PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 2, Pages 363–370
S 0002-9939(00)05568-4
Article electronically published on August 30, 2000

RESTRICTIONS ON ARRANGEMENTS OF OVALS OF PROJECTIVE ALGEBRAIC CURVES OF ODD DEGREE

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(Communicated by Leslie Saper)

ABSTRACT. This paper investigates the first part of Hilbert's 16th problem which asks about topology of the real projective algebraic curves. Using the Rokhlin-Viro-Fiedler method of complex orientation, we obtain new restrictions on the arrangements of ovals of projective algebraic curves of odd degree $d = 4k + 1$, $k \geq 2$, with nests of depth k.

1. PRELIMINARIES

A topological curve in the real projective plane $\mathbb{R}P^2$ is called *simple* if it consists of a finite number of disjoint connected components homeomorphic to a circle. The component is called an *oval* if it realizes the zero homological class in $H_1(\mathbb{R}P^2)$, and is called a *one-sided component* J otherwise. A simple curve is called *one-sided* if it has a one-sided component and is called *two-sided* otherwise. A simple curve can have no more than one one-sided component. The complement of each oval in $\mathbb{R}P^2$ consists of two connected components: an open disk and an open Mobius band. With respect to the oval, the disk is called the *inner region* of the oval.

A pair of disjoint topological ovals in $\mathbb{R}P^2$ is called an *injective pair* if one of them is contained in the inner region of the other one. In this case the complement of an injective pair of ovals in $\mathbb{R}P^2$ consists of three connected components: an open disk, an open annulus, and an open Mobius band. An injective pair of oriented ovals is called a *positive* injective pair if some orientation of the annulus induces the orientation of the ovals; otherwise this pair of ovals is called a *negative* injective pair.

A simple topological curve that consists of k components can be oriented in 2^k different ways. If the orientation of the curve is fixed, then the numbers of positive and negative injective pairs are denoted by Π^+ and Π^-, respectively. A set of h ovals, any two of which form an injective pair, is called a *nest of depth h*.

The ovals of a one-sided oriented topological curve can be divided into positive and negative ones. Let J be the one-sided component and S be an oval of the curve. The complement of the oval S in $\mathbb{R}P^2$ consists of two connected components one of which is homeomorphic to an open disk D, and another one $\mathbb{R}P^2 \setminus D$ is homeomorphic to a Mobius band. Let $[J]$ and $[S]$ be their homological classes in

Received by the editors April 14, 1998 and, in revised form, May 4, 1999.
2000 *Mathematics Subject Classification.* Primary 14P25; Secondary 14H50, 14P05.
Key words and phrases. Complex orientations of algebraic curves, positive and negative ovals, positive and negative injective pairs, chains of ovals.

$H_1(\mathbb{R}P^2 \setminus D)$. If $2[J] = -[S]$, then the oval S is called *positive*. If $2[J] = [S]$, then the oval S is called *negative*. It is clear that all ovals of an oriented curve are divided into positive and negative ones. The number of positive and negative ovals is denoted by Λ^+ and Λ^-, respectively.

To describe the arrangement of components of a simple curve in $\mathbb{R}P^2$, O. Ya. Viro [V2] has suggested the following system of notation. The curve consisting of only one oval is denoted by the symbol $\langle 1\rangle$, the empty curve by $\langle 0\rangle$, the curve consisting of only a one-sided component by $\langle J\rangle$. If a curve C without the one-sided component is denoted by the symbol $\langle \mathcal{A}\rangle$, then the curve that consists of the curve C and one oval encompassing the curve C is denoted by the symbol $\langle 1\langle \mathcal{A}\rangle\rangle$. If two disjoint curves are denoted by the symbols $\langle \mathcal{A}\rangle$ and $\langle \mathcal{B}\rangle$ and each oval of one curve does not encompass an oval of the other curve, then the curve that is the union of these two curves is denoted by the symbol $\langle \mathcal{A} \sqcup \mathcal{B}\rangle$. Moreover, there are three abbreviations:

$$\langle \underbrace{\mathcal{A} \sqcup \ldots \sqcup \mathcal{A}}_{\alpha \text{ times}} \rangle = \langle \alpha \times \mathcal{A}\rangle, \qquad \langle \alpha \times 1\rangle = \langle \alpha\rangle,$$

$$\langle \underbrace{1\langle 1\langle \ldots \langle 1}_{k \text{ times}} \langle \mathcal{A}\rangle\rangle \ldots\rangle\rangle\rangle = \langle 1^k\langle \mathcal{A}\rangle\rangle.$$

Let x_1, x_2, x_3 be homogeneous point coordinates in a complex projective plane $\mathbb{C}P^2$, $x = (x_1, x_2, x_3)$. An *algebraic curve* is a real homogeneous polynomial $f(x)$, considered up to a constant factor. The set $\mathbb{R}f = \{x \in \mathbb{R}P^2 \mid f(x) = 0\}$ is called the *set of real points*, and the set $\mathbb{C}f = \{x \in \mathbb{C}P^2 \mid f(x) = 0\}$ is called the *set of complex points* of the curve f. The degree d of the polynomial f is called the *degree* of the curve.

If the curve f is nonsingular, then the locus $\mathbb{C}f$ is an orientable smooth two-dimensional algebraic variety of genus $g = (d-1)(d-2)/2$, and the connected components of $\mathbb{R}f$ are homeomorphic to a circle. If a curve of odd degree is nonsingular, then it has exactly one one-sided component. If the degree of the curve is even, then the set $\mathbb{R}f$ consists only of ovals. According to A. Harnack [H], for any positive integer d and for any integer l that satisfies the inequalities

$$0 \le l \le \frac{(d-1)(d-2)}{2} + \frac{1+(-1)^d}{2} \tag{1}$$

there exists a curve of degree d with l ovals. The curves of degree d having $\frac{(d-1)(d-2)}{2} + \frac{1+(-1)^d}{2}$ ovals are called *M-curves*.

In this paper we consider only nonsingular (in $\mathbb{C}P^2$) algebraic curves.

Consider the set $\mathbb{C}f \setminus \mathbb{R}f$. The curve f is called *divisible* (or of *type* I) if the set $\mathbb{C}f \setminus \mathbb{R}f$ is not connected and is called *nondivisible* (or of *type* II) otherwise. The set $\mathbb{C}f \setminus \mathbb{R}f$ of a divisible curve consists of two connected halves with common boundary $\mathbb{R}f$. These halves are interchanged by the complex conjugation involution $\mathbb{C}P^2 \to \mathbb{C}P^2$: $(x_1, x_2, x_3) \mapsto (\bar{x}_1, \bar{x}_2, \bar{x}_3)$. The set of fixed points of the involution coincides with the set $\mathbb{R}f$ of real points of the curve. Then natural orientations of the halves induce two opposite orientations on $\mathbb{R}f$. These two orientations of $\mathbb{R}f$ are called *complex* orientations. Obviously, the numbers Π^+, Π^- and Λ^+, Λ^- do not depend on the choice of complex orientation.

According to Rokhlin [R] if the degree d of a divisible curve is odd, then

$$\Lambda^+ - \Lambda^- + 2(\Pi^+ - \Pi^-) = l - \frac{d^2-1}{4}. \tag{2}$$

The number l of ovals of a divisible curve of degree d satisfies the congruence

$$l \equiv g + \frac{1+(-1)^d}{2} \ (mod\, 2). \tag{3}$$

These preliminaries are needed for the following exposition; for detailed stories see the surveys [R], [V1], [V2] and the paper [F].

2. Technical lemmas

Let $\{S_1, \dots, S_l\}$ be the set of ovals of a one-sided oriented curve. Define a natural function $\omega : \{S_1, \dots, S_l\} \to \{-1, +1\}$ by the formula

$$\omega(S) = \begin{cases} +1, & \text{if the oval } S \text{ is positive,} \\ -1, & \text{if the oval } S \text{ is negative.} \end{cases}$$

Lemma 1. 1) *If a one-sided oriented curve has l ovals $S_1, \dots, S_l$ and $\omega(S_i) = \omega_i$, $i = 1, \dots, l$, then*

$$\Lambda^+ - \Lambda^- = \omega_1 + \dots + \omega_l.$$

2) *For any injective pair of oriented ovals S_1, S_2 with $\omega(S_i) = \omega_i$, $i = 1, 2$, we have*

$$\Pi^+ - \Pi^- = -\omega_1\omega_2.$$

The proof is obvious.

Lemma 2. *If a one-sided oriented curve is described by the symbol $\langle 1^h\langle 0\rangle \sqcup J\rangle$, then*
1) $2(\Pi^+ - \Pi^-) = h - (\Lambda^+ - \Lambda^-)^2$,
2) $2(\Pi^+ + \Pi^-) = h^2 - h$.

Proof. 1) Let $S_1, \dots, S_h$ be the ovals and $\omega(S_i) = \omega_i$, $i = 1, \dots, h$; then

$$\begin{aligned} 2(\Pi^+ - \Pi^-) &= 2(-\omega_1\omega_2 - \dots - \omega_1\omega_h - \omega_2\omega_3 - \dots - \omega_{h-1}\omega_h) \\ &= h - (\omega_1 + \dots + \omega_h)^2. \end{aligned}$$

2) is obvious. □

Consider curves which have nests of the form $\langle 1^h\langle\alpha\rangle\rangle$, where $h \geq 1$ and $\alpha \geq 0$. The ovals denoted by the symbol $\langle\alpha\rangle$ are called *inner* ovals; the ovals denoted by the symbol $\langle 1^h\rangle$ are called *principal* ovals. The principal ovals form a nest of depth h which is called the *principal* nest.

Lemma 3. *If a one-sided oriented curve is described by the symbol $\langle 1^h\langle\alpha\rangle \sqcup J\rangle$ and has Λ_p^+ positive and Λ_p^- negative principal ovals, Λ_{in}^+ positive and Λ_{in}^- negative inner ovals, then*
1) $2(\Pi^+ - \Pi^-) = h - (\Lambda_p^+ - \Lambda_p^-)^2 - 2(\Lambda_p^+ - \Lambda_p^-)(\Lambda_{in}^+ - \Lambda_{in}^-)$,
2) $2(\Pi^+ + \Pi^-) = h^2 - h + 2\alpha h$.

Proof. The proof is obvious.

Let f be a curve of degree $d=4k+1, k\geq 2$, described by the symbol

$$\langle\mathcal{A}\rangle=\langle 1\langle\alpha_1\sqcup 1\langle\alpha_2\ldots\sqcup 1\langle\alpha_{k-2}\sqcup 1\langle\alpha_{k-1}\rangle\rangle\ldots\rangle\rangle$$

$$\sqcup 1\langle\beta_1\sqcup 1\langle\beta_2\ldots\sqcup 1\langle\beta_{k-2}\sqcup 1\langle\beta_{k-1}\rangle\rangle\ldots\rangle\rangle$$

$$\sqcup 1\langle\gamma_1\sqcup 1\langle\gamma_2\ldots\sqcup 1\langle\gamma_{k-2}\sqcup 1\langle\gamma_{k-1}\rangle\rangle\ldots\rangle\rangle$$

$$\sqcup 1\langle\delta_1\sqcup 1\langle\delta_2\ldots\sqcup 1\langle\delta_{k-2}\sqcup 1\langle\delta_{k-1}\rangle\rangle\ldots\rangle\rangle\sqcup\varepsilon\sqcup J\rangle,$$

where $\alpha_i,\beta_i,\gamma_i,\delta_i\geq 0$ for $i=1,2,\ldots,k-2$, $\alpha_{k-1},\beta_{k-1},\gamma_{k-1},\delta_{k-1}\geq 1$, and $\alpha_{k-1}+\beta_{k-1}+\gamma_{k-1}+\delta_{k-1}+\varepsilon\geq 5$. The curve has ε outer ovals and $\alpha_{k-1}+\beta_{k-1}+\gamma_{k-1}+\delta_{k-1}$ nests of depth k. Choose one oval in each set of ovals described by symbols $\langle\alpha_{k-1}\rangle$, $\langle\beta_{k-1}\rangle$, $\langle\gamma_{k-1}\rangle$, $\langle\delta_{k-1}\rangle$, and denote these ovals as S_1,S_2,S_3,S_4, respectively. Let a_1,a_2,a_3,a_4 be 4 points in the inner regions of ovals S_1,S_2,S_3,S_4, respectively. We draw the 6 lines that connect each pair of points a_1,a_2,a_3,a_4. Denote these lines such that $a_1,a_3\in L_1$, $a_2,a_4\in L_2$, $a_1,a_4\in L_3$, $a_2,a_3\in L_4$, $a_1,a_2\in L_5$, $a_3,a_4\in L_6$. As a result there arise 3 more points a_5,a_6,a_7 of intersection of these 6 lines: $a_5=L_1\cap L_2$, $a_6=L_3\cap L_4$, $a_7=L_5\cap L_6$. One can see that these 6 lines form 15 quadruples $(L_{i_1},L_{i_2},L_{i_3},L_{i_4})$ of mutually distinct lines. There are 3 quadruples of them (L_1,L_2,L_3,L_4), (L_1,L_2,L_5,L_6), (L_3,L_4,L_5,L_6) that contain lines placed in general position. Each of these 3 quadruples divides the projective plane $\mathbb{R}P^2$ into 4 triangles and 3 quadrangles, and each quadruple forms some quadrangle with vertices a_1,a_2,a_3,a_4. Denote the quadrangles with the vertices a_1,a_2,a_3,a_4 that are formed by the quadruple (L_1,L_2,L_3,L_4), (L_1,L_2,L_5,L_6), (L_3,L_4,L_5,L_6) as Q_1,Q_2,Q_3, respectively. The closures of these quadrangles form a closed covering of $\mathbb{R}P^2$.

Lemma 4. *The one-sided component J of a nonsingular curve f of degree $d=4k+1,k\geq 2$, described by the symbol $\langle\mathcal{A}\rangle$ with $\alpha_{k-1},\beta_{k-1},\gamma_{k-1},\delta_{k-1}\geq 1$ and $\alpha_{k-1}+\beta_{k-1}+\gamma_{k-1}+\delta_{k-1}+\varepsilon\geq 5$, is placed in $\mathbb{R}P^2$ such that exactly one of the intersections $J\cap Q_1$, $J\cap Q_2$, $J\cap Q_3$ is empty.*

Proof. Assume the contrary, that either 0, or 2, or 3 of the intersections are empty.

In the first case, when all of the intersections are nonempty, the one-sided component J passes through each of the quadrangles Q_1,Q_2,Q_3. Consider the pencil of conics that pass through the points a_1,a_2,a_3,a_4. It is easy to check that the one-sided component intersects each conic from the pencil at least at 2 points. According to the hypotheses of this lemma, the set $\mathbb{R}f$ of real points of the curve consists of $4(k-1)$ principal ovals, at least of 4 inner ovals S_1,S_2,S_3,S_4, and at least of one more oval S, which can be either an inner or outer oval. The total number of ovals is at least $4k+1$. The conic from the pencil that passes through a point a in the inner region of oval S, intersects the curve f of degree $d=4k+1,k\geq 2$, at least at $8k+4$ points, $8k+2$ points of which lie on ovals, and 2 points lie on the one-sided component. According to Bezout's theorem it is impossible.

It is obvious that cases, when 2 or 3 of the intersections are empty, are impossible. □

Renumber the quadrangles, if it is necessary, such that $J\cap Q_1=\emptyset$, thus $J\subset Q_2\cup Q_3$, and consider the 4 triangles and 3 quadrangles into which the lines L_3,L_4,L_5,L_6 divide the projective plane $\mathbb{R}P^2$. Denote them such that the triangle T_1 has vertices a_1,a_2,a_6, the triangle T_2 has vertices a_2,a_3,a_7, the triangle T_3 has vertices a_3,a_4,a_6, the triangle T_4 has vertices a_4, a_1, a_7, the quadrangle Q_1 has vertices a_1, a_2, a_3, a_4, the quadrangle Q_4 has vertices a_1, a_6, a_3, a_7, and the quadrangle Q_5 has vertices a_2, a_6, a_4, a_7. The one-sided component J passes through the quadrangles Q_4, Q_5 and through two of the triangles T_1, T_2, T_3, T_4, namely, either through T_1 and T_2, or T_2 and T_3, or T_3 and T_4, or T_4 and T_1. Renumber the triangles and quadrangles, if it is necessary, such that J crosses T_1 and T_2. According to Bezout's theorem the one-sided component J intersects each of the lines L_3, L_4, L_5, L_6 at one point because another $4k$ points of intersection of the curve f with each line already lie on the ovals. Denote the points $a_8 = J \cap L_3$, $a_9 = J \cap L_4$, $a_{10} = J \cap L_5$, and $a_{11} = J \cap L_6$, and denote the topological triangles with vertices a_6, a_8, a_9 as $T_1^{'}$ and with vertices a_7, a_{10}, a_{11} as $T_2^{'}$. Denote the region $Q_1 \cup Q_4 \cup Q_5 \cup T_1^{'} \cup T_2^{'}$ as Q.

Lemma 5. *The nonsingular curve of degree $d = 4k+1, k \geq 2$, described by the symbol $\langle \mathcal{A} \rangle$ with $\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1} \geq 1$ and $\alpha_{k-1}+\beta_{k-1}+\gamma_{k-1}+\delta_{k-1}+\varepsilon \geq 5$, does not have ovals lying in Q.*

Proof. Let us consider the pencil of conics that passes through the points a_1, a_2, a_3, a_4. It is easy to check that every conic from the pencil that passes through the region Q intersects the curve f at least at $8k+2$ points: the one-sided component J at 2 points, the principal ovals at $8(k-1)$ points, and the inner ovals S_1, S_2, S_3, S_4 at 8 points. If we suppose that the curve f has an oval S in the region Q, then the conics from the pencil passing through a point a lying in the inner region of oval S intersects the curve f at least at $8k+4$ points. According to Bezout's theorem this is impossible. □

Lemma 6. *If a divisible curve f of degree $d = 4k+1, k \geq 2$, is described by one of symbols $\langle \mathcal{A} \rangle$ with $\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1} \geq 1$ and $\alpha_{k-1}+\beta_{k-1}+\gamma_{k-1}+\delta_{k-1}+\varepsilon \geq 5$, then $\Lambda^+ = \Lambda^-$.*

Proof. The curve f satisfies the hypotheses of Lemma 5. We preserve all notations of Lemma 5. Consider the segment of the pencil of lines $\{\lambda_t\}_{3 \leq t \leq 5}$, with the center at the point a_1, such that $\lambda_3 = L_3$ and $\lambda_5 = L_5$, and the lines of the segment intersect the quadrangle Q_1. According to Fiedler's theorem [F] the ovals which are contained in $(T_2 \setminus T_2^{'}) \cup T_3$ and the ovals which encompass the points a_2, a_3 and a_4 form k chains of ovals and the complex orientations of the ovals in each chain alternate. The latter fact is established by applying Fiedler's theorem to the segment of lines of the pencil throughout the region $T_2 \cup T_3$.

It is analogously proved that the ovals which are contained in $(T_1 \setminus T_1^{'}) \cup T_4$ and the ovals which encompass the points a_2, a_1 and a_4 form k chains of ovals and the complex orientations of ovals in each chain alternate.

The union of these $2k$ chains of ovals constructed in the two previous paragraphs forms some complete chains with alternation of the complex orientations of ovals in each complete chain. Thus, each complete chain contains an even number of ovals. All ovals of the curve are included in the chains. The curve f is divisible. According to congruence (3) the total number $l = \Lambda^+ + \Lambda^-$ of ovals of the curve f is even. Thus $\Lambda^+ = \Lambda^-$. □

Consider a curve which has a fragment described by the symbol $1\langle \alpha \rangle$, where $\alpha \geq 1$. Recall that the oval denoted by 1 is called principal. The inner region of the principal oval contains the inner ovals denoted by symbol $\langle \alpha \rangle$. A line L that intersects the inner region of the principal oval divides this inner region into some connected components. If all the inner ovals belong to the same component of the inner region, then we say that the line L *does not divide* the set of inner ovals; otherwise we say that the line L *divides* the set of inner ovals.

Let f be a curve of degree $d = 4k+1, k \geq 2$, described by the symbol

$$\langle \mathcal{B} \rangle = \langle 1\langle \alpha_1 \sqcup 1\langle \alpha_2 \ldots \sqcup 1\langle \alpha_{k-2} \sqcup 1\langle \alpha_{k-1}\rangle\rangle \ldots \rangle\rangle$$

$$\sqcup 1\langle \beta_1 \sqcup 1\langle \beta_2 \ldots \sqcup 1\langle \beta_{k-2} \sqcup 1\langle \beta_{k-1}\rangle\rangle \ldots \rangle\rangle$$

$$\sqcup 1\langle \gamma_1 \sqcup 1\langle \gamma_2 \ldots \sqcup 1\langle \gamma_{k-2} \sqcup 1\langle \gamma_{k-1}\rangle\rangle \ldots \rangle\rangle \sqcup \mathcal{C}\rangle$$

where $\langle \mathcal{C} \rangle$ is some fragment of the curve.

Lemma 7. *If a curve f of degree $d = 4k+1, k \geq 2$, is described by symbol $\langle \mathcal{B} \rangle$ with $\alpha_{k-1}, \beta_{k-1} \geq 2$, $\gamma_{k-1} \geq 1$, then every line that intersects any two inner ovals of the fragment $\langle 1\langle \alpha_{k-1}\rangle\rangle$ does not divide the set of inner ovals of the fragment $\langle 1\langle \beta_{k-1}\rangle\rangle$.*

Proof. Assume to the contrary that a line L that intersects two inner ovals S_1 and S_2 of the fragment $\langle 1\langle \alpha_{k-1}\rangle\rangle$ divides the set of inner ovals of the fragment $\langle 1\langle \beta_{k-1}\rangle\rangle$. Let S_3 and S_4 be inner ovals that belong to a distinct component of the fragment $\langle 1\langle \beta_{k-1}\rangle\rangle$, on which the line L divides the inner region of the fragment. Let S_5 be an inner oval of the fragment $\langle 1\langle \gamma_{k-1}\rangle\rangle$. Choose 5 points: 2 points on the line L inside ovals S_1, S_2, 2 points inside ovals S_3, S_4, and 1 point inside the oval S_5. Draw a conic passing through these points. One can calculate that the conic intersects the curve f at least at $8k+4$ points. According to Bezout's theorem it is impossible. □

3. Restrictions

Theorem 1. *If a divisible curve f of degree $d = 8p+1$, $p \geq 1$, is described by one of symbols $\langle 1^{2p-1}\langle \alpha \rangle \sqcup 1^{2p-1}\langle \beta \rangle \sqcup 1^{2p-1}\langle \gamma \rangle \sqcup 1^{2p-1}\langle \delta \rangle \sqcup \varepsilon \sqcup J\rangle$, where $\alpha, \beta, \gamma, \delta \geq 1$, and $\alpha+\beta+\gamma+\delta+\varepsilon \geq 5$, then $2\varepsilon \equiv l \ (mod\, 4)$.*

Proof. The total number of ovals of the curve F is $l = \alpha+\beta+\gamma+\delta+\varepsilon+4(2p-1)$. The curve satisfies Lemma 6 and so the formula (2) becomes $2(\Pi^+ - \Pi^-) = l - 16p^2 - 4p$. According to Lemma 3 we obtain $2(\Pi^+ + \Pi^-) = 4(2p-1)^2 - 4(2p-1) + 2(\alpha+\beta+\gamma+\delta)(2p-1) = -16p^2 + 8p + 4p(l-\varepsilon) - 2l + 2\varepsilon$. Thus we obtain the equality $4\Pi^+ = -32p^2 + 4p + 4p(l-\varepsilon) + 2\varepsilon - l$ which, taken modulo 4, gives the result we need. □

Theorem 2. *If an M-curve f of degree $d = 8p+1, p \geq 1$, is described by one of symbols*

$$\langle 1^{2p-1}\langle \alpha \rangle \sqcup 1^{2p-1}\langle \beta \rangle \sqcup 1^{2p-1}\langle \gamma \rangle \sqcup 1^{2p-1}\langle \delta \rangle \sqcup \varepsilon \sqcup J\rangle,$$

where $\alpha, \beta, \gamma, \delta \geq 1$, then the number ε of outer ovals is even.

Proof. According to Harnack's theorem (1) the total number of ovals of the curve f is $l = 32p^2 - 4p$. So the result we need immediately follows from Theorem 1. □

Theorem 3. 1) *If an M-curve f of degree 9 is described by one of symbols*

$$\langle 1\langle \alpha \rangle \sqcup 1\langle \beta \rangle \sqcup 1\langle \gamma \rangle \sqcup 1\langle \delta \rangle \sqcup J\rangle,$$

where $\alpha, \beta, \gamma, \delta \geq 1$, then the numbers $\alpha, \beta, \gamma, \delta$ of inner ovals are odd.

2) *There do not exist M-curves of degree* $4k+1, k \geq 3$*, described by symbols*

$$\langle 1^{k-1}\langle\alpha\rangle \sqcup 1^{k-1}\langle\beta\rangle \sqcup 1^{k-1}\langle\gamma\rangle \sqcup 1^{k-1}\langle\delta\rangle \sqcup J\rangle,$$

where $\alpha, \beta, \gamma, \delta \geq 1$.

Proof. 1) This part of the theorem was proven in [K].

2) Assume the contrary. Any M-curve under consideration satisfies the hypotheses of Lemma 6, so we have $\Lambda^+ = \Lambda^-$. According to Harnack's theorem (1) we have $l = 8k^2 - 2k$. Let A, B, Γ, Δ be the differences of the numbers of positive and negative principal ovals, and A_1, B_1, Γ_1, Δ_1 be the differences of the numbers of positive and negative inner ovals in the nests denoted by the symbols $\langle 1^{k-1}\langle\alpha\rangle\rangle, \langle 1^{k-1}\langle\beta\rangle\rangle, \langle 1^{k-1}\langle\gamma\rangle\rangle, \langle 1^{k-1}\langle\delta\rangle\rangle$, respectively. Using Lemmas 3 and 6 we may write the formula (2) in the form

(4)
$$4(k-1) - (\mathrm{A}^2 + 2\mathrm{A}\mathrm{A}_1 + \mathrm{B}^2 + 2\mathrm{B}\mathrm{B}_1 + \Gamma^2 + 2\Gamma\Gamma_1 + \Delta^2 + 2\Delta\Delta_1) = 4k^2 - 4k.$$

According to Fiedler's theorem [F], each of the integers A_1, B_1, Γ_1, Δ_1 may be equal either to −1, or to 0, or to +1. On the other hand it is clear that $-k+1 \leq$ A, B, Γ, Δ $\leq k-1$. It is not difficult to check that the formula (4) is not satisfied for this set of values because the left side is less than the right side. Thus the indicated M-curves do not exist. □

Theorem 4. *There do not exist M-curves of degree* $4k+1, k \geq 2$*, described by symbols* $\langle 1^{k-1}\langle\alpha\rangle \sqcup 1^{k-1}\langle\beta\rangle \sqcup 1^{k-1}\langle\gamma\rangle \sqcup J\rangle$*, where* $\alpha, \beta, \gamma \geq 1$.

Proof. Assume the contrary. According to Harnack's theorem (1) the total number of ovals of an M-curve under consideration is $l = 8k^2 - 2k$. Let A, B, Γ be the differences of the numbers of positive and negative principal ovals, and A_1, B_1, Γ_1 be the differences of the numbers of positive and negative inner ovals in the nests denoted by the symbols $\langle 1^{k-1}\langle\alpha\rangle\rangle, \langle 1^{k-1}\langle\beta\rangle\rangle, \langle 1^{k-1}\langle\gamma\rangle\rangle$, respectively. Using Lemma 3 we may write the formula (2) in the form

(5)
$$3(k-1) + \mathrm{A} + \mathrm{A}_1 + \mathrm{B} + \mathrm{B}_1 + \Gamma + \Gamma_1$$
$$-(\mathrm{A}^2 + 2\mathrm{A}\mathrm{A}_1 + \mathrm{B}^2 + 2\mathrm{B}\mathrm{B}_1 + \Gamma^2 + 2\Gamma\Gamma_1) = 4k^2 - 4k.$$

According to Lemma 7 and Fiedler's theorem [F], each of the integers A_1, B_1, Γ_1 may be equal either to −1, or to 0, or to +1. On the other hand it is clear that $-k+1 \leq$ A, B, Γ $\leq k-1$. It is not difficult to check that the formula (5) is not satisfied on this set of values because the left side is less than the right side. Thus the indicated M-curves do not exist. □

Theorems 1 and 2, for $p = 1$, and Theorems 3 and 4, for $k = 2$, were proven in [K].

For their helpful comments concerning this paper, my thanks to Patrick Gilmer, Louisiana State University, and David Weinberg, Texas Tech University.

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