$ should occur in $\omega(A) \cap \omega(B)$. This completes the proof. ☐

Corollary 7. *If $\omega(A) \cap \omega(B)$ has no interior points, then, for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$,*

$$\omega(M_C) = \omega(A) \cup \omega(B). \tag{7.1}$$

In particular if either $A \in \mathcal{L}(\mathcal{H})$ or $B \in \mathcal{L}(\mathcal{K})$ is a compact operator (more generally, a "Riesz operator"), then (7.1) *holds.*

Proof. The first assertion follows at once from Theorem 6. The second assertion follows from the fact that the Weyl spectrum of a Riesz operator is contained in $\{0\}$. ☐

Let $r(\cdot)$ and $r_\omega(\cdot)$ denote the spectral radius and the "Weyl spectral radius", respectively. Du and Jin [5, Proposition 4] have shown that for a given pair (A, B) of operators, $r(M_C)$ is a constant. We also have an analogue for r_ω:

Corollary 8. *For a given pair (A, B) of operators, $r_\omega(M_C)$ is a constant. Furthermore if $\pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \emptyset$, then, for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$,*

$$r\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r_\omega\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r_\omega\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}. \tag{8.1}$$

Proof. The first assertion follows at once from Theorem 6. For the second assertion we claim that

$$\eta\,\sigma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \setminus \eta\,\omega\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \subseteq \pi_{00}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{8.2}$$

Indeed if $\lambda \in \eta\,\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \setminus \eta\,\omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, then there exists $\epsilon > 0$ such that $\{\mu : |\lambda - \mu| < \epsilon\} \cap \eta\,\omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \emptyset$, which forces that $\lambda \in \operatorname{iso}\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ because if it were not so, then λ would be in $\eta\,\omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, a contradiction. This proves (8.2) and hence the second equality in (8.1). ☐

Remark 9. If $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is normaloid (i.e., norm equals spectral radius) and if $\pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \emptyset$, then

$$\left\| \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right\| \le r \begin{pmatrix} A & C \\ D & B \end{pmatrix} \quad \text{for every compact operator } D \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : \tag{9.1}$$

for we can also argue, by (8.1),

$$\left\| \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right\| = r \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = r_\omega \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = r_\omega \begin{pmatrix} A & C \\ D & B \end{pmatrix} \le r \begin{pmatrix} A & C \\ D & B \end{pmatrix}.$$

Note that (9.1) may, in general, fail for even finite dimensional matrices. For example,

$$\left\| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\| = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r \begin{pmatrix} 1 & 1 \\ \frac{1}{4} & 1 \end{pmatrix} = \frac{3}{2}.$$

H. Weyl [15] has shown that every hermitian operator $A \in \mathcal{L}(\mathcal{H})$ satisfies the equality

$$\sigma(A) \setminus \omega(A) = \pi_{00}(A). \tag{9.2}$$

Today we say that *Weyl's theorem holds for* $A \in \mathcal{L}(\mathcal{H})$ if A satisfies the equality (9.2). Weyl's theorem has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L. Coburn [4], to several classes of operators including seminormal operators by S. Berberian [1], [2], and to a few classes of Banach space operators [12]. But Weyl's theorem may or may not hold for a direct sum of operators for which Weyl's theorem holds. For example, if U is the unilateral shift on ℓ_2, then Weyl's theorem holds for both U and U^*, while it does not hold for $U \oplus U^*$. In this case note that $\omega(U) \cup \omega(U^*) = \mathbb{D}$ (the closed unit disk) and $\omega(U \oplus U^*) = \mathbb{T}$ (the unit circle). Recall ([2]) that an operator $A \in \mathcal{L}(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(A)$ is an eigenvalue of A. For example, every hyponormal operator is isoloid ([14, Theorem 2]). We then have:

Lemma 10. *Suppose Weyl's theorem holds for* $A \in \mathcal{L}(\mathcal{H})$ *and* $B \in \mathcal{L}(\mathcal{K})$.

(a) *If Weyl's theorem holds for* $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, *then*

$$\omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \omega(A) \cup \omega(B). \tag{10.1}$$

(b) *If A and B are isoloid, then the converse of* (a) *is true.*

Proof. The proof of the statement (a) is known from [11, Theorem 4]. For the statement (b) observe that if A and B are isoloid, then

$$\pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\pi_{00}(A) \cap \rho(B)) \cup (\rho(A) \cap \pi_{00}(B)) \cup (\pi_{00}(A) \cap \pi_{00}(B)), \tag{10.2}$$

where $\rho(\cdot)$ denotes the resolvent set. If Weyl's theorem holds for A and B, then the right-hand side of (10.2) must be just the set $\big(\sigma(A) \cup \sigma(B)\big) \setminus \big(\omega(A) \cup \omega(B)\big)$. Thus if (10.1) holds, then $\pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \setminus \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, which says that Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. $\square$

The assumption "A and B are isoloid" is essential in the statement (b) of Lemma 10. For example if $A, B : \ell_2 \to \ell_2$ are defined by

$$A(x_1, x_2, \cdots) = (0, x_2, x_3, x_4, \cdots) \quad \text{and} \quad B(x_1, x_2, \cdots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots),$$

then we have that (i) Weyl's theorem holds for A and B; (ii) $\omega(A) = \{1\}$ and $\omega(B) = \{0\}$; (iii) $\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \{0, 1\}$; (iv) $\pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \{0\}$; (v) B is not isoloid.

Corollary 11. *Suppose $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are isoloid. If Weyl's theorem holds for A and B, and if $\omega(A) \cap \omega(B)$ has no interior points, then Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.*

Proof. This follows from Lemma 10 together with applying Corollary 7 with $C = 0$. ☐

It is familiar ([6, p. 17]) that for given operators $A \in \mathcal{L}(\mathcal{H}), B \in \mathcal{L}(\mathcal{K})$ and $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, if the operator equation

$$AZ - ZB = C \qquad \text{(where } Z \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \text{ is the unknown)} \tag{11.1}$$

is solvable, then $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. In fact, $\left(\begin{smallmatrix} 1 & Z \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -Z \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. Also it is known (cf. [6, Theorem I.4.1]) that if $\sigma(A) \cap \sigma(B) = \emptyset$, then the operator equation (11.1) is solvable. Thus if $\sigma(A) \cap \sigma(B) = \emptyset$, then for most of the familiar kinds of spectrum ϖ there is equality

$$\varpi \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \varpi \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{11.2}$$

Note that (11.2) for $\varpi = \omega$ is a special case of Corollary 7. However, evidently the condition "$\sigma(A) \cap \sigma(B)$ has no interior points" does not imply the solvability of the operator equation (11.1). For example, take $\mathcal{H} = \mathcal{K}, A = B = 0$ and $C = I$.

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References

1. S.K. Berberian, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279. MR **40:**3335
2. S.K. Berberian, *The Weyl spectrum of an operator*, Indiana Univ. Math. J. **20** (1970), 529–544. MR **43:**5344
3. B. Chevreau, *On the spectral picture of an operator*, J. Operator Theory **4** (1980), 119–132. MR **81k:**47002
4. L.A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288. MR **34:**1846
5. Hong-Ke Du and Pan Jin, *Perturbation of spectrums of* 2×2 *operator matrices*, Proc. Amer. Math. Soc. **121** (1994), 761–766. MR **94i:**47004
6. I. Gohberg, S. Goldberg and M.A. Kaashoek, *Classes of Linear Operators (vol I)*, Birkhäuser, Basel, 1990. MR **93d:**47002
7. R.E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Royal Irish Acad. **85A (2)** (1985), 151–176. MR **87h:**47029
8. R.E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Dekker, New York, 1988. MR **89d:**47001
9. R.E. Harte and W.Y. Lee, *The punctured neighbourhood theorem for incomplete spaces*, J. Operator Theory **30** (1993), 217–226. MR **95h:**47022
10. R.E. Harte and W.Y. Lee, *An index formula for chains*, Studia Math. **116(3)** (1995), 283–294. MR **96i:**47021
11. R.E. Harte and W.Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), 2115–2124. MR **98j:**47024
12. K.K. Oberai, *On the Weyl spectrum*, Illinois J. Math. **18** (1974), 208–212. MR **48:**12086
13. C.M. Pearcy, *Some Recent Developments in Operator Theory*, CBMS 36, AMS, Providence, 1978. MR **58:**7120

14. J. Stampfli, *Hyponormal operators*, Pacific J. Math. **12** (1962), 1453–1458. MR **26:**6772
15. H. Weyl, *Über beschränkte quadratische Formen, deren Differenz vollsteig ist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.
16. K.-W. Yang, *Index of Fredholm operators*, Proc. Amer. Math. Soc. **41** (1973), 329–330. MR **47:**7492
17. K.-W. Yang, *The generalized Fredholm operators*, Trans. Amer. Math. Soc. **216** (1976), 313–326. MR **54:**11095

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