

## MIRROR SYMMETRY AND $\mathbb{C}^\times$

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ABSTRACT. We show that counting functions of covers of  $\mathbb{C}^\times$  are equal to sums of integrals associated to certain ‘Feynman’ graphs. This is an analogue of the mirror symmetry for elliptic curves.

### 1. INTRODUCTION AND MAIN RESULT

According to Mirror Symmetry, it is believed that various counting functions on Calabi-Yau manifolds are related to period integrals on their mirror families. One case where the relation is well understood is that of 1-dimensional compact Calabi-Yau manifolds, that is, elliptic curves. In [D], it is shown that the counting function of simply ramified finite coverings of an elliptic curve is equal to the partition functions in fermionic and bosonic field theories on the mirror family.

We can also consider non-compact but algebraic analogues of Calabi-Yau manifolds. The simplest one is  $\mathbb{C}^\times$ : we regard it as an ‘open Calabi-Yau manifold’ since it has a nowhere vanishing holomorphic 1-form  $dz/z$  that has only logarithmic poles at 0 and  $\infty$ . We consider the following enumerative problem and show that the resulting functions are equal to bosonic partition functions on  $\mathbb{C}^\times$ .

**Definition 1.1.** Let  $b$  be a nonnegative integer and  $k, l, d_1, \dots, d_k, e_1, \dots, e_l$  positive integers. We consider  $\mathbb{C}^\times$  as  $\mathbb{P}^1 \setminus \{0, \infty\}$  and take distinct points  $P_1, \dots, P_b \in \mathbb{C}^\times$  and sufficiently small circles  $\gamma$  and  $\gamma'$  around 0 and  $\infty$  respectively. Let  $\delta$  (resp.  $\delta'$ ) be the disjoint union of  $d_i$ -ple (resp.  $e_i$ -ple) covers of  $\gamma$  (resp.  $\gamma'$ ), and let  $p: \delta \rightarrow \mathbb{C}^\times$  (resp.  $p': \delta' \rightarrow \mathbb{C}^\times$ ) be the natural map.

Then we define  $n_{b; d_1, \dots, d_k; e_1, \dots, e_l}$  to be the number of isomorphism classes of quadruples  $(C, \pi, i, i')$  where:

- $C$  is a possibly non-connected smooth curve,
- $\pi: C \rightarrow \mathbb{C}^\times$  is a finite covering that is simply ramified over  $P_1, \dots, P_b$  (i.e., exactly one branch maps with degree 2) and unramified elsewhere, and
- $i: \delta \rightarrow C$  and  $i': \delta' \rightarrow C$  satisfy  $\pi \circ i = p$  and  $\pi \circ i' = p'$ .

Here, an isomorphism of  $(C_1, \pi_1, i_1, i'_1)$  and  $(C_2, \pi_2, i_2, i'_2)$  is an isomorphism  $f: C_1 \rightarrow C_2$  such that  $\pi_2 \circ f = \pi_1$ ,  $f \circ i_1 = i_2$  and  $f \circ i'_1 = i'_2$ .

**Definition 1.2.** Let  $F_{b, k, l}(z_1, \dots, z_k; w_1, \dots, w_l)$  be the generating function

$$\sum n_{b; d_1, \dots, d_k; e_1, \dots, e_l} z_1^{d_1} \cdots z_k^{d_k} w_1^{-e_1} \cdots w_l^{-e_l}.$$

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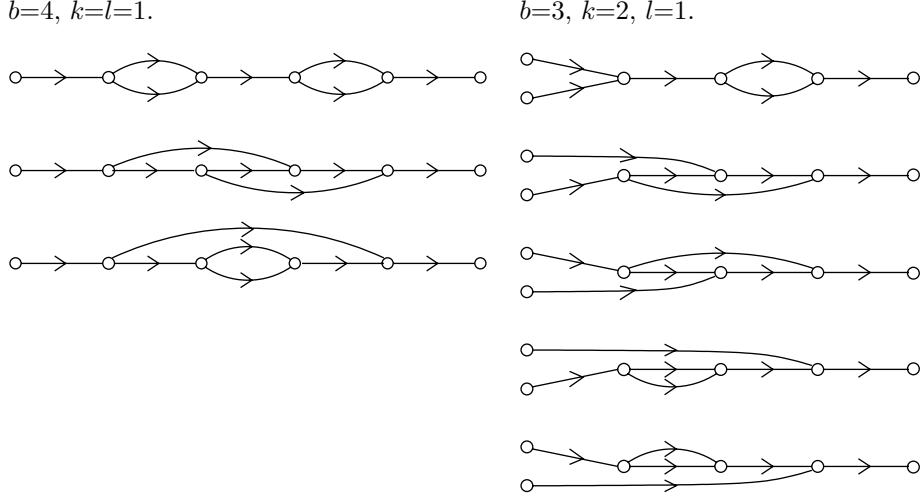


FIGURE 1.

*Remark 1.3.* (1) The number  $n_{b;d_1, \dots, d_k; e_1, \dots, e_l}$  is nonzero only if  $\sum d_i = \sum e_i$ . Thus,  $F_{b,k,l}$  is homogeneous of degree 0.

(2) If  $k + l + b$  is odd, we have  $F_{b,k,l} = 0$ .

For bosonic interpretation, we consider the following ‘Feynman’ graphs. Figure 1 shows examples.

**Definition 1.4.** Let  $b$  be a nonnegative integer and  $k, l$  positive integers. We consider sets  $V_1 = \{z_1, \dots, z_k\}$ ,  $V_2 = \{x_1, \dots, x_b\}$ ,  $V_3 = \{w_1, \dots, w_l\}$  and  $V = V_1 \amalg V_2 \amalg V_3$ .

We define a partial ordering on  $V$  as follows: for  $v, v' \in V$ ,  $v < v'$  if and only if  $v \in V_i, v' \in V_{i'}$  with  $i < i'$  or  $v = x_j, v' = x_{j'}$  and  $j < j'$ .

Let  $E$  be a set and  $v_{in}, v_{fin}$  maps  $E \rightarrow V$ . Then  $V$  and  $E$  defines an oriented graph. Now we define  $G_{b,k,l}$  as the set of graphs satisfying:

- for any  $e \in E$ ,  $v_{in}(e) < v_{fin}(e)$ ,
- $\#\{e \in E | v_{in}(e) = z_i\} = 1$ ,
- $\{\#\{e \in E | v_{in}(e) = x_i\}, \#\{e \in E | v_{fin}(e) = x_i\}\} = \{1, 2\}$ , and
- $\#\{e \in E | v_{fin}(e) = w_i\} = 1$ .

We define an isomorphism of graphs  $(E, v_{in}, v_{fin})$  and  $(E', v'_{in}, v'_{fin})$  to be a bijection of  $E$  and  $E'$  that commutes with  $v_{in}, v_{fin}$  and  $v'_{in}, v'_{fin}$ .

The following is our ‘Bosons’ formula.

**Theorem 1.5.** *In the notations of Definition 1.4, we regard elements of  $V$  as variables. Then, if  $\max\{|z_i|\} < r_1 < \dots < r_b < \min\{|w_i|\}$ , we have*

$$F_{b,k,l}(z_1, \dots, z_k; w_1, \dots, w_l) = \sum_{[\Gamma] \in G_{b,k,l}} \frac{I_\Gamma}{\#\text{Aut}(\Gamma)},$$

where, for  $\Gamma = (E, v_{in}, v_{fin})$ ,  $I_\Gamma$  is defined as

$$\prod_{i=1}^b \left( \oint_{|x_i|=r_i} \frac{dx_i}{2\pi\sqrt{-1}x_i} \right) \prod_{e \in E} \frac{v_{in}(e)v_{fin}(e)}{(v_{in}(e) - v_{fin}(e))^2}.$$

*Remark 1.6.* Consider a graph as in Definition 1.4 with the condition

$$\{\#\{e \in E | v_{in}(e) = x_i\}, \#\{e \in E | v_{fin}(e) = x_i\}\} = \{1, 2\}$$

replaced by

$$\{\#\{e \in E | v_{in}(e) = x_i\}, \#\{e \in E | v_{fin}(e) = x_i\}\} = \{0, 3\}.$$

For such a graph, the integral in Theorem 1.5 is 0 since all the poles of the integrand with respect to  $x_i$  are either all contained in  $|x_i| < r_i$  or in  $|x_i| > r_i$ .

Thus,  $F_{b,k,l}$  is also equal to the sum of the integrals as in Theorem 1.5 over the set of isomorphism classes of graphs  $\Gamma$  with ‘incoming’, ‘intermediate’ and ‘outgoing’ vertices  $z_1, \dots, z_k; x_1, \dots, x_b; w_1, \dots, w_l$  — ordering of intermediate vertices is essential — satisfying the following conditions:

- the graph has no loops, that is, the two end points of any edge are not the same vertex,
- each vertex  $x_i$  has three edges connected,
- each  $z_i$  is connected to exactly one of  $x_1, \dots, x_b, w_1, \dots, w_l$  and each  $w_i$  is connected to exactly one of  $x_1, \dots, x_b, z_1, \dots, z_k$ .

*Remark 1.7.* (1) As for ‘Fermions’ formula in [D], it is straightforward to modify the arguments in §5 of [D] to prove the following formula:

$$\sum_{b=0}^{\infty} F_{b,1,1}(q; 1) \frac{\lambda^b}{b!} = \left( \sum_{p \in \mathbb{Z}_{\geq 0+1/2}} q^p e^{\lambda p^2/2} \right) \left( \sum_{p \in \mathbb{Z}_{\geq 0+1/2}} q^p e^{-\lambda p^2/2} \right).$$

[O] shows that the generating function for all  $n_{b;d_1, \dots, d_k; e_1, \dots, e_l}$ ’s satisfies the Toda lattice hierarchy of Ueno and Takasaki.

(2) The related problem of counting the numbers of covers of  $\mathbb{P}^1$  with arbitrary ramification over  $\infty$ , called Hurwitz numbers, is also being actively studied (see, for example, [SSV], [V], [GJ1], [GJ2]). This can be seen as an analogue of enumeration of curves in Fano manifolds, and as in the case of Fano manifolds, some recursive formulas are known.

(3) In dimension 2, the author counted curves of degree up to 8 in an affine cubic surface which are images of morphisms from an affine line ([T1]), and interpreted those numbers from the viewpoint of mirror symmetry of log surfaces ([T2]).

## 2. PROOF

Now we prove Theorem 1.5. We use notations in Definitions 1.1 and 1.4.

Let  $d_1, \dots, d_k$  and  $e_1, \dots, e_l$  be positive integers such that  $\sum d_i = \sum e_i = d$ . We rewrite  $n_{b;d_1, \dots, d_k; e_1, \dots, e_l}$  as in §4 of [D].

**Definition 2.1.** We denote the group of permutations of  $\{1, \dots, d\}$  by  $S_d$ .

Let  $\tilde{d}_i = \sum_{j=1}^i d_j$ , and let  $\sigma_{d_1, \dots, d_k} \in S_d$  be the permutation

$$(\tilde{d}_0 + 1 \ \dots \ \tilde{d}_1)(\tilde{d}_1 + 1 \ \dots \ \tilde{d}_2) \cdots (\tilde{d}_{k-1} + 1 \ \dots \ \tilde{d}_k).$$

We define  $\tilde{e}_i$  and  $\sigma_{e_1, \dots, e_l}$  in the same way.

For  $\sigma, \tau \in S_d$ , we write  $\sigma^\tau = \tau^{-1} \sigma \tau = \tau \circ \sigma \circ \tau^{-1}$ .

**Lemma 2.2.**  $n_{b;d_1, \dots, d_k; e_1, \dots, e_l}$  is the number of sequences  $(g_1, \dots, g_b, \tau)$ , where  $g_i \in S_d$  are transpositions and  $\tau \in S_d$ , such that  $g_b \cdots g_1 \cdot \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_l})^\tau$ .

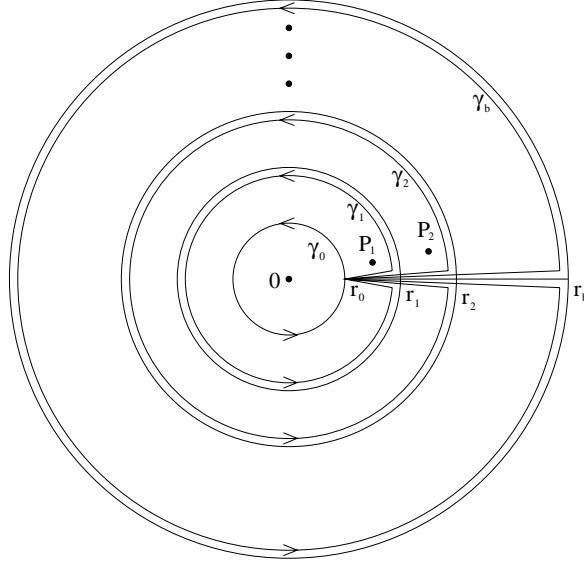


FIGURE 2.

*Proof.* We may take  $\gamma = \{|z| = r_0\}$ ,  $\gamma' = \{|z| = r_b\}$  and  $P_i = x_i e^{\epsilon\sqrt{-1}}$ , where  $0 < r_0 < x_1 < \dots < x_b < r_b$  and  $0 < \epsilon \ll 1$ . Also choose  $r_i$  with  $x_i < r_i < x_{i+1}$  for  $i = 1, \dots, b-1$ . Then, let  $\gamma_i$  be the cycle that starts at  $r_0$ , goes to  $r_i$ , then runs on  $|z| = r_i$  counterclockwise and goes back to  $r_0$  (see Figure 2).

We choose a numbering of  $p^{-1}(r_0)$ , where  $p : \delta \rightarrow \mathbb{C}^\times$  is as in Definition 1.1, such that the monodromy of  $p$  along  $\gamma_0$  is  $\sigma_{d_1, \dots, d_k}$ . Similarly, for  $p'^{-1}(r_b)$ , we give a numbering so that the monodromy along the circle  $|z| = r_b$ , counterclockwise, is  $\sigma_{e_1, \dots, e_l}$ .

Let  $(C, \pi, i, i')$  be given. Then, we can talk of monodromy  $\tilde{g}_i \in S_d$  of  $\pi$  along  $\gamma_i$ . Then  $g_i := \tilde{g}_i \tilde{g}_{i-1}^{-1}$  is a transposition, since it is the monodromy of the simple ramification over  $P_i$ . Also, let  $\tau \in S_d$  be the permutation defined by the path from  $r_b$  to  $r_0$  and numbering of  $p'^{-1}(r_b)$  and  $p^{-1}(r_0)$ . Then, we have  $g_b \cdots g_1 \sigma_{d_1, \dots, d_k} = \tilde{g}_b = (\sigma_{e_1, \dots, e_l})^\tau$ .

Conversely, such  $(g_1, \dots, g_b, \tau)$  defines  $(C, \pi, i, i')$ .  $\square$

We associate a graph to each  $(g_1, \dots, g_b, \tau)$  as in the above lemma.

**Definition 2.3.** Let  $(g_1, \dots, g_b, \tau)$  be as in the above lemma. For  $i = 0, \dots, b$ , let  $E_i$  be the set of cyclic components of  $g_i \cdots g_1 \sigma_{d_1, \dots, d_k}$ .

Let  $\sim$  be the equivalence relation on  $\coprod E_i$  generated by the following relation: for  $\sigma \in E_i$  and  $\sigma' \in E_{i+1}$ ,  $\sigma \sim \sigma'$  if  $\sigma = \sigma'$  as elements of  $S_d$ .

Then let  $E = \coprod E_i / \sim$ . For an element  $e = \{\sigma_{i_0}, \sigma_{i_0+1}, \dots, \sigma_{i_1}\}$  of  $E$ , where  $\sigma_i \in E_i$ , we define  $v_{in}(e)$  to be  $x_{i_0}$  if  $i_0 > 0$  and  $v_{fin}(e)$  to be  $x_{i_1+1}$  if  $i_1 < b$ . If  $i_0 = 0$ , we take  $v$  to satisfy  $\sigma_0 = (\tilde{d}_{v-1} + 1 \ \dots \ \tilde{d}_v)$  and let  $v_{in}(e) = z_v$ , and similarly if  $i_1 = b$ , we take  $v$  with  $\sigma_b = (\tilde{e}_{v-1} + 1 \ \dots \ \tilde{e}_v)^\tau$  and let  $v_{fin}(e) = w_v$ .

We say  $(g_1, \dots, g_b, \tau)$  belongs to (the isomorphism class of)  $(E, v_{in}, v_{fin})$ .

*Remark 2.4.* In the situation of the above lemma and definition,  $E_i$  is naturally in one-to-one correspondence with the set of connected components of  $\pi^{-1}(\{|z| = r_i\})$ . The graph defined describes how these cycles meet and break.

**Lemma 2.5.** *The isomorphism class of  $(E, v_{in}, v_{fin})$  defined in Definition 2.3 belongs to  $G_{b,k,l}$ .*

*Proof.* We use notations in Definition 2.3.

If  $\sigma_i \in E_i$  and  $e$  is the equivalence class of  $\sigma_i$ , then  $v_{in}(e) = x_i$  if and only if  $g_i$  and  $\sigma_i$  intersect and  $v_{fin}(e) = x_{i+1}$  if and only if  $\sigma_i$  and  $g_{i+1}$  intersect.

There are either one or two cyclic components of  $g_{i-1} \cdots g_1 \sigma_{d_1, \dots, d_k}$  that intersect  $g_i$ , since  $g_i$  is a transposition. If there is only one, say  $\sigma_{i-1}$ , then two components of  $g_i \cdots g_1 \sigma_{d_1, \dots, d_k}$  intersect  $g_i$ , i.e. the two components of  $g_i \sigma_{i-1}$ . In this case, we have  $\#\{e \in E | v_{in}(e) = x_i\} = 2$  and  $\#\{e \in E | v_{fin}(e) = x_i\} = 1$ . If there are two,  $\sigma_{i-1}$  and  $\sigma'_{i-1}$ , then  $g_i \sigma_{i-1} \sigma'_{i-1}$  is the sole component of  $g_i \cdots g_1 \sigma_{d_1, \dots, d_k}$  that intersects  $g_i$ , and we have  $\#\{e \in E | v_{in}(e) = x_i\} = 1$ ,  $\#\{e \in E | v_{fin}(e) = x_i\} = 2$ .

The other properties are easy to see.  $\square$

**Definition 2.6.** For a graph  $\Gamma = (E, v_{in}, v_{fin})$ , let  $n_{\Gamma; d_1, \dots, d_k; e_1, \dots, e_l}$  be the number of sequences  $(g_1, \dots, g_b, \tau)$ , where  $g_i$  are transpositions and  $\tau \in S_d$ , such that  $g_b \cdots g_1 \cdot \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_l})^\tau$  and that the associated graph is isomorphic to  $\Gamma$ .

Also, let

$$F_\Gamma(z_1, \dots, z_k; w_1, \dots, w_l) = \sum n_{\Gamma; d_1, \dots, d_k; e_1, \dots, e_l} z_1^{d_1} \cdots z_k^{d_k} \cdot w_1^{-e_1} \cdots w_l^{-e_l}.$$

From the two lemmas above, it suffices to prove the following to prove Theorem 1.5.

**Proposition 2.7.** *Let  $\Gamma$  be a graph whose isomorphism class belongs to  $G_{b,k,l}$ . Then, if  $\max\{|z_i|\} < \min\{|w_i|\}$ , we have*

$$F_\Gamma(z_1, \dots, z_k; w_1, \dots, w_l) = \frac{I_\Gamma}{\#\text{Aut}(\Gamma)},$$

where, as in Theorem 1.5,

$$I_\Gamma := \prod_{i=1}^b \left( \oint_{|x_i|=r_i} \frac{dx_i}{2\pi\sqrt{-1}x_i} \right) \prod_{e \in E} \frac{v_{in}(e)v_{fin}(e)}{(v_{in}(e) - v_{fin}(e))^2},$$

with  $\max\{|z_i|\} < r_1 < \cdots < r_b < \min\{|w_i|\}$ .

We prove the proposition by induction on  $b$ .

*Claim 2.8.* Proposition 2.7 holds for  $b = 0$ .

*Proof.* In this case, we have  $k = l$  and  $\Gamma$  connects  $z_i$  to  $w_{\sigma(i)}$  for some  $\sigma \in S_k$ . By symmetry, we may assume  $\sigma = id$ . Then,  $\tau \in S_d$  satisfies  $\sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_k})^\tau$  and the associated graph is isomorphic to  $\Gamma$  if and only if  $d_i = e_i$  and

$$(\tilde{d}_{i-1} + 1 \quad \dots \quad \tilde{d}_i) = (\tilde{e}_{i-1} + 1 \quad \dots \quad \tilde{e}_i)^\tau$$

for  $i = 1, \dots, k$ . When  $d_i = e_i$ , the number of such  $\tau$  is  $\prod d_i$ . Now

$$\sum_{d_1, \dots, d_k=1}^{\infty} \prod d_i \prod (z_i^{d_i} w_i^{-d_i}) = \prod \frac{z_i w_i}{(z_i - w_i)^2}$$

proves the claim.  $\square$

*Claim 2.9.* If Proposition 2.7 holds for  $b = n$ , it holds for  $b = n + 1$ .

*Proof.* Let  $\Gamma = (E, v_{in}, v_{fin})$  be an element of  $G_{n+1,k,l}$ .

Case (1):  $\#\{e \in E \mid v_{in}(e) = x_{n+1}\} = 1$ .

By symmetry, we may assume  $v_{fin}(e) = w_l$ . There are two edges  $e', e''$  such that  $v_{fin}(e') = v_{fin}(e'') = x_{n+1}$ . Let  $\Gamma' \in G_{n,k,l+1}$  be the graph obtained by removing  $x_{n+1}$  and connecting  $e'$  to  $w_l$  and  $e''$  to  $w_{l+1}$ .

Write  $\tilde{e}_i = \sum_{j=1}^i e_j$ . If  $(g_1, \dots, g_{n+1}, \tau)$  belongs to  $\Gamma$  and  $g_{n+1} \cdots g_1 \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_l})^\tau$ , then there exist  $\tau' \in S_d$  with  $\tau'(i) = \tau(i)$  for  $0 \leq i \leq \tilde{e}_{l-1}$  and positive integers  $e'_l, e'_{l+1}$  with  $e'_l + e'_{l+1} = e_l$  such that  $g_n \cdots g_1 \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_{l-1}, e'_l, e'_{l+1}})^{\tau'}$  and that  $(g_1, \dots, g_n, \tau')$  belongs to  $\Gamma'$ , as is easily seen from the definition.

Let  $e_1, \dots, e_{l-1}, e'_l, e'_{l+1}$  be positive integers with  $\sum_{i=1}^{l-1} e_i + e'_l + e'_{l+1} = d$ . Then, let  $X_{e'_l, e'_{l+1}}$  be the set of  $(g_1, \dots, g_n, t_1, \dots, t_{\tilde{e}_{l-1}})$  such that there exists  $\tau' \in S_d$  with the property that  $\tau'(i) = t_i$  ( $0 \leq i \leq \tilde{e}_{l-1}$ ),  $g_n \cdots g_1 \cdot \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_{l-1}, e'_l, e'_{l+1}})^{\tau'}$  and that the graph associated to  $(g_1, \dots, g_n, \tau')$  is isomorphic to  $\Gamma'$ .

If  $v_{in}(e') \neq v_{in}(e'')$  or  $e'_l \neq e'_{l+1}$ ,  $\tau'$  maps  $(\tilde{e}'_{l-1} + 1 \dots \tilde{e}'_l)$  to the cyclic component of  $g_n \cdots g_1 \cdot \sigma_{d_1, \dots, d_k}$  connected to  $v_{in}(e')$ , and  $(\tilde{e}'_l + 1 \dots \tilde{e}'_{l+1})$  to the one connected to  $v_{in}(e'')$ . If  $e'_l \neq e'_{l+1}$ ,  $\tau'$  maps  $(\tilde{e}'_{l-1} + 1 \dots \tilde{e}'_l)$  and  $(\tilde{e}'_l + 1 \dots \tilde{e}'_{l+1})$  to cyclic components of lengths  $e'_l$  and  $e'_{l+1}$ , respectively. Thus there are  $e'_l \cdot e'_{l+1}$  choices for  $\tau'$  in these cases and we have  $n_{\Gamma'; d_1, \dots, d_k; e_1, \dots, e_{l-1}, e'_l, e'_{l+1}} = \#X_{e'_l, e'_{l+1}} \cdot e'_l \cdot e'_{l+1}$ . If  $v_{in}(e') = v_{in}(e'')$  and  $e'_l = e'_{l+1}$  hold, then it is  $2\#X_{e'_l, e'_{l+1}} \cdot e_l^2$ .

On the other hand, for each  $(g_1, \dots, g_n, t_1, \dots, t_{\tilde{e}_{l-1}}) \in X_{e'_l, e'_{l+1}}$ , there are  $e'_l e'_{l+1}$  transpositions  $g_{n+1}$  for which there exists  $\tau$  with  $\tau(i) = t_i$  for  $0 \leq i \leq \tilde{e}_{l-1}$  such that  $(g_1, \dots, g_{n+1}, \tau)$  belongs to  $\Gamma$ . The number of such  $\tau$  is  $e_l := e'_l + e'_{l+1}$ , and we have

$$n_{\Gamma; d_1, \dots, d_k; e_1, \dots, e_l} = \sum \#X_{e'_l, e'_{l+1}} \cdot e_l \cdot e'_l \cdot e'_{l+1}.$$

Here the sum is taken over *different*  $X_{e'_l, e'_{l+1}}$ 's with  $e'_l + e'_{l+1} = e_l$ .

Noting that  $X_{e'_l, e'_{l+1}}$  and  $X_{e'_{l+1}, e'_l}$  are the same for  $e'_l \neq e'_{l+1}$  if and only if  $v_{in}(e') = v_{in}(e'')$ , we see

$$n_{\Gamma; d_1, \dots, d_k; e_1, \dots, e_l} = \sum_{e'_l + e'_{l+1} = e_l} e_l n_{\Gamma'; d_1, \dots, d_k; e_1, \dots, e_{l-1}, e'_l, e'_{l+1}}$$

if  $v_{in}(e') \neq v_{in}(e'')$  and

$$n_{\Gamma; d_1, \dots, d_k; e_1, \dots, e_l} = \frac{1}{2} \sum_{e'_l + e'_{l+1} = e_l} e_l n_{\Gamma'; d_1, \dots, d_k; e_1, \dots, e_{l-1}, e'_l, e'_{l+1}}$$

if  $v_{in}(e') = v_{in}(e'')$ .

Finally, since

$$I_\Gamma = \oint \frac{dx}{2\pi\sqrt{-1}x} \frac{xw_l}{(x-w_l)^2} I_{\Gamma'}(z_1, \dots, z_k; w_1, \dots, w_{l-1}, x, x)$$

holds,

$$\oint \frac{dx}{2\pi\sqrt{-1}x} \frac{xw_l}{(x-w_l)^2} x^{-e'_l} x^{-e'_{l+1}} = e_l w_l^{-e_l}$$

proves the assertion.

Case (2):  $\#\{e \in E \mid v_{in}(e) = x_{n+1}\} = 2$ .

By symmetry, we may assume that  $w_{l-1}$  and  $w_l$  appear as  $v_{fin}(e)$ . Let  $e'$  be the edge such that  $v_{fin}(e') = x_{n+1}$ . Let  $\Gamma' \in G_{n,k,l-1}$  be the graph obtained by removing  $x_{n+1}$  and connecting  $e'$  to  $w_{l-1}$ .

As in Case (1), if  $(g_1, \dots, g_{n+1}, \tau)$  belongs to  $\Gamma$  and  $g_{n+1} \cdots g_1 \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_l})^\tau$ , then there exists  $\tau' \in S_d$  such that  $g_n \cdots g_1 \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_{l-2}, e'_l})^{\tau'}$ , where  $e'_l = e_{l-1} + e_l$ , and that  $(g_1, \dots, g_n, \tau')$  belongs to  $\Gamma'$ .

For positive integers  $e_1, \dots, e_{l-2}, e'_{l-1}$  with  $\sum_{i=1}^{l-2} e_i + e'_{l-1} = d$ , let  $X$  be the set of  $(g_1, \dots, g_n, t_1, \dots, t_{\tilde{e}_{l-2}})$  such that there exists  $\tau' \in S_d$  with the property that  $\tau'(i) = t_i (0 \leq i \leq \tilde{e}_{l-2})$ ,  $g_n \cdots g_1 \sigma_{d_1, \dots, d_k} = (\sigma_{e_1, \dots, e_{l-2}, e'_{l-1}})^{\tau'}$  and that the graph associated to  $(g_1, \dots, g_n, \tau')$  is isomorphic to  $\Gamma'$ .

Then there are  $e'_{l-1}$  such  $\tau'$ , and  $n_{\Gamma'; d_1, \dots, d_k; e_1, \dots, e_{l-2}, e'_{l-1}}$  is  $\#X \cdot e'_{l-1}$ .

On the other hand, for  $e_{l-1}$  and  $e_l$  with  $e_{l-1} + e_l = e'_{l-1}$ , the number of pairs  $(g_{n+1}, \tau)$  with  $\tau(i) = t_i (0 \leq i \leq \tilde{e}_{l-2})$  such that  $(g_1, \dots, g_{n+1}, \tau)$  belongs to  $\Gamma$  is  $e'_{l-1}$  times  $e_{l-1}e_l$ : let  $(a_1 \ \dots \ a_{e'_{l-1}})$  be the cyclic permutation corresponding to  $e'$ . Choose  $1 \leq r \leq e'_{l-1}$  and let  $s$  be the residue of  $r + e_l$  modulo  $e'_{l-1}$ . We take  $g_{n+1}$  to be  $(a_r \ a_s)$ , and then  $g_{n+1} \cdot (a_1 \ \dots \ a_{e'_{l-1}})$  has cyclic components of lengths  $e_{l-1}$  and  $e_l$ , containing  $a_r$  and  $a_s$  respectively. We can choose  $\tau$  so that the two components equal  $(\tilde{e}_{l-2} + 1 \ \dots \ \tilde{e}_{l-1})^\tau$  and  $(\tilde{e}_{l-1} + 1 \ \dots \ \tilde{e}_l)^\tau$ , respectively.

Thus we have

$$n_{\Gamma; d_1, \dots, d_k; e_1, \dots, e_l} = \#X \cdot e'_{l-1} \cdot e_{l-1} \cdot e_l = e_{l-1} e_l n_{\Gamma'; d_1, \dots, d_k; e_1, \dots, e'_{l-1}}.$$

Since

$$I_\Gamma = \oint \frac{dx}{2\pi\sqrt{-1}x} \frac{xw_{l-1}}{(x-w_{l-1})^2} \frac{xw_l}{(x-w_l)^2} I_{\Gamma'}(z_1, \dots, z_k; w_1, \dots, w_{l-2}, x)$$

holds in this case,

$$\oint \frac{dx}{2\pi\sqrt{-1}x} \frac{xw_{l-1}}{(x-w_{l-1})^2} \frac{xw_l}{(x-w_l)^2} x^{-e'_{l-1}} = \sum_{e_{l-1}+e_l=e'_{l-1}} e_{l-1}e_l w_{l-1}^{-e_{l-1}} w_l^{-e_l}$$

proves the assertion.  $\square$

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