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A MODEL FORM FOR EXACT b-METRICS

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ABSTRACT. Any manifold with boundary can be equipped with a b-metric which takes the form $\frac{dx^2}{x^2} + h(x, y, dx, dy)$ with respect to some product decomposition near the boundary, and h positive definite on restriction to the tangent space of the boundary. Here we show the existence of a product decomposition such that h is independent of dx modulo terms vanishing to infinite order at the boundary. The uniqueness of this decomposition is also examined.

1. INTRODUCTION

Scattering theories have recently been developed for compact manifolds with various sorts of complete metrics. In particular there has been interest in manifolds with asymptotically hyperbolic ends or zero-metrics, asymptotically Euclidean ends or scattering metrics, and asymptotically cylindrical ends or b-metrics. For a discussion of these cases, we refer the reader to Melrose's overview, [4]. In this note, we show that techniques developed to prove the existence of model forms for zero and scattering metrics, [2], [3], can also be applied to b-metrics. As well as simplifying the analysis, these model forms are crucial if one wishes to prove inverse results. For a discussion of scattering on manifolds with b-metrics we refer the reader to [1] and for a general discussion of analysis on manifolds with b-metrics to [5].

Let $(X, \partial X)$ be a compact manifold with boundary. An exact b-metric is a smooth metric in the interior of X which for some boundary defining function x takes the form

$$g = \frac{dx^2}{x^2} + g', \tag{1.1}$$

where g' is a smooth 2-cotensor which is smooth up to the boundary and its restriction, h, to the boundary is positive definite on the tangent space of the boundary.

If we take a product decomposition of some open neighbourhood, U, of the boundary,

$$p \mapsto (x, y) \in [0, \epsilon) \times \partial X,$$

then we can write

$$g = \frac{dx^2}{x^2} + g'(x, y, dx, dy).$$

An interesting question is whether there exists a product decomposition such that g' is independent of dx. From the results of [2], [3], one would expect this to be the

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case. Here we show the existence of such a product decomposition modulo an error vanishing to infinite order at the boundary.

We also show that the boundary defining function is determined up to a scalar constant, modulo terms vanishing to infinite order at the boundary, by the metric and the smooth structure.

It is likely that there exists a product decomposition which removes the order infinity error but it is not clear how to proceed and we leave this to other authors.

2. Construction of the model

In this section, we prove our model theorem and then discuss its rigidity. We prove the model theorem by constructing a sequence of diffeomorphisms which approximate to succeeding levels of precision and then apply the Borel Lemma to obtain our final diffeomorphism.

Theorem 2.1. *Let $(X, \partial X)$ be a compact manifold with boundary. Let g be an exact b-metric on X. Then there exists a diffeomorphism,*

$$\psi : [0, \epsilon) \times \partial X \to U$$

with U a neighbourhood of the boundary such that

$$\psi^* g = \frac{dx^2}{x^2} + h(x, y, dy) + \mathcal{O}(x^\infty).$$

Proof. Picking an arbitrary product decomposition (which exists by taking geodesic normal coordinates with respect to a metric which is smooth up to the boundary) and using our definition of a b-metric we have the existence of a diffeomorphism ϕ such that

$$\phi^* g = \frac{dx^2}{x^2} + \gamma(x, y, dy) + \alpha(x, y)dx^2 + \beta(x, y, dy)dx, \tag{2.1}$$

with α, β, γ smooth up to $x = 0$ and $\gamma(0, y, dy)$ positive definite.

Our objective is to eliminate α and β. We do this by computing a Taylor series term by term. We work purely in local coordinates on the boundary—we shall see that all choices are forced and so natural and there is therefore no problem with patching these local germs together.

Consider putting $x = X, y_j = Y_j + b_j(Y)X$. We then have

$$\begin{aligned} dx =& dX, \\ dy_j =& dY_j + b_j(Y)dX + X\frac{\partial b_j}{\partial Y}dY. \end{aligned}$$

If $\gamma(0, y, dy) = \sum_{ij} \gamma_{ij}(y)dy_i dy_j$, then the metric becomes

$$\frac{dX^2}{X^2} + \sum_{ij} \gamma_{ij}(Y)dY_i dY_j + 2\sum_{ij} \gamma_{ij}(Y)b_j dY_i dX + \sum_{ij} \gamma_{ij}(Y)b_i b_j dX^2 \\ + \tilde{\alpha}(X, Y)dX^2 + \beta(X, Y, dY)dX + \mathcal{O}(X).$$

As $\gamma_{ij}(y)$ is positive definite, there is a unique choice of b_j such that

$$\sum_{ij} \gamma_{ij}(Y)b_j dY_i dX + \beta(0, Y, dY)dX$$

is equal to zero. We take this choice.

We now have the metric in the form (2.1) with the added condition that $\beta(0, y, dy) = 0$. Now put $x = X + a(y)X^3, y = Y$. Then we have

$$dx = dX + a(y)3X^2dX + \frac{\partial a}{\partial y}X^3dY,$$

and the metric becomes

$$\frac{dX^2}{X^2} + \gamma(X, Y, dY) + 4a(y)dX^2 + \tilde{\alpha}(X, Y)dX^2 + \beta(X, Y, dY)dX + \mathcal{O}(X).$$

So if we put $a = -\frac{1}{4}\tilde{\alpha}(0, y)$, we achieve the form (2.1) with both α and β vanishing at $x = 0$.

We have shown the existence of a diffemorphism ψ_0 of the form $\psi_0(X, Y) = (X + a(Y)X^3, Y + b(Y)X)$ such that

$$\psi_0^*\phi^*g = \frac{dX^2}{X^2} + h(X, Y, dY) + X\alpha_1(X)dX^2 + X\beta_1(X, Y, dY)dX.$$

Now suppose we have constructed a sequence of diffeomorphisms, ψ_r, for $r < k$, such that

$$\psi_r(X, Y) = (X + a_r(Y)X^{r+3}, Y + b_r(Y)X^{r+1}), \tag{2.2}$$

and such that ϕ_r, the composition of all the first r maps with ϕ, has the property

$$\phi_r^*g = \frac{dx^2}{x^2} + \gamma_r(x, y, dy) + x^{r+1}a_r(x, y)dx^2 + x^{r+1}b_r(x, y, dy)dx. \tag{2.3}$$

We now show how to construct ϕ_k. As above, first we choose the next Y term and then the next X term.

So let $y = Y + X^{k+1}b_k, x = X$. The metric, ϕ_{k-1}^*g, then becomes

$$\begin{aligned}\frac{dX^2}{X^2} &+ \gamma_{k-1}(X, Y, dY) + X^k\alpha_{k-1}(X, Y)dX^2 \\ &+ X^k\beta_{k-1}(X, Y, dY)dX + 2(k+1)\sum_{ij}\gamma_{ij}(Y)b_{i,k}dY_jdX + \mathcal{O}(X^{k+1}),\end{aligned}$$

where $b_k = (b_{1,k}, \dots, b_{n-1,k})$. As before there is then a unique choice of b_k cancelling with β_{k-1}. We now have the metric in the form

$$\frac{dx^2}{x^2} + \gamma_{k-1}(x, y, dy) + x^k\alpha_{k-1}(x, y)dx^2 + \mathcal{O}(x^{k+1}).$$

Putting $x = X + X^{k+3}a_k, y = Y$, the metric becomes

$$\frac{dX^2}{X^2} + \gamma_{k-1}(X, Y, dY) + X^k\alpha_{k-1}(X, Y)dX^2 + 2(k+2)X^ka_k(Y)dX^2 + \mathcal{O}(X^{k+1}).$$

So picking a appropriately, we eliminate the error at order k.

So we have shown that we can construct ψ_r for all r. Now as ψ_r fixes the boundary to order $k+1$, the first k terms of the Taylor series of ϕ_r will be independent of r for $r > k$. Using the Borel lemma, we therefore choose ψ to have these values as its Taylor series.

By construction ψ has the requisite properties. □

To what extent is this model unique? In [3], for scattering metrics it is noted that x is determined modulo x^2 by the metric and that the x^2 term in the Taylor series is arbitrary but that all succeeding terms are forced by the metric. In [2], for zero metrics it is observed that the term x can be replaced by $c(y)x$ with an arbitrary positive smooth function c but all succeeding terms are determined. What occurs with exact b-metrics?

If we put $x = c(y)X$, we get an additional term, $2X^{-1}c^{-1}\frac{\partial c}{\partial y}dydX$. This has to be zero so we conclude that c is constant. So the lead term of x is determined up to constants by the smooth structure and by the metric.

Now suppose $x = X + e(y)X^2$. We then gain an extra term, $2X^{-1}e(Y)dX^2$. This has to be zero so $e = 0$.

All further terms have been seen to be forced above. So we conclude that modulo terms vanishing to infinite order at the boundary, the choice of x modulo a constant scalar is unique. This is a greater amount of rigidity than in the zero and scattering cases where one term of the Taylor series was arbritrary.

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