PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 2, Pages 515–524
S 0002-9939(00)05608-2

ABSOLUTELY CONTINUOUS MEASURES ON NON QUASI-ANALYTIC CURVES WITH INDEPENDENT POWERS

MATS ANDERS OLOFSSON

(Communicated by Christopher D. Sogge)

Abstract. We prove that every non quasi-analytic Carleman class contains functions whose graph supports measures that are absolutely continuous with respect to arc length measure and yet they have independent convolution powers in the measure algebra $M(\mathbb{R}^2)$. The proof relies on conditions which ensure that the canonical map between two Cantor sets can be extended to a function in an arbitrary prescribed non quasi-analytic Carleman class.

1. Introduction

Our main technical object with this note is to prove certain sufficient conditions on pairs of Cantor sets K_0 and K to ensure that the so-called canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ (see Definition 1 in Section 3) can be extended to a function in $C\{M_j\}$, where $C\{M_j\}$ is an arbitrary prescribed non quasi-analytic Carleman class of C^∞-functions (see below). In Section 3 our extension result for Cantor set maps is given in three different forms in Theorem 3, Corollary 1 and Theorem 4. The content of those results is that extension is possible if the image Cantor set K is sufficiently small relative to the domain Cantor set K_0. In some sense these extension results are rather sharp (see Remark 3 in Section 3). Our proofs of those results rely on the methodology used to to prove Whitney's extension theorem (see Stein [10], Chapter VI).

The motivation for this work comes from Björk [1], which was inspired by [2] and [11]. Let A be a closed subalgebra (with unit) of the measure algebra $M(\mathbb{R}^n)$. The Fourier transform allows us to consider $\mathbb{R}^n$ as a subset of the maximal ideal space of A. A is called a Wiener algebra if $\mathbb{R}^n$ is dense in the maximal ideal space of A; this is known to imply that every measure in A whose Fourier transform is bounded away from zero is invertible in A. It is proved in [1] that if A is the closed subalgebra of $M(\mathbb{R}^n)$ generated by all Dirac measures and all measures that are absolutely continuous with respect to Lebesgue surface area measure on some real analytic submanifold of $\mathbb{R}^n$, then A is a Wiener algebra. See also Remark 8

Received by the editors April 29, 1999.

2000 *Mathematics Subject Classification.* Primary 43A10; Secondary 26E10.

Key words and phrases. Measure algebras, Wiener-Pitt phenomenon, independent powers.

The author was supported by the G. S. Magnusson Fund of the Royal Swedish Academy of Sciences.

in Section 4. In contrast to this we shall prove the following Wiener-Pitt type theorem:

Theorem 1. *Let $C\{M_j\}$ be non quasi-analytic. Then there exists a curve*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : \ y = f(x)\},$$

where $f \in C\{M_j\}$, and a compactly supported measure $\nu \in M(\mathbb{R}^2)$ absolutely continuous with respect to arc length measure on Γ such that ν has independent powers (Remark 6: Section 4) and is not an element in some Wiener subalgebra A of $M(\mathbb{R}^2)$.

This result sharpens the analogous result in [1] stating the same as Theorem 1 except that the curve Γ was only required to be of the class C^∞ there. The proof of Theorem 1 is given in Section 4. In the construction used to prove Theorem 1 our extension result Theorem 4 is used in an essential way.

Quasi-analytic classes. Let $\{M_j\}_{j=0}^\infty$ be a sequence of positive numbers. We denote by $C\{M_j\}$ the Carleman class consisting of all C^∞-functions f on $\mathbb{R}$ satisfying the estimates

$$|f^{(j)}(x)| \le \alpha A^j M_j, \quad x \in \mathbb{R}, \ j \ge 0,$$

for some constants $0 < \alpha, A < \infty$. It is known that every such class $C\{M_j\}$ can be defined by means of a logarithmically convex sequence. In fact, $C\{M_j\} = C\{\underline{M}_j\}$, where $\{\underline{M}_j\}_{j=0}^\infty$ is the largest logarithmically convex minorant sequence of $\{M_j\}_{j=0}^\infty$ (see [6], Chapter V, Section E). Using logarithmic convexity of $\{M_j\}_{j=0}^\infty$ one proves that $C\{M_j\}$ is an algebra of functions (see [6], Chapter IV, Section E).

The class $C\{M_j\}$ is said to be quasi-analytic if the zero function $f \equiv 0$ is the only function f in $C\{M_j\}$ satisfying $f^{(j)}(0) = 0$ for all $j \ge 0$. The celebrated theorem of Denjoy-Carleman (see [6], Chapter IV, Section B) states that $C\{M_j\}$ is quasi-analytic if and only if

$$\sum_{j=1}^\infty \frac{\underline{M}_{j-1}}{\underline{M}_j} = \infty.$$

A closely related result which will also be referred to as the theorem of Denjoy-Carleman is that

$$\sum_{j=1}^\infty \frac{M_{j-1}}{M_j} < \infty$$

implies that $C\{M_j\}$ is non quasi-analytic. In fact, this implication follows from the well-known construction of test functions by means of repeated convolutions of normalized characteristic functions (see [4], Theorem 1.3.2).

2. Whitney extension of a flat map

The aim with this section is to prove a variant of Whitney's extension theorem (see [10], Chapter VI, Section 2 or [4], Theorem 2.3.6), Theorem 2 below. The novelty of Theorem 2 is that it is formulated in terms of flatness (see assumption (3) below) and Carleman classes. See also Remark 1 below.

Lemma 1. *Let $F \subset \mathbb{R}$, $F \neq \mathbb{R}$, be a non-empty closed set and $C\{M_j\}$ a non quasi-analytic Carleman class. Then there exists a locally finite partition of unity $\{\varphi_k\}_{k=1}^\infty$ on $\mathbb{R} \setminus F$ such that:*

1) *Every point in* $\mathbb{R}\setminus F$ *has a neighborhood that intersects at most two of the* $\operatorname{supp}(\varphi_k)$*'s.*
2) *There exists a positive real constant* C *not depending on* k *such that:*

$$\operatorname{diam}(\operatorname{supp}(\varphi_k)) \le C \operatorname{dist}(\operatorname{supp}(\varphi_k), F).$$

3) *There exist constants* $0<\alpha, A<\infty$ *such that*

$$|\varphi_k^{(j)}(x)| \le \alpha A^j M_j \operatorname{dist}(x,F)^{-j}, \quad x\in\mathbb{R}, \quad j=0,1,2,\dots, \tag{1}$$

holds for all k.

We mention that partitions of unity similar to that in Lemma 1 are constructed in much more generality (arbitrary finite dimensional vector space) by Hörmander in [4] (see Theorem 1.4.10). For the sake of completeness we include a sketch of the proof of Lemma 1.

Sketch of the proof of Lemma 1. Write $\mathbb{R}\setminus F = \bigcup_{k=1}^{\infty} I_k$, where the I_k's are closed intervals whose interiors are pairwise disjoint and such that

$$c_1 \operatorname{diam}(I_k) \le \operatorname{dist}(I_k, F) \le c_2 \operatorname{diam}(I_k), \quad k=1,2,\dots,$$

for some constants c_1 and c_2. Let I_k^* be the interval with the same center as I_k but expanded by a factor $1+\varepsilon$ for some $\varepsilon>0$. We now choose $\varepsilon>0$ sufficiently small to ensure that I_k^* and I_j^* intersect only if the intervals I_k and I_j have a common endpoint.

Let $I=[-1,1]$ and $I^*=[-(1+\varepsilon),1+\varepsilon]$. Let $\psi\in C\{M_j\}$ be such that $\psi=1$ on I, $\operatorname{supp}(\psi)\subset I^*$ and $0\le\psi\le 1$. (Such a test function ψ can be constructed by convolving a non-negative compactly supported function in $C\{M_j\}$ with integral $=1$ and small support with the characteristic function for an interval.) We now transfer ψ to I_k^* by defining $\psi_k(x)=\psi((x-a_k)/s_k)$, where $I_k=[a_k-s_k, a_k+s_k]$. We now define the φ_k's by

$$\varphi_1(x)=\psi_1(x), \quad \varphi_k(x)=\psi_k(x)\prod_{j=1}^{k-1}(1-\psi_j(x)), \quad k\ge 2.$$

One checks that $\{\varphi_k\}$ is a partition of unity on $\mathbb{R}\setminus F$. 1) and 2) hold by the construction of the I_k and I_k^*'s. By the construction there are constants β, B such that

$$|\psi_k^{(j)}(x)| \le \beta B^j M_j \operatorname{dist}(x,F)^{-j}, \quad x\in\mathbb{R}, \quad j=0,1,\dots, \tag{2}$$

holds for all $k\ge 1$. Since locally φ_k is a product of at most two functions satisfying (2) it follows from the Leibniz formula for the derivative of a product that (1) holds. □

Lemma 2. *Let* F *and* $\{\varphi_k\}$ *be as in Lemma 1. For every* k, *let* p_k *be a point in* F *closest to* $\operatorname{supp}(\varphi_k)$. *Then*

$$|p_k - y| \le (2+C)|x-y|$$

for all $x\in\operatorname{supp}(\varphi_k)$ *and* $y\in F$, *where* C *is as in Lemma 1.*

Proof. Choose $x^*\in\operatorname{supp}(\varphi_k)$ such that $|x^*-p_k| = \operatorname{dist}(\operatorname{supp}(\varphi_k),F)$. By part 2) of Lemma 1 we have

$$|p_k-y| \le |p_k-x^*| + |x^*-x| + |x-y| \le (2+C)|x-y|.$$

□

Theorem 2. *Let $C\{M_j\}$ be non quasi-analytic. Let $F \subset \mathbb{R}$, $F \neq \mathbb{R}$, be a non-empty closed set and ψ a continuous function on F such that*

$$|\psi(x) - \psi(y)| \leq \Lambda_j |x-y|^j, \quad x, y \in F, \quad j = 1, 2, \ldots, \tag{3}$$

for some constants $0 < \Lambda_j < \infty$.

Then ψ has an extension to a C^∞-function Ψ on $\mathbb{R}$ satisfying the estimates

$$|\Psi^{(j)}(x)| \leq 2\alpha A^j (2+C)^j \Lambda_j M_j, \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, \tag{4}$$

where α, A and C are as in Lemma 1. If ψ is bounded, then $\Psi \in C\{\Lambda_j M_j\}$.

Proof. Choose $\{\varphi_k\}_{k=1}^\infty$ and $\{p_k\}_{k=1}^\infty$ as in the preceding lemmas. The extension Ψ of ψ is now defined by

$$\Psi(x) = \begin{cases} \sum_{k=1}^\infty \psi(p_k)\varphi_k(x), & x \notin F, \\ \psi(x), & x \in F. \end{cases}$$

It is clear that Ψ is C^∞ in $\mathbb{R} \setminus F$ and that $\Psi^{(j)}$ exists and is $= 0$ in the interior of F for all $j \geq 1$. For proving that Ψ is C^∞ it suffices to check that every $\Psi^{(j)}$ extends continuously to $\mathbb{R}$.

The case $j = 0$. It suffices to prove that $F \not\ni x \to y \in F$ implies $\Psi(x) \to \psi(y)$. We have that

$$\Psi(x) - \psi(y) = \sum_k \left(\psi(p_k) - \psi(y)\right) \varphi_k(x).$$

By (3) and Lemma 2,

$$|\Psi(x) - \psi(y)| \leq \sum_k \Lambda_1 |p_k - y| \varphi_k(x) \leq (2+C)\Lambda_1 |x-y| \to 0$$

as $x \to y$.

The case $j > 0$. Assume $x \notin F$. Choose $y \in F$ as close to x as possible. Since $\sum_k \varphi_k^{(j)}(x) = 0$ we have that

$$\Psi^{(j)}(x) = \sum_k \psi(p_k)\varphi_k^{(j)}(x) = \sum_k \left(\psi(p_k) - \psi(y)\right) \varphi_k^{(j)}(x). \tag{5}$$

By (3) and Lemma 2,

$$|\Psi^{(j)}(x)| \leq \sum_k \Lambda_{j+1} |p_k - y|^{j+1} |\varphi_k^{(j)}(x)| \leq \Lambda_{j+1}(2+C)^{j+1}|x-y|^{j+1} \sum_k |\varphi_k^{(j)}(x)|.$$

By (1) and part 1) of Lemma 1 we can estimate $\sum_k |\varphi_k^{(j)}(x)|$ by $2\alpha A^j M_j |x-y|^{-j}$. We conclude that

$$|\Psi^{(j)}(x)| \leq 2\alpha A^j (2+C)^{j+1} \Lambda_{j+1} M_j |x-y| \to 0$$

as $x \to F$.

We now prove (4). Assume $x \notin F$. Choose $y \in F$ as close to x as possible. By (5) we have

$$|\Psi^{(j)}(x)| \leq \sum_k \Lambda_j |p_k - y|^j |\varphi_k^{(j)}(x)| \leq \Lambda_j (2+C)^j |x-y|^j \sum_k |\varphi_k^{(j)}(x)|.$$

Estimating $\sum_k |\varphi_k^{(j)}(x)|$ by $2\alpha A^j M_j |x-y|^{-j}$ we get (4). $\square$

Remark 1. The case of interest in Theorem 2 is when the set F is highly irregular. For instance, if $F = [0,1]$, then the assumption (3) for $j = 2$ implies that ψ is constant. In the proof of Theorem 3 in Section 3 we apply Theorem 2 to the case when F is a Cantor set and ψ is a so-called canonical map (Definition 1: Section 3).

3. Extension of maps between Cantor sets

A class of Cantor sets on $[0,1]$. For every $n \geq 1$ let there be given 2^{n-1} open intervals

$$\Delta_j^{(n)} = (\alpha_j^{(n)}, \beta_j^{(n)}), \quad j = 1, \dots, 2^{n-1},$$

such that

$$0 < \alpha_1^{(n)} < \beta_1^{(n)} < \alpha_2^{(n)} < \beta_2^{(n)} < \cdots < \alpha_{2^{n-1}}^{(n)} < \beta_{2^{n-1}}^{(n)} < 1.$$

Write $S^{(0)} = [0,1] = S_1^{(0)}$ and set

$$S^{(n)} = S^{(n-1)} \setminus \bigcup_{j=1}^{2^{n-1}} \Delta_j^{(n)} = \bigcup_{j=1}^{2^n} S_j^{(n)}, \quad n \geq 1,$$

where the $S_j^{(n)}$, $1 \leq j \leq 2^n$, are pairwise disjoint closed intervals. We demand further in the construction that the intervals $\Delta_j^{(n)}$ are such that

$$\Delta_j^{(n+1)} \subset S_j^{(n)}, \quad 1 \leq j \leq 2^n,\ n \geq 1.$$

The corresponding Cantor set is now defined by

$$K = \bigcap_{n=0}^{\infty} S^{(n)} = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} \Delta_j^{(n)}.$$

We define

$$\lambda_K(n) = \max_{1 \leq j \leq 2^n} |S_j^{(n)}| \quad (n \geq 0),$$

$$\delta_K(n) = \min_{1 \leq j \leq 2^{n-1}} |\Delta_j^{(n)}| \quad (n \geq 1)$$

and

$$\mathcal{B}_K = \left\{0, 1, \alpha_j^{(n)}, \beta_j^{(n)} :\ 1 \leq j \leq 2^{n-1},\ n \geq 1\right\}.$$

We call K an admissible Cantor set if K is constructed as above and $\lambda_K(n) \to 0$ as $n \to \infty$.

Remark 2. Note that $\mathcal{B}_K$ is dense in K if K is an admissible Cantor set. Let us also notice that there exist admissible Cantor sets with positive Lebesgue measure. For example when every $\Delta_j^{(n)}$, $1 \leq j \leq 2^{n-1}$, has length $1/4^n$ and the $S_j^{(n)}$, $1 \leq j \leq 2^n$, are of equal length, then the above construction gives an admissible Cantor set of measure $1/2$.

For easy reference we record the following lemma whose proof is trivial.

Lemma 3. *Let K be an admissible Cantor set on $[0,1]$. Let $x, y \in K$, $x \neq y$, and denote by $n(x,y)$ the largest integer $n \geq 0$ such that both $x \in S_j^{(n)}$ and $y \in S_j^{(n)}$ for some j. Then,*

$$\delta_K(n(x,y)+1) \leq |x - y| \leq \lambda_K(n(x,y)).$$

Definition 1. Let K_0 and K be admissible Cantor sets on $[0,1]$, where K is constructed as above and K_0 is obtained at step $n \geq 1$ by removing the intervals

$$(a_j^{(n)}, b_j^{(n)}), \quad j = 1, \ldots, 2^{n-1}.$$

The canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ is defined by $\psi(0) = 0$, $\psi(1) = 1$, $\psi(a_j^{(n)}) = \alpha_j^{(n)}$, $\psi(b_j^{(n)}) = \beta_j^{(n)}$.

Note that ψ is strictly increasing and thereby preserves the construction.

Proposition 1. *Let K_0 and K be admissible Cantor sets on $[0,1]$ and let $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ be the corresponding canonical map (Definition 1). Then the following estimate holds:*

$$|\psi(x) - \psi(y)| \leq \Lambda_j |x-y|^j, \quad x, y \in \mathcal{B}_{K_0}, \quad j = 0, 1, 2, \ldots, \tag{6}$$

where

$$\Lambda_j = \sup_{n \geq 0} \frac{\lambda_K(n)}{\delta_{K_0}(n+1)^j}. \tag{7}$$

Proof. Since ψ is strictly increasing, $n(\psi(x), \psi(y)) = n(x, y)$, where $n(\cdot, \cdot)$ is as in Lemma 3. For $x, y \in \mathcal{B}_{K_0}$, $x \neq y$, we have

$$|\psi(x) - \psi(y)| \leq \lambda_K(n(\psi(x), \psi(y))) = \frac{\lambda_K(n(x,y))}{|x-y|^j} |x-y|^j$$

$$\leq \frac{\lambda_K(n(x,y))}{\delta_{K_0}(n(x,y)+1)^j} |x-y|^j \leq \Lambda_j |x-y|^j.$$

□

We are mainly interested in the case when $\Lambda_j < \infty$ for all j. Note however that ψ extends uniquely to a Lipschitz continuous function $\psi : K_0 \to K$ if $\Lambda_1 < \infty$.

Remark 3. Let the notation be as in Proposition 1 and assume that $\Lambda_j < \infty$ for every j. If Ψ is a C^∞-extension of ψ, then $\Psi^{(j)} \equiv 0$ on K_0 for every $j \geq 1$. In particular, such an extension Ψ (if it exists) cannot be in some quasi-analytic class. In Theorem 4 below we show that if $\lambda_K(n) \to 0$ sufficiently fast, then there exists an extension $\Psi \in C\{M_j\}$ of ψ, where $C\{M_j\}$ is an arbitrary prescribed non quasi-analytic Carleman class.

Also, since $\Psi^{(j)}(0) = 0$ for all $j \geq 0$, Ψ can be glued together with any C^∞-function, all of whose derivatives vanish at 0 to form a C^∞-extension of ψ.

Applying Theorem 2 to the canonical map between two Cantor sets we obtain:

Theorem 3. *Let $C\{M_j\}$ be non quasi-analytic and let $\{\Lambda_j\}_{j=0}^\infty$ be a sequence of positive real numbers with $\lim_{j\to\infty} \Lambda_j^{1/j} = \infty$. Let K_0 and K be admissible Cantor sets on $[0,1]$ and $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ be the corresponding canonical map (Definition 1).*

If the set K is constructed in such a way that

$$\lambda_K(n) \leq \min_{j \geq 0} \delta_{K_0}(n+1)^j \Lambda_j \tag{8}$$

for all n sufficiently large, then ψ has an extension to a function Ψ in $C\{\Lambda_j M_j\}$.

Proof. By (8) and Proposition 1 we can uniquely extend ψ to a continuous function $\psi : K_0 \to K$ such that for some $C > 0$,

$$|\psi(x) - \psi(y)| \leq C^{j+1}\Lambda_j |x - y|^j, \quad j \geq 0,$$

for all $x, y \in K_0$. By Theorem 2, ψ has an extension to a function Ψ in the class $C\{C^{j+1}\Lambda_j M_j\} = C\{\Lambda_j M_j\}$. □

Remark 4. The assumption on $\{\Lambda_j\}_{j=0}^{\infty}$ that $\lim_{j\to\infty} \Lambda_j^{1/j} = \infty$ is needed to guarantee that the minima in (8) are attained.

Corollary 1. *Let $C\{M_j\}$ be non quasi-analytic. Let K_0, K and ψ be as in Theorem 3. Let*

$$\Lambda_j = \sup_{n\geq 0} \frac{\lambda_K(n)}{\delta_{K_0}(n+1)^j}.$$

If $C\{M_j/\Lambda_j\}$ is non quasi-analytic, then ψ has an extension to a function Ψ in $C\{M_j\}$.

Proof. Let $L_j = M_j/\Lambda_j$. By our choice of Λ_j,

$$\lambda_K(n) \leq \inf_{j\geq 0} \delta_{K_0}(n+1)^j \Lambda_j.$$

By Theorem 3 we see that ψ extends to a function in $C\{\Lambda_j L_j\} = C\{M_j\}$. □

The assumption in Corollary 1 that $C\{M_j/\Lambda_j\}$ is non quasi-analytic is, by the Denjoy-Carleman theorem, satisfied if

$$\sum_{j=1}^{\infty} \frac{\Lambda_j}{\Lambda_{j-1}} \frac{M_{j-1}}{M_j} < \infty.$$

This condition is fulfilled if $\lambda_K(n) \to 0$ sufficiently fast. We state this explicitly in the next theorem.

Theorem 4. *Let $C\{M_j\}$ be non quasi-analytic and let K_0 be an admissible Cantor set on* $[0, 1]$.

If K is an admissible Cantor set on $[0, 1]$ *such that $\lambda_K(n) \to 0$ sufficiently fast (the rate of convergence depends only on $\{\delta_{K_0}(n)\}_{n=1}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$), then the canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ can be extended to a function Ψ in $C\{M_j\}$.*

Proof. We may assume that $\{M_j\}_{j=0}^{\infty}$ is logarithmically convex. Since, by the theorem of Denjoy-Carleman, $\sum M_{j-1}/M_j < \infty$, we can choose $0 < \omega_j \to \infty$ such that

$$\sum_{j=1}^{\infty} \omega_j \frac{M_{j-1}}{M_j} < \infty. \tag{9}$$

Set $\Lambda_0 = 1$ and

$$\Lambda_j = \prod_{l=1}^{j} \omega_l, \quad j \geq 1.$$

By the theorem of Denjoy-Carleman (the non log-convex case, see the introductory remark on quasi-analytic classes), (9) implies that the class $C\{M_j/\Lambda_j\}$ is non quasi-analytic.

If K is such that

$$\lambda_K(n) \leq \min_{j\geq 0} \delta_{K_0}(n+1)^j \Lambda_j, \quad n \text{ large},$$

then by Theorem 3, ψ extends to a function in $C\{\Lambda_j M_j/\Lambda_j\} = C\{M_j\}$. ☐

4. Application to Wiener algebras

In this section we give the proof of Theorem 1 in the introduction. First we state some preliminary results.

Lemma 4. *There exists a Cantor set K on $[0,1]$ such that $K \setminus \{0\}$ is linearly independent over $\mathbb{Q}$ and $\lambda_K(n) \to 0$ faster than any prescribed quantity.*

Proof. By translating and dilating, this lemma is an immediate consequence of the construction of a linearly independent Cantor set (see [3], pages 187-189 or [5], pages 20-21). ☐

Lemma 5 (Theorem 5.3.2 in [8])**.** *Let G be a non-discrete locally compact abelian group and E an independent (see Remark 5) compact subset of G. If $\mu \in M(G)$ is a continuous measure concentrated on $E \cup (-E)$, then the measures $\delta_0, \mu, \mu^{*2}, \mu^{*3}, \dots$ are mutually singular.*

Remark 5. A set $E \subset G$ is said to be independent if for every choice of distinct points $x_1, \dots, x_k$ in E and $n_1, \dots, n_k \in \mathbb{Z}$, $n_1x_1 + \dots + n_kx_k = 0$ implies $n_1x_1 = \dots = n_kx_k = 0$.

Remark 6. Recall that $\mu \in M(G)$ is said to have independent powers if the measures $\delta_0, \mu, \mu^{*2}, \dots$ are mutually singular.

Below we use the notation $\sigma_A(x)$ for the spectrum of the element x in the Banach algebra A.

Lemma 6. *Let A be a closed subalgebra of a Banach algebra B. Let $x \in A$. Then the boundary of $\sigma_A(x)$ is contained in $\sigma_B(x)$.*

Proof. This result is well known and can be found as part of Theorem 10.18 in [9]. ☐

Proof of Theorem 1. Let $C\{M_j\}$ be non quasi-analytic. Let K_0 be an admissible Cantor set on $[0,1]$ with positive Lebesgue measure (see Remark 2 in Section 3). By Lemma 4 we can find an admissible Cantor set K on $[0,1]$ such that $K \setminus \{0\}$ is linearly independent over $\mathbb{Q}$ and $\lambda_K(n) \to 0$ sufficiently fast in the sense of Theorem 4. By Theorem 4, the canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ (Definition 1: Section 3) has an extension to a function Ψ in the class $C\{M_j\}$. Let $P = K_0 \times K$ and define

$$f(x) = \begin{cases} \Psi(x) & \text{if} \quad x \geq 0, \\ -\Psi(-x) & \text{if} \quad x \leq 0. \end{cases}$$

Then, by Remark 3, f is a function in $C\{M_j\}$. Let

$$\Gamma = \left\{(x,y) \in \mathbb{R}^2 : \ y = f(x), \ x \in \mathbb{R}\right\}$$

be the graph of f. Let σ denote the arc length measure on Γ. Define the measure ν by

$$\nu(E) = \sigma\left(E \cap (P \cup (-P))\right)$$

for every Borel set E in $\mathbb{R}^2$. It is obvious that ν is absolutely continuous with respect to σ. The proof is now completed by the following theorem:

Theorem 5. *In the above situation, ν has independent powers (Remark 6: Section 4) and is not an element in some Wiener subalgebra A of $M(\mathbb{R}^2)$.*

Proof. By Lemma 5 with $G = \mathbb{R}^2$ and $E = P \cap \Gamma = \{(x, f(x)) : \ x \in K_0\}$, ν has independent powers. Using Remark 3 a computation shows that the action of ν on test functions is given by

$$\int \varphi d\nu = \int_{K_0 \cup (-K_0)} \varphi(x, f(x)) dx. \tag{10}$$

Since K_0 has positive Lebesgue measure, $\nu \neq 0$. Assume for reaching a contradiction that $\nu \in A$ for some Wiener algebra A in $M(\mathbb{R}^2)$. Since f is odd it follows from (10) that the Fourier transform of ν is real-valued. Since A is Wiener, $\sigma_A(\nu) \subset \mathbb{R}$.

Let A_0 be the closed subalgebra of $M(\mathbb{R}^2)$ generated by ν. Since ν has independent powers, A_0 consists of all measures of the form

$$\mu = \sum_{j=0}^{\infty} a_j \nu^{*j}, \tag{11}$$

with

$$\|\mu\| = \sum_{j=0}^{\infty} |a_j| \|\nu\|^j < \infty.$$

It is well known that the multiplicative linear functionals on such an algebra are of the form

$$\mu \mapsto \sum_{j=0}^{\infty} a_j \lambda^j,$$

where λ is a complex number such that $|\lambda| \leq \|\nu\|$ and μ is given by (11). In particular,

$$\sigma_{A_0}(\nu) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq \|\nu\|\}.$$

By Lemma 6 this implies that

$$\{\lambda \in \mathbb{C} : \ |\lambda| = \|\nu\|\} \subset \sigma_A(\nu),$$

which contradicts $\sigma_A(\nu) \subset \mathbb{R}$. □

Remark 7. By (10), ν is the push-forward measure $F_*(dx|K_0)$, where $F : x \mapsto (x, f(x))$.

Remark 8. The strongest result in [1] states that if A is the closed subalgebra of $M(\mathbb{R}^n)$ generated by all discrete measures and all measures that are absolutely continuous with respect to Lebesgue surface area measure on some C^1-submanifold of $\mathbb{R}^n$ in the so-called generic position, then A is a Wiener algebra. A consequence of this result is that if B is the closed subalgebra of $M(\mathbb{R}^2)$ generated by all discrete measures, $L^1(\mathbb{R}^2)$ and all measures that are absolutely continuous with respect to arc length measure on some quasi-analytic curve γ in $\mathbb{R}^2$, then B is a Wiener algebra. Here the assertion that γ is a quasi-analytic curve is to be understood in the sense that the components of γ, considered as functions on the parameter interval, are both members in some linear quasi-analytic class of C^∞-functions. An exposition of the above results by Björk and some related matters will appear in [7].

Acknowledgement

The author is grateful to Jan-Erik Björk and Jan Boman for their valuable suggestions regarding this work.

References

1. J-E. Björk, *Wiener Subalgebras of* $M(\mathbb{R}^n)$ *Generated by Smooth Measures Carried by Smooth Submanifolds of* $\mathbb{R}^n$, preprint, Stockholm University, 1979.
2. J. Boman, Equivalence of Generalized Moduli of Continuity, *Ark. Mat.* **18** (1980) 73-100. MR **82m:**42012
3. I. Gelfand, D. Raikov and G. Shilov, *Commutative Normed Rings*, Chelsea Publishing Company, 1964. MR **34:**4940
4. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, second edition, Springer, 1990.
5. J-P Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, 1963. MR **28:**3279
6. P. Koosis, *The Logarithmic Integral I*, Cambridge University Press, 1988. MR **99j:**30001
7. A. Olofsson, Nilpotent Measures and Wiener Subalgebras of $M(\mathbb{R}^n)$, paper IV in *Topics in Real and Complex Analysis*, doctoral thesis, Stockholm University, 2000.
8. W. Rudin, *Fourier Analysis on Groups*, John Wiley & Sons, 1990. MR **91b:**43002
9. W. Rudin, *Functional Analysis*, McGraw-Hill, 1973. MR **51:**1315
10. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
11. N. Varopoulos, Studies in Harmonic Analysis, *Proc. Camb. Phil. Soc.* **60** (1964) 465-516. MR **29:**1284

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: `anderso@matematik.su.se`