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TWIST DECOMPOSITIONS OF GLUING HOMEOMORPHISMS OF PLANAR HEEGAARD DIAGRAMS OF GENUS TWO

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Abstract. We will give a very simple algorithm to decompose a gluing homeomorphism of a planar Heegaard diagram of genus two into Dehn twists associated with the canonical base.

1. Introduction

Let H be a handlebody of genus g standardly embedded in the standard Euclidean 3-space R^3. Let $v = \{v_1, v_2, ..., v_g\}$ be a standard complete system of meridians of $\partial H = F_g$ and let $x = \{x_1, x_2, ..., x_g\}$ be a collection of mutually disjoint circles on F_g such that each circle x_i intersects v_j transversely at only one point if $i = j$ and it is disjoint from v_j if $i \neq j$. Moreover assume that each x_i bounds a disk in the complement of H in R^3. The pair $\{v, x\}$ is called a standard meridian-longitude system of H. Let $L = \{K_1, K_2, ..., K_g\}$ be a collection of mutually disjoint circles on F_g. Then L is called a g-bridge link if each connected component K_i of L always intersects x_i transversely at only one point, and $K_i \cap x_j = \emptyset$ if $i \neq j$ $(i, j = 1, 2, ..., g)$.

Let M be a connected orientable 3-manifold obtained by an integral Dehn surgery along a g-bridge link L with g-components. Then M has a Heegaard diagram $(F_g; v, w)$, where $\{v, x\}$ (resp. $\{w, y\}$) is a standard meridian-longitude system of H_1 (resp. H_2). The diagram has the following properties:

(1) each meridian w_i is identified with K_i,

(2) each longitude x_j is identified with y_j as a set,

where $1 \leq i, j \leq g$.

Let h be a self-homeomorphism of F_g induced from a homeomorphism $\partial H_2 \to \partial H_1$ such that $h(w_i) = K_i$ $(i = 1, 2, ..., g)$. This one h gives a gluing homeomorphism of a Heegaard splitting of M. We call $(F_g; v, L)$ a planar Heegaard diagram ([O]). In this paper, we study a method to decompose h into a product of canonical Dehn twists. Generally speaking, such a method was first given by Lickorish [L1, L2, L3] but, given a gluing homeomorphism, to decompose it into such a product is very difficult. We can simplify the method for a gluing homeomorphism of a planar Heegaard diagram.

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2. Dehn surgeries along a 2-bridge link with 2-components

Let F_2 be a closed connected orientable surface of genus two, and a_1, b_1, a_2, b_2, and c be the circles illustrated in Figure 1. Moreover, let c_1 (resp. c_2) be the front segment (resp. back segment) of the circle c as illustrated in Figure 1.

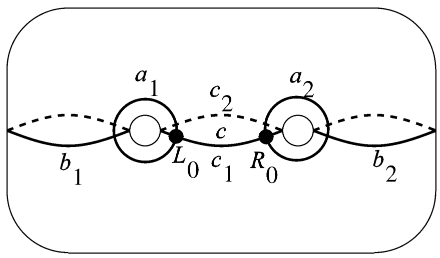

Figure 1.

Let $L = \{K_1, K_2\}$ be a 2-bridge link with 2-components with the canonical form (p, q), where p and q are relatively prime integers such that $p = 2m$ and $q = 2n + 1 (m, n \geq 0)$. Note that we also consider even a case of $p < q$ as the canonical form for the sake of convenience, because the case results with an intermediate stage under series of decomposing deformations. We denote the point $c \cap a_1$ (resp. $c \cap a_2$) by L_0 (resp. R_0). Then we can assume that K_1 (resp. K_2) always intersects a_1 (resp. a_2) transversely at only the point L_0 (resp. R_0), $K_1 \cap a_2 = \emptyset$, $K_2 \cap a_1 = \emptyset$, $K_{L_0} \cap b_1 = \emptyset$ and $K_{R_0} \cap b_2 = \emptyset$, where K_{L_0} (resp. K_{R_0}) is the closure of the connected component including L_0 (resp. R_0) among $K_1 - (b_1 \cup b_2)$ (resp. $K_2 - (b_1 \cup b_2)$).

Next, let $\{\tilde{K}_1, \tilde{K}_2\}$ be a collection of surgery circles which are given by an integral Dehn surgery along L. That is, we can assume that $\tilde{K}_1 = \sigma_2^r(K_1)$ and $\tilde{K}_2 = \sigma_4^s(K_2)$ for certain integers r and s (see later definitions for σ_2 and σ_4). Then the Dehn surgery is denoted by $L((p, q)^{+1}; r, s)$, where $\tilde{K}_1$ (resp. $\tilde{K}_2$) is a circle on F_2 such that it departs from L_0 (resp. R_0), goes around a_1 (resp. a_2) only r (resp. s) times, goes in the upward (resp. downward) direction on the front surface of F_2 , and comes back to L_0 (resp. R_0) from the back surface of F_2. Moreover, with the intermediate stage $\tilde{K}_1$ (resp. $\tilde{K}_2$) also intersects b_1 $|m - 1|$ (resp. m) times, b_2 m (resp. $|m - 1|$) times, c_1 n times, and c_2 n times. When counting the numbers of intersection, we ignore the end points L_0 (resp. R_0). If $r, s > 0$ (resp. $r, s < 0$), then they go around a_1 and a_2 clockwise (resp. counterclockwise).

For the sake of simplicity, later we denote $L((p, q)^\epsilon; 0, 0)$ simply by $L(p, q)^\epsilon$.

Let Ψ be the straight line in R^3 which intersects F_2 at 6 points including L_0 and R_0 in Figure 2. Then $L(p, q)^{-1}$ denotes the Dehn surgery induced by surgery circles obtained from $\tilde{K}_1$ and $\tilde{K}_2$ by rotating F_2 around Ψ only π and then by reflecting F_2.

We note that when $\epsilon = \pm 1$ and $m = n = 0$, $L(0, 1)^\epsilon$ is as shown in Figure 5 (5). As shown in the figure, $\tilde{K}_1$ (resp. $\tilde{K}_2$) does not intersect c_1 and c_2, namely it intersects c_1 0 times and c_2 0 times.

Moreover we define $L(0, 1)$ as the last figure in Figure 6.

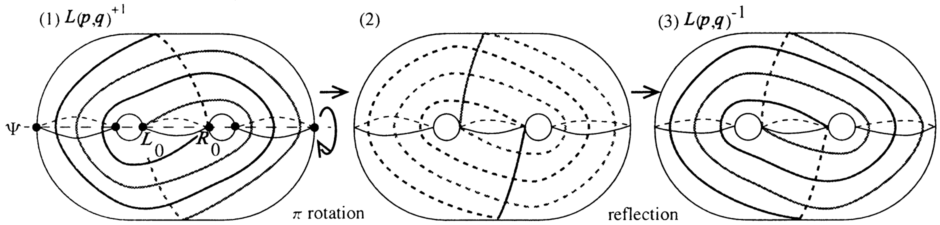

Figure 2. $L(p,q)^{-1}$.

3. Decomposition of a homeomorphism into canonical Dehn twists

Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be Dehn twists along b_1, a_1, c, a_2, and b_2 respectively.

Let d_0, d_1, d_{-1}, d_2 be circles illustrated in Figure 3, and let D_0, D_1, D_{-1}, D_2 be Dehn twists along d_0, d_1, d_{-1}, d_2 respectively. Then we have the following lemma using Lemmata in [L1, L2, L3]:

Lemma 1. $D_0 = \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_5\sigma_4\sigma_5^{-1}\sigma_3\sigma_2\sigma_1$,
$D_1 = \sigma_5\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_4^{-1}\sigma_5^{-1}$,
$D_{-1} = \sigma_5^{-1}\sigma_4^{-1}\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_5$,
$D_2 = \sigma_5^2\sigma_4\sigma_5^2\sigma_4^2\sigma_5^2\sigma_4\sigma_5^2$.

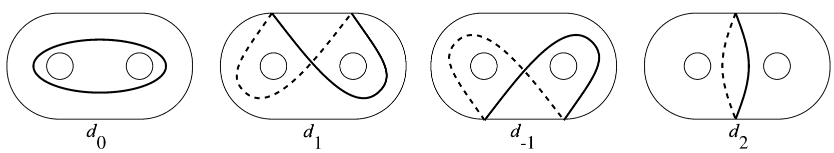

Figure 3.

Let α be any product of Dehn twists on F_2 and $L((p,q)^\epsilon; r, s)$ be any integral Dehn surgery along L, where $\epsilon = \pm 1$. Then we can get new surgery circles $\tilde{K}_1'$ and $\tilde{K}_2'$ such that $\tilde{K}_1' = \alpha(\tilde{K}_1)$ and $\tilde{K}_2' = \alpha(\tilde{K}_2)$ and denote the resulting Dehn surgery by $L((p',q')^\epsilon; r', s')$. The notation $L((p,q)^\epsilon; r, s) \to \alpha \to L((p',q')^\epsilon; r', s')$ denotes such the changing process. Then we have:

Lemma 2. $L((p,q)^\epsilon; r, s) \to \sigma_2^{-r}\sigma_4^{-s} \to L(p,q)^\epsilon$.

From now on, for any $L(p,q)^\epsilon$ let $\tilde{L} = \tilde{K}_1 \cup \tilde{K}_2$, and let $A = b_1 \cap \tilde{L}, B = b_2 \cap \tilde{L}$, $C_1 = (c_1 - \{L_0, R_0\}) \cap \tilde{L}$, and $C_2 = (c_2 - \{L_0, R_0\}) \cap \tilde{L}$. Then it is easily seen that $|A| = p-1, |B| = p-1$, and $|C_1| = |C_2| = q-1$.

Lemma 3. *If* $p - 2q > 0$, $L(p,q)^{\pm 1} \to D_0^{\mp 1} \to L(p-2q,q)^{\pm 1}$.

Proof. (See Figure 4.) Let $D = \tilde{L} \cap d_0^+ \cup L_0$, and $E = \tilde{L} \cap d_0^- \cup R_0$, where $d_0 = d_0^+ \cup d_0^-$ and d_0^+ (resp. d_0^-) is the upper (resp. lower) half of d_0 in (1). Then we have that $|D| = q$ and $|E| = q$. Next let $F = \tilde{L}' \cap b_1$ and $G = \tilde{L}' \cap b_2$ in (5). We can see $|F| = |A| - |D| - |E|, |G| = |B| - |D| - |E|$ and get $|F| = |G| = (p-2q) - 1$ in (2). $|F|$ and $|G|$ are held from (2) to (5). Then $|F| = |G| = (p-2q) - 1 (\geq 1)$ in (5) because $p - 2q > 0$ and p is even. Furthermore we can easily verify that $|C_1|$ and $|C_2|$ are invariant. Hence we can get that $p' = p - 2q$ and $q' = q$. ∎

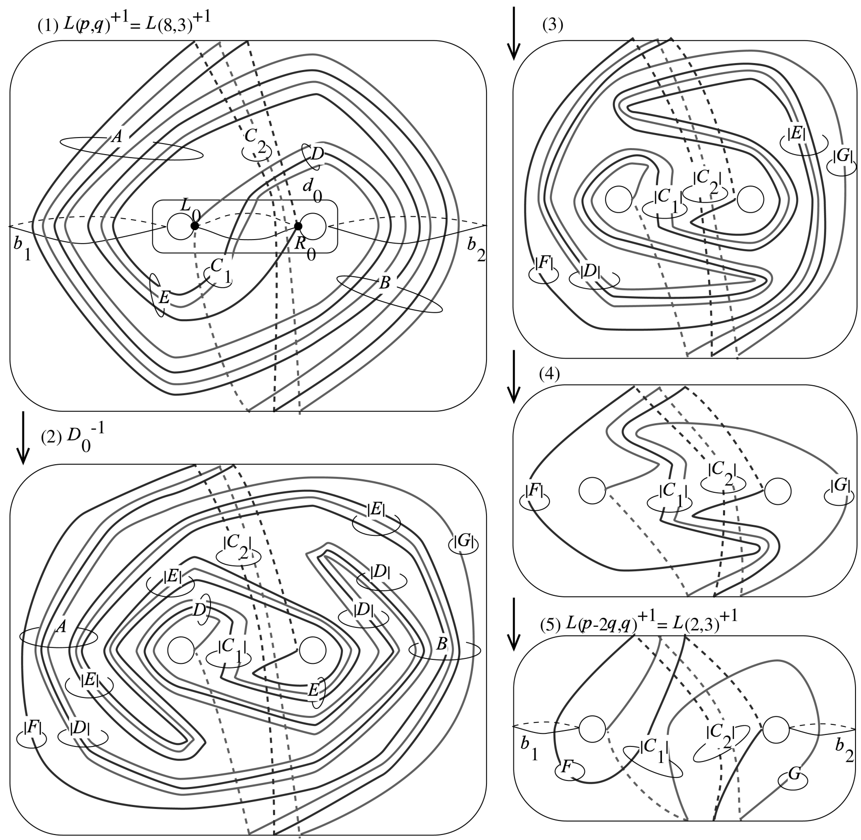

FIGURE 4.

Lemma 4. *If* $\frac{p}{2} < q < p-1$, $L(p,q)^{\pm 1} \to \sigma_2^{\pm 2}\sigma_4^{\pm 2}D_0^{\mp 1} \to L(2q-p,q)^{\mp 1}$.

Lemma 5. *If* $q = p-1$ *and* $p-2 > 0$, $L(p,q)^{\pm 1} \to D_{\mp 1}^{\pm 1} \to L(p-2,q-2)^{\pm 1}$.

Lemma 6. *If* $q = p+1$ *and* $p-2 \geq 0$, $L(p,q)^{\pm 1} \to D_{\mp 1}^{\mp 1} \to L(p-2,q-2)^{\pm 1}$.

Proof. General cases are abbreviated. Especially in $p = 2$, we get $L(0,1)^{\pm 1}$ from $L(2,3)^{\pm 1}$ in Figure 5. □

Lemma 7. *If* $p+1 < q < 2p$, $L(p,q)^{\pm 1} \to D_2^{\pm 1} \to L(p,2p-q)^{\mp 1}$.

Lemma 8. *If* $2p < q$, $L(p,q)^{\pm 1} \to D_2^{\pm 1} \to L(p,q-2p)^{\pm 1}$.

Lemma 9. $L(2,1)^{\pm 1} \to D_{\mp 1}^{\pm 1} \to L(0,1)^{\mp 1}$ *and* $L(0,1)^{\pm 1} \to \sigma_2^{\pm 1}\sigma_4^{\pm 1} \to L(0,1)$.

By this lemma, we get the last figure $L(0,1)$. Finally, we have the following:

Main Theorem. *Let M be a closed connected orientable 3-manifold given by an integral Dehn surgery $L((p,q)^{\epsilon};r,s)$,where $\epsilon = \pm 1$, along a 2-bridge link with 2-components. Then there exists a simple algorithm to get Π such that $L((p,q)^{\epsilon};r,s)$ $\to \Pi \to L(0,1)$, where Π is a product of canonical Dehn twists. The algorithm is given by the following procedure:*

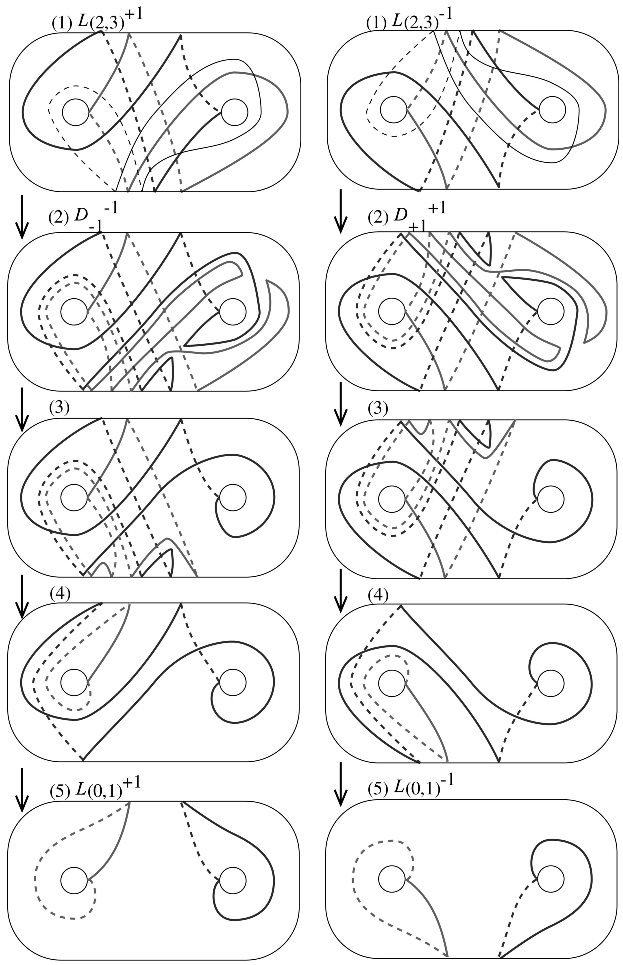

FIGURE 5.

$L((p,q)^\epsilon; r, s) \to \sigma_2^{-r}\sigma_4^{-s} \to L(p,q)^\epsilon$.
while $(p > 0)$ $\{$
  if $(q == p-1)$ *then* $\{L(p,q)^\epsilon \to D^\epsilon_{-\epsilon} \to L(p-2, q-2)^\epsilon; p' = p-2; q' = q-2\}$
  else if $(q == p+1)$ *then* $\{ L(p,q)^\epsilon \to D^{-\epsilon}_{-\epsilon} \to L(p-2, q-2)^\epsilon; p' = p-2; q' = q-2\}$
  else if $(q < p-1)$ *then* $\{$
    if $(q < \frac{p}{2})$ $\{ L(p,q)^\epsilon \to D_0^{-\epsilon} \to L(p-2q, q)^\epsilon; p' = p-2q; q' = q\}$
    else $\{ L(p,q)^\epsilon \to \sigma_2^{2\epsilon}\sigma_4^{2\epsilon}D_0^{-\epsilon} \to L(p-2q, q)^\epsilon; p' = p-2q; q' = q\}$
  $\}$ *else* $\{ L(p,q)^\epsilon \to D_2^\epsilon \to L(p, q-2p)^\epsilon; p' = p; q' = q-2p\}$
  if $(p' < 0)$ *then* $\{ p = |p'|; q = q'; \epsilon = -\epsilon \}$
  else if $(q' < 0)$ *then* $\{ p = p'; q = |q'|; \epsilon = -\epsilon \}$
  else $\{ p = p'; q = q' \}$
$\}$
$L(0,1)^\epsilon \to \sigma_2^\epsilon\sigma_4^\epsilon \to L(0,1)$

4. Examples and final remarks

Example 1. Poincaré homology 3-sphere Σ is given by $L(8,3)^{+1}$ and so we can decompose the gluing homeomorphism h of Σ. The decomposing process is as follows:

$$L(8,3)^{+1} \to D_0^{-1} \to L(2,3)^{+1} \to D_{-1}^{-1} \to L(0,1)^{+1} \to \sigma_2\sigma_4 \to L(0,1).$$

Then we have that $L(8,3)^{+1} \to \sigma_2\sigma_4 D_{-1}^{-1} D_0^{-1} \to L(0,1)$ (see Figure 6). Hence, we have the following decomposition of h given by $h(w_i) = K_i (i = 1, 2)$ such that

$$\begin{aligned} h &= D_0 D_{-1} \sigma_4^{-1} \sigma_2^{-1} \\ &= \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_5\sigma_4\sigma_5^{-1}\sigma_3\sigma_2\sigma_1\sigma_5^{-1}\sigma_4^{-1}\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_5\sigma_4^{-1}\sigma_2^{-1}. \end{aligned}$$

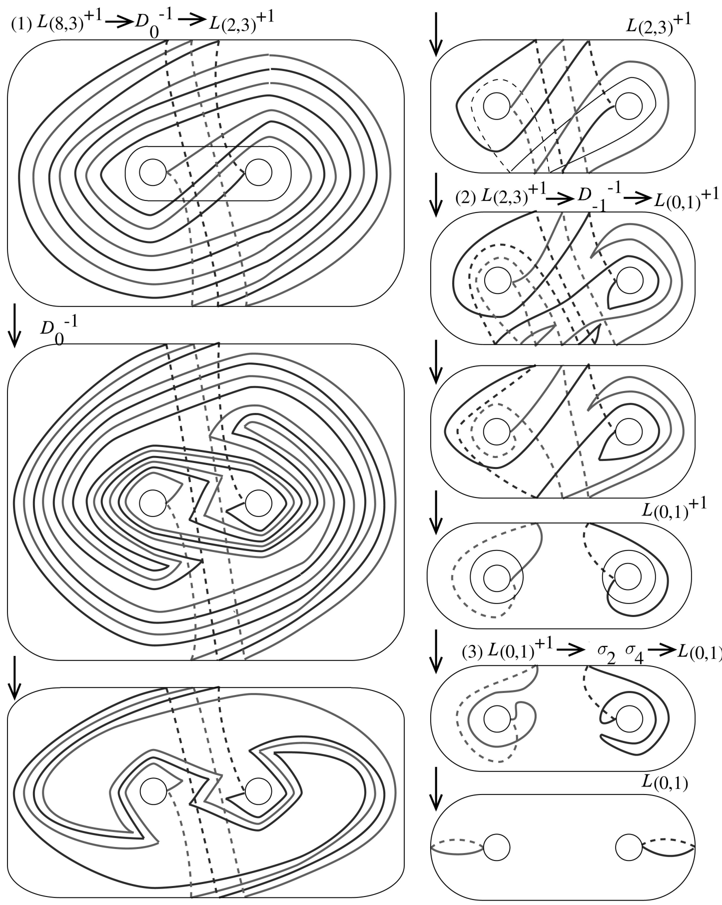

Figure 6. Poincaré homology 3-sphere.

Example 2. We give another example with the same p, $L(8,11)^{+1}$. In this case, note that $p < q$. The decomposing process is as follows:

$$\begin{aligned} &L(8,11)^{+1} \to D_2^{+1} \to L(8,5)^{-1} \to \sigma_2^{-2}\sigma_4^{-2}D_0^{+1} \to L(2,5)^{+1} \to D_2^{+1} \\ &\to L(2,1)^{+1} \to D_{-1}^{+1} \to L(0,1)^{-1} \to \sigma_2^{-1}\sigma_4^{-1} \to L(0,1). \end{aligned}$$

Then we have that

$$L(8,11)^{+1} \to \sigma_2^{-1}\sigma_4^{-1}D_{-1}^{+1}D_2^{+1}\sigma_2^{-2}\sigma_4^{-2}D_0^{+1}D_2^{+1} \to L(0,1)$$

(see Figure 7). Hence, we have the following decomposition of h given by $h(w_i) = K_i (i = 1, 2)$ such that

$$\begin{aligned} h &= D_2^{-1}D_0^{-1}\sigma_2^2\sigma_4^2D_2^{-1}D_{-1}^{-1}\sigma_2\sigma_4 \\ &= (\sigma_5^{-2}\sigma_4^{-1}\sigma_5^{-2}\sigma_4^{-2}\sigma_5^{-2}\sigma_4^{-1}\sigma_5^{-2})(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_5\sigma_4^{-1}\sigma_5^{-1}\sigma_3\sigma_2\sigma_1)\sigma_2^2\sigma_4^2 \\ &\quad (\sigma_5^{-2}\sigma_4^{-1}\sigma_5^{-2}\sigma_4^{-2}\sigma_5^{-2}\sigma_4^{-1}\sigma_5^{-2})(\sigma_5^{-1}\sigma_4^{-1}\sigma_3\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_5)\sigma_2\sigma_4. \end{aligned}$$

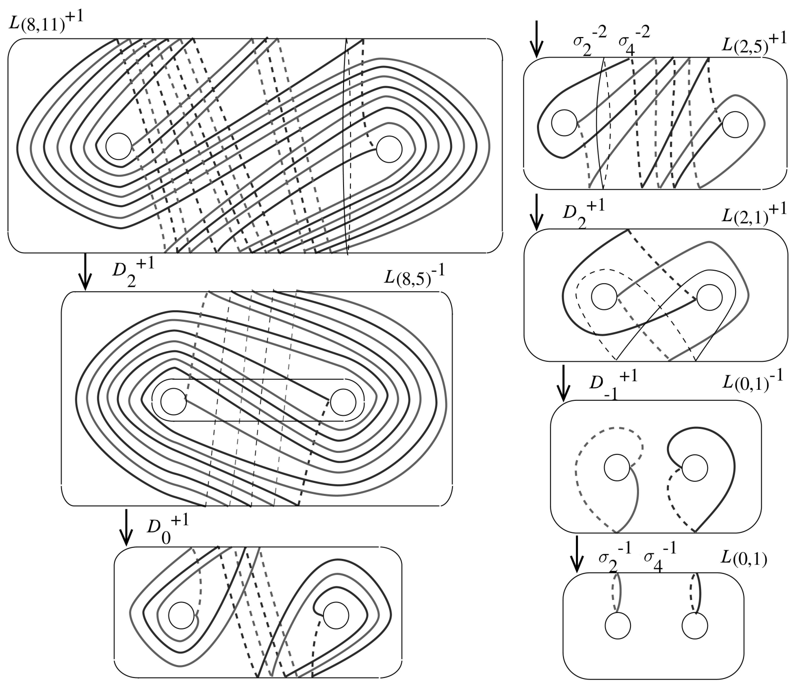

FIGURE 7. $L(8, 11)^{+1}$.

We made a computer program to decompose such homeomorphisms above mentioned into canonical Dehn twists and the program can be obtained by anonymous ftp from ics.nara-wu.ac.jp, in directory /export2/ftp/pub/ochiai. Furthermore, we are now developing computer software to analyze the case of higher genus and the results will be published in a forthcoming paper.

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