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SIMPLE COMPLETE BOOLEAN ALGEBRAS

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ABSTRACT. For every regular cardinal κ there exists a simple complete Boolean algebra with κ generators.

1. INTRODUCTION

A complete Boolean algebra is *simple* if it is atomless and has no nontrivial proper atomless complete subalgebra. The problem of the existence of simple complete Boolean algebras was first discussed in 1971 by McAloon in [8]. Previously, in [7], McAloon constructed a *rigid* complete Boolean algebra; it is easily seen that a simple complete Boolean algebra is rigid. In fact, it has no non-trivial one-to-one complete endomorphism [1]. Also, if an atomless complete algebra is not simple, then it contains a non-rigid atomless complete subalgebra [2].

McAloon proved in [8] that an atomless complete algebra B is simple if and only if it is rigid and *minimal*, i.e. the generic extension by B is a minimal extension of the ground model. Since Jensen's construction [5] yields a definable real of minimal degree over L, it shows that a simple complete Boolean algebra exists under the assumption $V = L$. McAloon then asked whether a rigid minimal algebra can be constructed without such an assumption.

In [10], Shelah proved the existence of a rigid complete Boolean algebra of cardinality κ for each regular cardinal κ such that $\kappa^{\aleph_0} = \kappa$. Neither McAloon's nor Shelah's construction gives a minimal algebra.

In [9], Sacks introduced perfect set forcing, to produce a real of minimal degree. The corresponding complete Boolean algebra is minimal, and has $\aleph_0$ generators. Sacks' forcing generalizes to regular uncountable cardinals κ (cf. [6]), thus giving a minimal complete Boolean algebra with κ generators. The algebras are not rigid however.

Under the assumption $V = L$, Jech constructed in [3] a simple complete Boolean algebra of cardinality κ, for every regular uncountable cardinal that is not weakly compact (if κ is weakly compact, or if κ is singular and GCH holds, then a simple complete Boolean algebra does not exist).

In [4], we proved the existence of a simple complete Boolean algebra (in ZFC). The algebra is obtained by a modification of Sacks' forcing, and has $\aleph_0$ generators

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(the forcing produces a definable minimal real). The present paper gives a construction of a simple complete Boolean algebra with κ generators, for every regular uncountable cardinal κ.

Main Theorem. *Let κ be a regular uncountable cardinal. There exists a forcing notion P such that the complete Boolean algebra $B = B(P)$ is rigid, P adds a subset of κ without adding any bounded subsets, and for every $X \in V[G]$ (the P-generic extension), either $X \in V$ or $G \in V[X]$. Consequently, B is a simple complete Boolean algebra with κ generators.*

The forcing P is a modification of the generalization of Sacks' forcing described in [6].

2. Forcing with perfect κ-trees

For the duration of the paper let κ denote a regular uncountable cardinal, and set $\mathrm{Seq} = \bigcup_{\alpha<\kappa} {}^{\alpha}2$.

Definition 2.1. (a) If $p \subseteq \mathrm{Seq}$ and $s \in p$, say that s *splits* in p if $s^\frown 0 \in p$ and $s^\frown 1 \in p$.

(b) Say that $p \subseteq \mathrm{Seq}$ is a *perfect tree* if:

(i) If $s \in p$, then $s{\restriction}\alpha \in p$ for every α.
(ii) If $\alpha < \kappa$ is a limit ordinal, $s \in {}^{\alpha}2$, and $s{\restriction}\beta \in p$ for every $\beta < \alpha$, then $s \in p$.
(iii) If $s \in p$, then there is a $t \in p$ with $t \supseteq s$ such that t splits in p.

Our definition of perfect trees follows closely [6], with one exception: unlike [6], Definition 1.1(b)(iv), the splitting nodes of p need not be closed.

We consider a notion of forcing P that consists of (some) perfect trees, with the ordering $p \le q$ iff $p \subseteq q$. Below we formulate several properties of P that guarantee that the proof of minimality for Sacks' forcing generalizes to forcing with P.

Definition 2.2. (a) If p is a perfect tree and $s \in p$, set

$$p_s = \{t \in p : s \subseteq t \text{ or } t \subseteq s\};$$

p_s is a *restriction* of p. A set P of perfect trees is *closed under restrictions* if for every $p \in P$ and every $s \in p$, $p_s \in P$. If $p_s = p$, then s is a *stem* of p.

(b) For each $s \in \mathrm{Seq}$, let $o(s)$ denote the domain (length) of s. If $s \in p$ and $o(s)$ is a successor ordinal, s is a *successor node* of p; if $o(s)$ is a limit ordinal, s is a *limit node* of p. If s is a limit node of p and $\{\alpha < o(s) : s \restriction \alpha \text{ splits in } p\}$ is cofinal in $o(s)$, s is a *limit of splitting nodes*.

(c) Let p be a perfect tree and let A be a nonempty set of mutually incomparable successor nodes of p. If, for each $s \in A$, $q(s)$ is a perfect tree with stem s and $q(s) \le p_s$, let

$$q = \{t \in p : \text{ if } t \supseteq s \text{ for some } s \in A, \quad \text{then } t \in q(s)\}$$

We call the perfect tree q the *amalgamation* of $\{q(s) : s \in A\}$ into p. A set P of perfect trees is *closed under amalgamations* if, for every $p \in P$, every set A of incomparable successor nodes of p and every $\{q(s) : s \in A\} \subset P$ with $q(s) \le p_s$, the amalgamation is in P.

Definition 2.3. (a) A set P of perfect trees is *κ-closed* if for every $\gamma < \kappa$ and every decreasing sequence $\langle p_\alpha : \alpha < \gamma\rangle$ in P, $\bigcap_{\alpha<\gamma} p_\alpha \in P$.

(b) If $\langle p_\alpha : \alpha < \kappa\rangle$ is a decreasing sequence of perfect trees such that

(i) if δ is a limit ordinal, then $p_\delta = \bigcap_{\alpha<\delta} p_\alpha$, and

(ii) for every α, $p_{\alpha+1} \cap {}^{\alpha}2 = p_\alpha \cap {}^{\alpha}2$,

then $\langle p_\alpha : \alpha < \kappa\rangle$ is called a *fusion sequence.* A set P is *closed under fusion* if, for every fusion sequence $\langle p_\alpha : \alpha < \kappa\rangle$ in P, $\bigcap_{\alpha<\kappa} p_\alpha \in P$.

The following theorem is a generalization of Sacks' Theorem from [9] to the uncountable case:

Theorem 2.4. *Let P be a set of perfect trees and assume that P is closed under restrictions and amalgamations, κ-closed, and closed under fusion. If G is P-generic over V, then G is minimal over V; namely if $X \in V[G]$ and $X \notin V$, then $G \in V[X]$. Moreover, $V[G]$ has no new bounded subsets of κ, and G can be coded by a subset of κ.*

Proof. The proof follows as much as in [9]. Given a name $\dot{X}$ for a set of ordinals and a condition $p \in P$ that forces $\dot{X} \notin V$, one finds a condition $q \le p$ and a set of ordinals $\{\gamma_s : s \text{ splits in } q\}$ such that $q_{s^\frown 0}$ and $q_{s^\frown 1}$ both decide $\gamma_s \in \dot{X}$, but in opposite ways. The generic branch can then be recovered from the interpretation of $\dot{X}$.

To construct q and $\{\gamma_s\}$ one builds a fusion sequence $\{p_\alpha : \alpha < \kappa\}$ as follows. Given p_α, let $Z = \{s \in p_\alpha : o(s) = \alpha \text{ and } s \text{ splits in } p_\alpha\}$. For each $s \in Z$, let γ_s be an ordinal such that $(p_\alpha)_s$ does not decide $\gamma_s \in \dot{X}$. Let $q(s^\frown 0) \le (p_\alpha)_{s^\frown 0}$ and $q(s^\frown 1) \le (p_\alpha)_{s^\frown 1}$ be conditions that decide $\gamma_s \in \dot{X}$ in opposite ways. Then let $p_{\alpha+1}$ be the amalgamation of $\{q(s^\frown i) : s \in Z \text{ and } i = 0, 1\}$ into p_α. Finally, let $q = \bigcap_{\alpha<\kappa} p_\alpha$. □

In [6] it is postulated that the splitting nodes along any branch of a perfect tree form a closed unbounded set. This guarantees that the set of all such trees is κ-closed and closed under fusion (Lemmas 1.2 and 1.4 in [6]). It turns out that a less restrictive requirement suffices.

Definition 2.5. Let $S \subset \kappa$ be a stationary set. A perfect tree $p \in P$ is S-*perfect* if whenever s is a limit of splitting nodes of p such that $o(s) \in S$, then s splits in p.

Lemma 2.6. *(a) If $\langle p_\alpha : \alpha < \gamma\rangle$, $\gamma < \kappa$, is a decreasing sequence of S-perfect trees, then $\bigcap_{\alpha<\gamma} p_\alpha$ is a perfect tree.*

(b) If $\langle p_\alpha : \alpha < \kappa\rangle$ is a fusion sequence of S-perfect trees, then $\bigcap_{\alpha<\kappa} p_\alpha$ is a perfect tree.

Proof. (a) Let $p = \bigcap_{\alpha<\gamma} p_\alpha$. The only condition in Definition 2.1 (b) that needs to be verified is (iii): for every $s \in p$ find $t \supseteq s$ that splits in p. First it is straightforward to find a branch $f \in {}^{\kappa}2$ through p such that s is an initial segment of f.

Second, it is equally straightforward to see that, for each $\alpha < \gamma$, the set of all β such that $f \restriction \beta$ splits in p_α is unbounded in κ. Thus for each $\alpha < \gamma$ let C_α be the closed unbounded set of all δ such that $f \restriction \delta$ is a limit of splitting nodes in p_α. Let $\delta \ge o(s)$ be an ordinal in $\bigcap_{\alpha<\gamma} C_\alpha \cap S$. Then for each $\alpha < \gamma$, $t = f \restriction \delta$ is a limit of splitting nodes in p_α, and hence t splits in p_α. Therefore t splits in p.

(b) Let $p = \bigcap_{\alpha<\kappa} p_\alpha$ and again, check (iii). Let $s \in p$, and let $f \in {}^{\kappa}2$ be a branch trough p. For each $\alpha < \gamma$ let C_α be the club of all δ such that $f \restriction \delta$ is a limit of splitting nodes in p_α. Let $\delta \ge o(s)$ be an ordinal in $\Delta_{\alpha<\kappa} C_\alpha \cap S$ and let $t = f \restriction \delta$.

If $\alpha < \delta$, then t splits in p_α, and therefore t splits in p_δ. Since $p_{\delta+1} \cap {}^{\delta}2 = p_\delta \cap {}^{\delta}2$, we have $t \in p_{\delta+1}$, and since $p_{\delta+1}$ is S-perfect, t splits in $p_{\delta+1}$. If $\alpha > \delta + 1$, then $p_\alpha \cap {}^{\delta+1}2 = p_{\delta+1} \cap {}^{\delta+1}2$, and so t splits in p_α. Hence t splits in p. □

This is trivial, but note that the limit condition p (in both (a) and (b)) is not only perfect but S-perfect as well.

3. The notion of forcing for which $B(P)$ is rigid

We now define a set P of perfect κ-trees that is closed under restrictions and amalgamations, κ-closed, and closed under fusion, with the additional property that the complete Boolean algebra $B(P)$ is rigid. That completes a proof of the Main Theorem.

Let S and S_ξ, $\xi < \kappa$, be mutually disjoint stationary subsets of κ, such that for all $\xi < \kappa$, if $\delta \in S_\xi$, then $\delta > \xi$.

Definition 3.1. The forcing notion P is the set of all $p \subseteq \mathrm{Seq}$ such that

(1) p is a perfect tree;
(2) p is S-perfect, i.e. if s is a limit of splitting nodes of p and $o(s) \in S$, then s splits in p;
(3) For every $\xi < \kappa$, if s is a limit of splitting nodes of p with $o(s) \in S_\xi$ and if $s(\xi) = 0$, then s splits in p.

The set P is ordered by $p \leq q$ iff $p \subseteq q$.

Clearly, P is closed under restrictions and amalgamations. By Lemma 2.6, the intersection of either a decreasing short sequence or of a fusion sequence in P is a perfect tree, and since both properties (2) and (3) are preserved under arbitrary intersections, we conclude that P is also κ-closed and closed under fusion.

We conclude the proof by showing that $B(P)$ is rigid.

Lemma 3.2. *If π is a nontrivial automorphism of $B(P)$, then there exist conditions p and q with incomparable stems such that $\pi(p)$ and q are compatible (in $B(P)$).*

Proof. Let π be a nontrivial automorphism. It is easy to find a nonzero element $u \in B$ such that $\pi(u) \cdot u = 0$. Let $p_1 \in P$ be such that $p_1 \leq u$, and let $q_1 \in P$ be such that $q_1 \leq \pi(p_1)$. As p_1 and q_1 are incompatible, there exists some $t \in q_1$ such that $t \notin p_1$. Let $q = (q_1)_t$. Then let $p_2 \in P$ be such that $p_2 \leq \pi^{-1}(q)$, and again, there exists some $s \in p_2$ such that $s \notin q$. Let $p = (p_2)_s$. Now s and t are incomparable stems of p and q, and $\pi(p) \leq q$. □

To prove that $B(P)$ has no nontrivial automorphism, we introduce the following property $\varphi(\xi)$.

Definition 3.3. Let $\xi < \kappa$; we say that ξ has property φ if and only if for every function $f : \kappa \to 2$ there exist a function $F : \mathrm{Seq} \to 2$ in V and a club $C \subset \kappa$ such that, for every $\delta \in C \cap S_\xi$, $f(\delta) = F(f \restriction \delta)$.

Lemma 3.4. *Let $t_0 \in Seq$ and let $\xi = o(t_0)$.*

(a) *Every condition with stem $t_0{}^\frown 0$ forces $\neg\varphi(\xi)$.*
(b) *Every condition with stem $t_0{}^\frown 1$ forces $\varphi(\xi)$.*

Proof. (a) Let $\dot{f}$ be the name for the generic branch $f_G : \kappa \to 2$ (i.e. $f_G = \bigcup\{s \in \text{Seq} : s \in p \text{ for all } p \in G\}$); this will be the counterexample for $\varphi(\xi)$. Let F be a function, $F : \text{Seq} \to 2$, let $\dot{C}$ be a name for a club and let $p \in P$ be such that $t_0^\frown 0$ is a stem of p. We shall find a $\delta \in S_\xi$ and $q \leq p$ such that $q \Vdash (\delta \in \dot{C} \text{ and } \dot{f}(\delta) \neq F(\dot{f} \restriction \delta))$.

We construct a fusion sequence $\langle p_\alpha : \alpha < \kappa \rangle$, starting with p, so that for each α, if $s \in p_{\alpha+1}$ and $o(s) = \alpha+1$, then $(p_{\alpha+1})_s$ decides the value of the αth element of $\dot{C}$; we call this value γ_s. (We obtain $p_{\alpha+1}$ by amalgamation into p_α.) Let $r = \bigcap_{\alpha<\kappa} p_\alpha$.

Let b be a branch through r, and let $s_\alpha = b \restriction \alpha$ for all α. There exists a $\delta \in S$ such that s_δ is a limit of splitting nodes of r, and such that, for every $\alpha < \delta$, $\gamma_{s_{\alpha+1}} < \delta$. Since $s_\delta(\xi) = 0$, s_δ splits in r, and $r_{s_\delta} \Vdash \delta \in \dot{C}$.

Now if $F(s_\delta) = i$, it is clear that $g = r_{s_\delta^\frown(1-i)}$ forces $\dot{f} \restriction \delta = s_\delta$ and $\dot{f}(\delta) = 1 - i$.

(b) Let $\dot{f}$ be a name for a function from κ to 2, and let p be a condition with stem $t_0^\frown 1$ that forces $\dot{f} \notin V$ ($\varphi(\xi)$ holds trivially for those f that are in V). We shall construct a condition $q \leq p$ and collections $\{h_s : s \in Z\}$ and $\{i_s : s \in Z'\}$, where Z is the set of all limits of splitting nodes in q and $Z' = \{s \in Z : o(s) \in S_\xi\}$, such that

(3.5)

(i) For each $s \in Z$, $h_s \in \text{Seq}$ and $o(h_s) = o(s)$; if $o(s) = \alpha$, then $q_s \Vdash \dot{f} \restriction \alpha = h_s$.
(ii) If $s, t \in Z$, $o(s) = o(t) = \alpha$, and $s \neq t$, then $h_s \neq h_t$.
(iii) For each $s \in Z'$, $i_s = 0$ or $i_s = 1$; if $o(s) = \delta$, then $q_s \Vdash \dot{f}(\delta) = i_s$.

Then we define F by setting $F(h_s) = i_s$, for all $s \in Z'$ (and $F(h)$ arbitrary for all other $h \in \text{Seq}$); this is possible because of (ii). We claim that q forces that, for some club C, $\dot{f}(\delta) = F(\dot{f} \restriction \delta)$ for all $\delta \in C \cap S_\xi$. (This will complete the proof.)

To prove the claim, let G be a generic filter with $q \in G$, let g be the generic branch ($g = \bigcup\{s : s \in p \text{ for all } p \in G\}$), and let f be the G-interpretation of $\dot{f}$. Let C be the set of all α such that $g \restriction \alpha$ is the limit of splitting nodes in q. If $\delta \in C \cap S_\xi$, let $s = g \restriction \delta$; then $s \in Z'$, $f \restriction \delta = h_s$ and $f(\delta) = i_s$. It follows that $f(\delta) = F(f \restriction \delta)$.

To construct q, h_s and i_s, we build a fusion sequence $\langle p_\alpha : \alpha < \kappa \rangle$ starting with p_0. We take $p_\alpha = \bigcap_{\beta<\alpha} p_\beta$ when α is a limit ordinal, and construct $p_{\alpha+1} \leq p_\alpha$ such that $p_{\alpha+1} \cap {}^\alpha 2 = p_\alpha \cap {}^\alpha 2$. For each α, we satisfy the following requirements:

(3.6) For all $s \in p_\alpha$, if $o(s) < \alpha$, then:

(i) If s is a limit of splitting nodes in p_α and $o(s) \in S_\xi$, then s does not split in p_α.
(ii) If s does not split in p_α, then $(p_\alpha)_s$ decides the value of $\dot{f}(o(s))$.
(iii) If s splits in p_α, let γ_s be the least γ such that $(p_\alpha)_s$ does not decide $\dot{f}(\gamma)$. Then $(p_\alpha)_{s^\frown 0}$ and $(p_\alpha)_{s^\frown 1}$ decide $\dot{f}(\gamma_s)$ in opposite ways, and both $(p_\alpha)_{s^\frown 0}$ and $(p_\alpha)_{s^\frown 1}$ have stems of length greater than γ_s.

Note that if p_α satisfies (iii) for a given s, then every p_β, $\beta > \alpha$, satisfies (iii) for this s, with the same γ_s. Also (by induction on $o(s)$), we have $\gamma_s \geq o(s)$. Clearly, if α is a limit ordinal and each p_β, $\beta < \alpha$, satisfies (3.6), then p_α also satisfies (3.6). We show below how to obtain $p_{\alpha+1}$ when we have already constructed p_α.

Now let $q = \bigcap_{\alpha<\kappa} p_\alpha$, and let us verify that q satisfies (3.5). So let α be a limit ordinal, and let $Z_\alpha = \{t \in q : t \text{ is a limit of splitting nodes in } q \text{ and } o(t) = \alpha\}$. If

$t \in Z_\alpha$, then t is a limit of splitting nodes of p_α. It follows from (3.6) (ii) and (iii) that $(p_\alpha)_t$ decides $\dot{f} \restriction \alpha$, and we let h_t be this sequence. If $t_1 \neq t_2$ are in Z_α, let $s = t_1 \cap t_2$. By (3.6) (iii) we have $\gamma_s < \alpha$ (because there exist s_1 and s_2 such that $s \subset s_1 \subset t_1$, $s \subset s_2 \subset t_2$ and both s_1 and s_2 split in p_α). It follows that $h_{t_1} \neq h_{t_2}$. If $\alpha \in S_\xi$ and $s \in Z_\alpha$, then by (3.6) (i), s does not split in $p_{\alpha+1}$ and so $(p_{\alpha+1})_s$ decides $\dot{f}(\alpha)$; we let i_s be this value. These h_t and i_s satisfy (3.5) for the condition q.

It remains to show how to obtain $p_{\alpha+1}$ from p_α. Thus assume that p_α satisfies (3.6). First let $r \leq p_\alpha$ be the following condition such that $r \cap {}^\alpha 2 = p_\alpha \cap {}^\alpha 2$: If $\alpha \notin S_\xi$, let $r = p_\alpha$; if $\alpha \in S_\xi$, consider all $s \in p_\alpha$ with $o(s) = \alpha$ that are limits of splitting nodes, and replace each $(p_\alpha)_s$ by a stronger condition $r(s)$ such that s does not split in $r(s)$. For all other $s \in p_\alpha$ with $o(s) = \alpha$, let $r(s) = (p_\alpha)_s$. Let r be the amalgamation of the $r(s)$; the tree r is a condition because $s(\xi) = 1$ for all $s \in p_\alpha$ with $o(s) = \alpha$.

Now consider all $s \in r$ with $o(s) = \alpha$. If s does not split in r, let t be the successor of s and let $q(t) \leq r_t$ be some condition that decides $\dot{f}(\alpha)$. If s splits in r, let t_1 and t_2 be the two successors of s, and let γ_s be the least γ such that $\dot{f}(\gamma)$ is not decided by r_s. Let $q(t_1) \leq r_{t_1}$ and $q(t_2) \leq r_{t_2}$ be conditions that decide $\dot{f}(\gamma_s)$ in opposite ways, and such that they have stems of length greater than γ_s.

Now we let $p_{\alpha+1}$ be the amalgamation of all the $q(t)$, $q(t_1)$, $q(t_2)$ into r. Clearly, $p_{\alpha+1} \cap {}^\alpha 2 = r \cap {}^\alpha 2 = p_\alpha \cap {}^\alpha 2$. The condition $p_{\alpha+1}$ satisfies (3.6) (i) because $p_\alpha \leq r$. It satisfies (ii) because if s does not split and $o(s) = \alpha$, then $(p_{\alpha+1})_s = q(t)$ where t is the successor of s. Finally, it satisfies (iii), because if s splits and $o(s) = \alpha$, then $(p_{\alpha+1})_{s^\frown 0} = q(t_1)$ and $(p_{\alpha+1})_{s^\frown 1} = q(t_2)$ where t_1 and t_2 are the two successors of s. □

We now complete the proof that $B(P)$ is rigid.

Theorem 3.7. *The complete Boolean algebra $B(P)$ has no nontrivial automorphism.*

Proof. Assume that π is a nontrivial automorphism of $B(P)$. By Lemma 3.2 there exist conditions p and q with incomparable stems s and t such that $\pi(p)$ and q are compatible. Let $t_0 = s \cap t$ and let $\xi = o(t_0)$. Hence $t_0^\frown 0$ and $t_0^\frown 1$ are stems of the two conditions and by Lemma 3.4, one forces $\varphi(\xi)$ and the other forces $\neg\varphi(\xi)$. This is a contradiction because $\pi(p)$ forces the same sentences that p does, and $\pi(p)$ is compatible with q. □

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