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POSITIVE SOLUTIONS OF A DEGENERATE ELLIPTIC EQUATION WITH LOGISTIC REACTION

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ABSTRACT. The degenerate elliptic equation $\lambda\Delta_p u + u^{q-1}(1-u^r) = 0$ with zero Dirichlet boundary condition, where λ is a positive parameter, $2 < p < q$ and $r > 0$, is studied in three aspects: existence of maximal solution, λ-dependence of maximal solution and multiplicity of solutions.

1. INTRODUCTION AND RESULTS

Let Ω be a connected, bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with $C^{2,\alpha}$-boundary $\partial\Omega$ for some $\alpha \in (0,1)$. We consider the following degenerate elliptic equation:

$$(\mathrm{P})_{\lambda,\Omega} \begin{cases} \lambda\Delta_p u + f(u) = 0 & \text{in } \Omega, \\ u \geq 0, \ \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a positive parameter and Δ_p is the p-Laplace operator defined by

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

with $p > 2$ and f is given by

$$f(u) = u^{q-1}(1-u^r)$$

with $q \geq 2$ and $r > 0$. We often write '$(\mathrm{P})_\lambda$' instead of '$(\mathrm{P})_{\lambda,\Omega}$'. A function $u = u_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *solution of* $(\mathrm{P})_\lambda$ if $u \geq 0$ a.e. in Ω, u does not vanish in a set of positive measure, and

$$(1.1) \qquad -\lambda\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx + \int_\Omega f(u)\varphi dx = 0$$

for all $\varphi \in W_0^{1,p}(\Omega)$. A solution u of $(\mathrm{P})_\lambda$ is called a *maximal solution of* $(\mathrm{P})_\lambda$ if $u \geq v$ a.e. in Ω for all solutions v of $(\mathrm{P})_\lambda$. Obviously, a maximal solution is decided uniquely. If a function $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies $u \geq 0$ (resp. $u \leq 0$) on $\partial\Omega$ and

$$-\lambda\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx + \int_\Omega f(u)\varphi dx \leq 0 \quad (\text{resp. } \geq 0)$$

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for all $\varphi \in W_0^{1,p}(\Omega)$ satisfying $\varphi \geq 0$ a.e. in Ω, then it is called an *upper* (resp. a *lower*) *solution of* $(\mathrm{P})_\lambda$.

With respect to $(\mathrm{P})_\lambda$, there are a few works on the *equidiffusive case* $p = q$ as follows. Let λ_1 be the first eigenvalue of $-\Delta_p$ under zero Dirichlet boundary condition. In the one-dimensional case $N = 1$, Guedda and Véron [10] have shown by phase-plane analysis that if $\lambda < 1/\lambda_1$, then $(\mathrm{P})_\lambda$ has a unique solution u_λ, and that a set called the *flat core of* u_λ,

$$\mathcal{O}_\lambda = \mathcal{O}_\lambda(u_\lambda) := \{x \in \Omega; u_\lambda(x) = 1\}, \tag{1.2}$$

is non-empty for sufficiently small λ. Since the length of $\mathcal{O}_\lambda$ can be indicated explicitly, we can see that as $\lambda \to 0$, $\mathcal{O}_\lambda$ spreads out toward the whole of Ω with the growth as

$$\lim_{\lambda \to 0} \lambda^{-1/p} \operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) = C(f, p), \tag{1.3}$$

where $C(f,p) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 (F(1) - F(s))^{-1/p} ds$ and $F(s) = \int_0^s f(t)dt$. In the higher-dimensional case $N \geq 2$, phase-plane analysis is no longer useful and one has to use other methods. Constructing a suitable lower solution by using the eigenfunction for λ_1, Kamin and Véron [12] have proved that the unique solution of $(\mathrm{P})_\lambda$ has a flat core for sufficiently small λ and extended the results of [10]. However, they have given only an estimate $\operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq C\lambda^{1/p}$ as $\lambda \to 0$, where C is a constant independent of λ, without explicit information about C and any estimate of $\operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega)$ from below. In virtue of an exact estimate for $\mathcal{O}_\lambda$, García-Melián and Sabina de Lis [9] have utilized the solutions for $N = 1$, whose dependence on λ is understood well, to make upper and lower solutions and concluded that (1.3) also holds true in the case $N \geq 2$. The *subdiffusive case* $p > q$ can also be investigated in the same way as the equidiffusive case. One can observe that there exists a unique solution u_λ for every $\lambda > 0$ and that as the equidiffusive case, $\mathcal{O}_\lambda(u_\lambda)$ is nonempty for sufficiently small $\lambda > 0$ and it grows as in (1.3). See the author and Yamada [19] for $N = 1$ and [9] with its Remarks 2.2 b for $N \geq 2$. For uniqueness, see also Diaz and Saa [5].

On the other hand, the structure of solution set in the *superdiffusive case* $p < q$ is essentially different from those in the other cases. For $N = 1$, using time-map, the author and Yamada [19] have shown that $(\mathrm{P})_\lambda$ produces a spontaneous bifurcation for λ. That is, there exists $\Lambda > 0$ such that if $\lambda > \Lambda$, then $(\mathrm{P})_\lambda$ has no solution; if $\lambda = \Lambda$, then $(\mathrm{P})_\lambda$ has a unique solution; if $\lambda < \Lambda$, then $(\mathrm{P})_\lambda$ has exactly two distinct solutions $\overline{u}_\lambda$ and $\underline{u}_\lambda$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$ in Ω. It also follows from our analysis that as $\lambda \to 0$, $\mathcal{O}_\lambda(\overline{u}_\lambda)$ spreads out toward the whole of Ω with (1.3) and $\underline{u}_\lambda \to 0$ uniformly in Ω. For $N \geq 2$, Guo [11] has studied the case that there exists $\beta > 0$ such that $f(0) = f(\beta) = 0$, $(\beta - x)f(x) > 0$ in $(0, \beta) \cup (\beta, +\infty)$, $\lim_{s \to 0} f(s)/s^{p-1} = 0$ and $(f(s)/s^{p-1})'' < 0$ in $(0, \beta)$ (the condition '$f''(x) < 0$' in [11, Theorem 3.3] is a misprint and should be replaced by '$(f(x)/x^{p-1})'' < 0$'), and have found two distinct solutions. This is the case $p < q < p + 1$ in our problem and no information about the shape of solutions is given. In the present paper, we will discuss $(\mathrm{P})_\lambda$ in the case $2 < p < q$, $N \geq 2$, and study $(\mathrm{P})_\lambda$ in three aspects: (a) existence of solution, especially maximal solution; (b) λ-dependence of maximal solution; and (c) multiplicity of solutions. As for (a), we can prove the following theorem by the method of upper and lower solutions:

Theorem 1.1. *Let $2 < p < q$ and $r > 0$. Then there exists a positive number $\overline{\lambda} > 0$ such that*

(i) *if $\lambda > \overline{\lambda}$, then* $(\mathrm{P})_\lambda$ *has no solution*;
(ii) *if $\lambda \le \overline{\lambda}$, then* $(\mathrm{P})_\lambda$ *has a maximal solution* $\overline{u}_\lambda$;
(iii) *if $\lambda_1 < \lambda_2 \le \overline{\lambda}$, then $\overline{u}_{\lambda_2} \le \overline{u}_{\lambda_1}$*;

(iv) *the mapping $\lambda \mapsto \overline{u}_\lambda$ is left-continuous on $(0, \overline{\lambda}]$ in $C^{1,\beta'}(\overline{\Omega})$ for any $\beta' \in (0, \beta)$, where β is the constant appearing in Proposition 2.1.*

Remark 1.1. Theorem 1.1 (i) has been obtained by Véron [21, Theorem 3] for the p-Laplace operator on a compact Riemannian manifold without boundary.

We will state our result on (b). The proof essentially consists of constructing suitable upper and lower solutions by the idea of García-Melián and Sabina de Lis [8, 9] and the one-dimensional result in [19].

Theorem 1.2. *Let $2 < p < q$ and $r > 0$. There exists a positive number $\lambda^* \in (0, \overline{\lambda}]$ such that*

(i) *if $\lambda \le \lambda^*$, then $\mathcal{O}_\lambda = \mathcal{O}_\lambda(\overline{u}_\lambda)$ is non-empty*;
(ii) *if $\lambda_1 < \lambda_2 \le \lambda^*$, then $\mathcal{O}_{\lambda_2} \subset \mathcal{O}_{\lambda_1}$*;
(iii) *for sufficiently small $\varepsilon > 0$, there exists $\lambda \le \lambda^*$ such that $\Omega \setminus \Omega_\varepsilon \subset \mathcal{O}_\lambda$, where $\Omega_\varepsilon := \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$.*

Furthermore, $\mathcal{O}_\lambda$ satisfies (1.3) *as $\lambda \to 0$.*

Remark 1.2. From the last assertion of Theorem 1.2, we can see that the growth order of maximal solution of $(\mathrm{P})_\lambda$ when $\lambda \to 0$ is same as that of case $p \ge q$.

To mention (c), we define the functional Φ on $W_0^{1,p}(\Omega)$ corresponding with $(\mathrm{P})_\lambda$:

$$\Phi(u) = \frac{\lambda}{p}\|\nabla u\|_p^p - \int_\Omega \overline{F}(u)dx, \tag{1.4}$$

where $\overline{F}(u) = \int_0^u \overline{f}(s)ds$ and $\overline{f}(s) := f(s)$ in $[0,1]$, $:= 0$ in $\mathbb{R} \setminus [0,1]$. By the Mountain Pass Theorem (cf. [1, 17]) for Φ, we can find a distinct solution from $\overline{u}_\lambda$ for small λ ($< \overline{\lambda}$), and consequently deduce the multiplicity of solutions. At this time, it plays an important role that $\Phi(\overline{u}_\lambda)$ becomes negative if λ is sufficiently small. In other words, the larger $\mathcal{O}_\lambda(\overline{u}_\lambda)$ spreads out, the more $\Phi(\overline{u}_\lambda)$ decreases.

Theorem 1.3. *Let $2 < p < q$ and $r > 0$. There exists a positive number $\Lambda \in (0, \overline{\lambda}]$ such that if $\lambda < \Lambda$, then $(\mathrm{P})_\lambda$ has another solution $u_\lambda \le \overline{u}_\lambda$, $\not\equiv \overline{u}_\lambda$.*

Remark 1.3. We expect that a solution distinct from $\overline{u}_\lambda$ exists for all $\lambda \in (0, \overline{\lambda})$. Theorem 1.3, whose proof directly utilizes the growth of flat hat, gives a partial result for this problem. It will be discussed in the forthcoming paper [18]. (See also Remark 2.2.) In connection with multiplicity for the p-Laplace operator, we can refer to Ambrosetti, Garcia Azorero and Peral [2], Drábek and Pohozaev [7].

2. Proofs of results

The following proposition is fundamental in this paper.

Proposition 2.1. *Let u be a solution of* $(\mathrm{P})_\lambda$. *Then $u \in C_0^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(\overline{\Omega_\varepsilon})$ for some $\beta \in (0,1)$ and sufficiently small $\varepsilon > 0$, and $0 < u(x) \le 1$ for all $x \in \Omega$.*

Proof of Proposition 2.1. Let u be any solution of $(\mathrm{P})_\lambda$. Putting $\varphi = (u-1)^+ := \max\{u-1, 0\} \in W_0^{1,p}(\Omega)$ in (1.1), we have

$$\lambda \int_\Omega |\nabla (u-1)^+|^p dx = \int_{\{u>1\}} f(u)(u-1)dx \le 0.$$

Hence $(u-1)^+ = 0$ a.e. in Ω; so $u(x) \le 1$ a.e. in Ω. This boundedness and a regularity result of Lieberman [13, Theorem 1] deduce that $u \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Thus, it follows from Vázquez's maximum principle [20, Theorem 5] that $0 < u$ (≤ 1) in Ω and

$$\frac{\partial u}{\partial \boldsymbol{n}}(x) < 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where $\boldsymbol{n}$ denotes an outer normal at $\partial\Omega$. By (2.1) and the fact that $|\nabla u| \in C^{0,\beta}(\overline{\Omega})$, there exists $\varepsilon_0 > 0$ such that $|\nabla u| \ge \delta > 0$ in Ω_{ε_0} for some $\delta > 0$. Therefore, since the equation of $(\mathrm{P})_\lambda$ in Ω_{ε_0} becomes a strictly elliptic one, we can conclude from classical theory that $u \in C^{2,\alpha}(\overline{\Omega_\varepsilon})$ for all $\varepsilon \in (0, \varepsilon_0)$. □

Lemma 2.1. *For sufficiently small $\lambda > 0$, there exists a maximal solution $\overline{u}_\lambda$ such that $\mathcal{O}_\lambda$ is non-empty and*

$$\limsup_{\lambda \to 0} \lambda^{-1/p} \operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \le C(f,p). \tag{2.2}$$

Proof. Take $R > 0$ and $x_0 \in \Omega$ satisfying $B_R(x_0) \subset \Omega$, where $B_R(x_0)$ is the ball with radius R and center at x_0. To obtain a lower solution of $(\mathrm{P})_{\lambda,\Omega}$, we will construct a lower solution v_{R,x_0} of $(\mathrm{P})_{\lambda,B_R(x_0)}$. It suffices to find a radially symmetric one, i.e., $v(\rho) = v_{R,x_0}(x)$ satisfying

$$\begin{cases} \lambda(\rho^{N-1}|v_\rho|^{p-2}v_\rho)_\rho + \rho^{N-1} f(v) \ge 0 & \text{in } (0,R), \\ v_\rho(0) = v(R) = 0, \end{cases} \tag{2.3}$$

where $\rho = |x - x_0|$. By a change of variable $\xi = g(\rho)$ such that

$$\xi = g(\rho) = \begin{cases} \frac{R^{1-\theta} - \rho^{1-\theta}}{1-\theta} & \text{if } \theta \ne 1, \\ \log \frac{R}{\rho} & \text{if } \theta = 1, \end{cases}$$

where $\theta := (N-1)/(p-1)$, (2.3) can be rewritten as follows:

$$\begin{cases} \lambda(|w_\xi|^{p-2}w_\xi)_\xi + g^{-1}(\xi)^{p\theta} f(w) \ge 0 & \text{in } (0,T), \\ w(0) = w_\xi(T) = 0, \end{cases} \tag{2.4}$$

where $w(\xi) = v(g^{-1}(\xi))$ and $T = +\infty$ if $\theta \ge 1$, $= \frac{R^{1-\theta}}{1-\theta}$ if $\theta < 1$. In order to find a function w satisfying (2.4), we take any $b \in (0,T)$ and consider the following auxiliary boundary value problem:

$$\begin{cases} \lambda(|\phi_\xi|^{p-2}\phi_\xi)_\xi + g^{-1}(b)^{p\theta} f(\phi) = 0 & \text{in } (0,b), \\ \phi(0) = \phi(b) = 0. \end{cases} \tag{2.5}$$

A change of scale $\xi = b\eta$ gives

$$\begin{cases} \lambda(|\psi_\eta|^{p-2}\psi_\eta)_\eta + \{b g^{-1}(b)^\theta\}^p f(\psi) = 0 & \text{in } (0,1), \\ \psi(0) = \psi(1) = 0, \end{cases} \tag{2.6}$$

where $\psi(\eta) = \phi(b\eta)$. Take λ sufficiently small as

$$\lambda \leq \left\{ \frac{bg^{-1}(b)^\theta}{2C(f,p)} \right\}^p .$$

Then, we already know from [19, Theorem 3.3] that (2.6) has a solution ψ such that $\psi(x) = 1$ in $[C_{\lambda,b}/b, 1 - C_{\lambda,b}/b]$, $0 \leq \psi(x) < 1$ otherwise, where

$$C_{\lambda,b} = \frac{C(f,p)}{g^{-1}(b)^\theta} \lambda^{1/p} \ (\leq b/2). \tag{2.7}$$

Thus, (2.5) also has a solution ϕ such that $\phi(x) = 1$ in $[C_{\lambda,b}, b - C_{\lambda,b}]$ and $0 \leq \phi(x) < 1$ otherwise. Using ϕ, we construct a function w satisfying (2.4) as follows: $w = \phi$ in $[0, C_{\lambda,b})$, $= 1$ in $[C_{\lambda,b}, T)$. Indeed, since g^{-1} is monotone decreasing,

$$\lambda(|w_\xi|^{p-2} w_\xi)_\xi + g^{-1}(\xi)^{p\theta} f(w) = \{g^{-1}(\xi)^{p\theta} - g^{-1}(b)^{p\theta}\} f(\phi) \geq 0 \quad \text{in } [0, C_{\lambda,b})$$

and the boundary conditions are obviously satisfied. Therefore $v(\rho) = w(g(\rho))$ satisfies (2.3); hence the function

$$v_{R,x_0}(x) = \begin{cases} 1 & \text{if } 0 \leq |x - x_0| \leq g^{-1}(C_{\lambda,b}), \\ \phi(g(|x - x_0|)) & \text{if } g^{-1}(C_{\lambda,b}) < |x - x_0| \leq R \end{cases} \tag{2.8}$$

is a lower solution of $(\mathrm{P})_{\lambda, B_R(x_0)}$. Now, we define $\tilde{v}_{R,x_0}(x) = v_{R,x_0}(x)$ in $B_R(x_0)$, $= 0$ in $\Omega \backslash B_R(x_0)$. Then, one can observe that $\tilde{v}$ is a lower solution of $(\mathrm{P})_{\lambda,\Omega}$. Taking the function $u \equiv 1$ as an upper solution, we obtain a maximal solution $\overline{u}_\lambda$ of $(\mathrm{P})_\lambda$ such that $\tilde{v}_{R,x_0}(x) \leq \overline{u}_\lambda(x) \leq 1$ for all $x \in \Omega$ by Diaz's book [4, Theorem 4.14] (see also Deuel and Hess [3], and Puel [15, Theorem 4.2]). In particular, it follows from (2.8) that $\overline{u}_\lambda(x) = 1$ in $B_{g^{-1}(C_{\lambda,b})}(x_0)$. By the arbitrariness of x_0 satisfying $B_R(x_0) \subset \Omega$ and the uniqueness of maximal solution, it holds that $\overline{u}_\lambda(x) = 1$ in $\Omega \setminus \Omega_{R'}$, where $R' = R'(\lambda, b) = R - g^{-1}(C_{\lambda,b})$. Thus $\mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq R'$. It follows from (2.7) and l'Hospital's theorem that $R'(\lambda, b) = R^\theta C_{\lambda,b} + o(\lambda^{1/p})$ as $\lambda \to 0$; so we obtain

$$\limsup_{\lambda \to 0} \lambda^{-1/p} \,\mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq \lim_{\lambda \to 0} \lambda^{-1/p} R'(\lambda, b) = \left\{ \frac{R}{g^{-1}(b)} \right\}^\theta C(f,p).$$

Passing to the limit as $b \to 0$, we conclude (2.2). □

Proof of Theorem 1.1. Define

$$\overline{\lambda} = \sup\{\lambda > 0; (\mathrm{P})_\lambda \text{ has a solution}\}.$$

Since Lemma 2.1 implies $\overline{\lambda} > 0$, we will show $\overline{\lambda} < +\infty$ to see (i). Suppose that there exists a sequence $\{\lambda_m\}_{m=1}^\infty$ such that $\lambda_m \to \infty$ as $m \to \infty$ and $(\mathrm{P})_{\lambda_m}$ has a solution $u_m = u_{\lambda_m}$. Putting $\lambda = \lambda_m$ and $u = \varphi = u_m$ in (1.1), we have $\lambda_m \|\nabla u_m\|_p^p = \int_\Omega u_m f(u_m) dx$. Since $sf(s) \leq s^p$ for $s \in [0,1]$ if $p < q$, it follows that $\lambda_m \|\nabla u_m\|_p^p \leq \|u_m\|_p^p$. Combining this inequality and the Poincaré inequality, we obtain $C\lambda_m \|u_m\|_p^p \leq \|u_m\|_p^p$, where C is a positive constant. Since $\|u_m\|_p^p > 0$, the inequality is a contradiction for sufficiently large m. Next, we will prove (ii) and (iii). Consider the case $\lambda < \overline{\lambda}$. From the definition of $\overline{\lambda}$, for $\lambda < \overline{\lambda}$ there exists $\mu \in (\lambda, \overline{\lambda}]$ such that $(\mathrm{P})_\mu$ has a solution u_μ. By an easy calculation, u_μ is a lower solution of $(\mathrm{P})_\lambda$. Since $u \equiv 1$ is an upper solution of $(\mathrm{P})_\lambda$, it follows from [4, Theorem 4.14] that $(\mathrm{P})_\lambda$ admits a maximal solution $\overline{u}_\lambda$ satisfying $\overline{u}_\lambda \geq u_\mu$. (Note that the same arguments give the proof of (iii).) The case $\lambda = \overline{\lambda}$ is treated as follows. Let $\{\lambda_m\}_{m=1}^\infty$ be a positive increasing sequence satisfying $0 < \lambda_m < \overline{\lambda}$ and $\lambda_m \to \overline{\lambda}$

as $m \to \infty$, and let $\overline{u}_m$ be the maximal solution of $(\mathrm{P})_{\lambda_m}$. From [13, Theorem 1], we know that $\{\overline{u}_m\}$ is uniformly bounded in $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Thus, Ascoli-Arzelà's theorem assures that there exist $u_{\overline{\lambda}}$ and a subsequence of $\{\overline{u}_m\}$, still denoted by $\{\overline{u}_m\}$, such that $\overline{u}_m \to u_{\overline{\lambda}}$ in $C^{1,\beta'}(\overline{\Omega})$ for each $\beta' \in (0,\beta)$. It is easy to see that $u_{\overline{\lambda}} \geq 0$ in Ω and that $u_{\overline{\lambda}}$ satisfies (1.1). To observe that $u_{\overline{\lambda}} \not\equiv 0$, we assume $u_{\overline{\lambda}} \equiv 0$. Since $\{\overline{u}_m\}$ converges to 0 uniformly in Ω as $m \to \infty$, it follows from $p < q$ that for sufficiently large m

$$C\|\overline{u}_m\|_p^p \leq \|\nabla \overline{u}_m\|_p^p = \frac{1}{\lambda_m}\int_\Omega \overline{u}_m f(\overline{u}_m)dx \leq \frac{C}{2}\|\overline{u}_m\|_p^p,$$

which contradicts to $\|\overline{u}_m\|_p^p > 0$. Therefore, $u_{\overline{\lambda}}$ is a solution of $(\mathrm{P})_{\overline{\lambda}}$. We have to show the maximality of $u_{\overline{\lambda}}$. Suppose that $u_{\overline{\lambda}}$ is not maximal. Then, $(\mathrm{P})_{\overline{\lambda}}$ has a maximal solution $v_{\overline{\lambda}} \geq u_{\overline{\lambda}}$, $(\not\equiv u_{\overline{\lambda}})$ and there exists $x_0 \in \Omega$ such that $u_{\overline{\lambda}}(x_0) < v_{\overline{\lambda}}(x_0)$. By (iii), since $\overline{u}_m$ decreases toward $u_{\overline{\lambda}}$ as $m \to \infty$, it holds that $u_{\overline{\lambda}}(x_0) \leq \overline{u}_m(x_0) < v_{\overline{\lambda}}(x_0)$ for sufficiently large m. On the other hand, it follows from (iii) and the fact $\lambda_m < \overline{\lambda}$ that $v_{\overline{\lambda}}(x_0) \leq \overline{u}_m(x_0)$. These inequalities contradict each other; so $u_{\overline{\lambda}}$ is maximal, which can be written as $\overline{u}_{\overline{\lambda}}$. Finally, one can observe (iv) in the similar way as the proof for maximality of $u_{\overline{\lambda}}$. □

Proof of Theorem 1.2. The existence of λ^* satisfying (i) is directly induced from Lemma 2.1 and (ii) follows from (iii) of Theorem 1.1. From the proof of Lemma 2.1, (iii) is obvious for sufficiently small $\varepsilon > 0$ such that $\Omega \setminus \Omega_\varepsilon \neq \emptyset$. It remains to show (1.3), i.e., growth-order of $\mathcal{O}_\lambda$ as $\lambda \to 0$ near $\partial\Omega$.

Take any $x_0 \in \partial\Omega$. Let $a > 0$ (resp. $R > 0$) be sufficiently small (resp. large) such that the annulus $A := \{x \in \mathbb{R}^N; a < |x - y_0| < R\}$, where $y_0 := x_0 + a\boldsymbol{n}$ and $\boldsymbol{n}$ denotes the outer normal at x_0, satisfies $\Omega \subset A$. Define $\tilde{u}_\lambda$ by $\tilde{u}_\lambda := \overline{u}_\lambda$ in Ω, $= 0$ in $A \setminus \Omega$. Then $\tilde{u}_\lambda$ is a lower solution of $(\mathrm{P})_{\lambda,A}$; so a maximal solution $\overline{v}_{\lambda,A}$ of $(\mathrm{P})_{\lambda,A}$ exists, in particular

$$\overline{u}_\lambda(x) \leq \overline{v}_{\lambda,A}(x) \quad \text{in } \Omega. \tag{2.9}$$

From the maximality, $\overline{v}_{\lambda,A}$ is radially symmetric on A; hence $v(\rho) = \overline{v}_{\lambda,A}(x)$ satisfies

$$\begin{cases} \lambda(\rho^{N-1}|v_\rho|^{p-2}v_\rho)_\rho + \rho^{N-1}f(v) = 0 & \text{in } (a,R), \\ v(a) = v(R) = 0, \end{cases} \tag{2.10}$$

where $\rho = |x - y_0|$. As in the proof of Lemma 2.1, we introduce a change of variable

$$\xi = h(\rho) = \begin{cases} \frac{\rho^{1-\theta} - a^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1, \\ \log\frac{\rho}{a} & \text{if } \theta = 1, \end{cases}$$

where $\theta := (N-1)/(p-1)$; then (2.10) can be rewritten as

$$\begin{cases} \lambda(|w_\xi|^{p-2}w_\xi)_\xi + h^{-1}(\xi)^{p\theta}f(w) = 0 & \text{in } (0,T), \\ w(0) = w(T) = 0, \end{cases}$$

where $w(\xi) = v(h^{-1}(\xi))$ and $T = h(R)$. It is easy to see that w is a lower solution of

$$\begin{cases} \lambda(|\phi_\xi|^{p-2}\phi_\xi)_\xi + h^{-1}(b)^{p\theta}f(\phi) = 0 & \text{in } (0,b), \\ \phi(0) = 0,\ \phi(b) = 1, \end{cases} \tag{2.11}$$

for any $b \in (0, T)$. Thus, (2.11) has a maximal solution $\overline{\phi}$ such that

$$w(\xi) \leq \overline{\phi}(\xi) \quad \text{in } (0, b). \tag{2.12}$$

In fact, we know from [19, Theorem 3.3] that $0 < \overline{\phi}(\xi) < 1$ in $(0, D_{\lambda,b})$, $\overline{\phi}(\xi) = 1$ otherwise, where $D_{\lambda,b} = C(f,p)\lambda^{1/p}/h^{-1}(b)^{\theta}$ $(\leq b/2)$. Hence, it follows from (2.9) and (2.12) that $\overline{u}_\lambda(x) \leq \phi(h(|x - y_0|)) < 1$ if $x \in \Omega$ and $a < |x - y_0| < h^{-1}(D_{\lambda,b})$. This means that $\operatorname{dist}(x_0, \mathcal{O}_\lambda) \geq h^{-1}(D_{\lambda,b}) - a$ for each $x_0 \in \partial\Omega$. Making $a > 0$ (resp. $R > 0$) sufficiently small (resp. large), one can get an uniform estimate $\operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq h^{-1}(D_{\lambda,b}) - a$. Since $h^{-1}(D_{\lambda,b}) - a = a^{\theta} D_{\lambda,b} + o(\lambda^{1/p})$ as $\lambda \to 0$, it is possible to obtain that

$$\liminf_{\lambda \to 0} \lambda^{-1/p} \operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq \left\{ \frac{a}{h^{-1}(b)} \right\}^{\theta} C(f,p).$$

Passing to the limit $b \to 0$, we have

$$\liminf_{\lambda \to 0} \lambda^{-1/p} \operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq C(f,p); \tag{2.13}$$

so combining (2.13) and (2.2) of Lemma 2.1, we conclude (1.3). □

Remark 2.1. From (2.13) and more delicate analyses of (2.2), we can see

$$\lim_{\lambda \to 0} \lambda^{-1/p} \sup_{x \in \partial\Omega} \operatorname{dist}(x, \mathcal{O}_\lambda) = C(f,p),$$

which implies that $\mathcal{O}_\lambda$ uniformly spreads out toward the whole of Ω as the order of $\lambda^{1/p}$.

Proof of Theorem 1.3. In virtue of Proposition 2.1, it is well known that u is a solution of $(\mathrm{P})_\lambda$ if and only if u is a critical point of the C^1-functional Φ, defined by (1.4) (cf. Rabinowitz's book [17, Proposition B.10]). We will check all conditions of the Mountain Pass Theorem (cf. [1, 17]). Take any $q^* \in (p, p^*)$, where $p^* := Np/(N-p)$ if $p < N$, $:= +\infty$ if $p \geq N$, and fix it. Since $p < q$, for any $\delta > 0$ there exists $C_\delta > 0$ such that $|\overline{f}(s)| \leq \delta s^{p-1} + C_\delta s^{q^*-1}$. First, it is easy to see that Φ satisfies the Palais-Smale condition. Indeed, let $\{u_n\}$ be any sequence in $W_0^{1,p}(\Omega)$ such that $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$. Then, it follows from the boundedness of $\overline{F}$ that $\{\|\nabla u_n\|_p\}$ is bounded; namely $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, a result of Dinca, Jebelean and Mawhin [6, Lemma 2.1] yields the assertion. In addition, the Sobolev inequality assures that there exist constants $\gamma,\ \rho > 0$ such that $\Phi(u) \geq \gamma$ if $\|\nabla u\|_p = \rho$, because

$$\begin{aligned}
\Phi(u) &\geq \frac{\lambda}{p}\|\nabla u\|_p^p - \frac{\delta}{p}\|u\|_p^p - \frac{C_\delta}{q^*}\|u\|_{q^*}^{q^*} \\
&\geq \left(\frac{\lambda - C_1\delta}{p} - \frac{C_2 C_\delta}{q^*}\|\nabla u\|_p^{q^*-p} \right) \|\nabla u\|_p^p \geq \gamma > 0,
\end{aligned}$$

where C_1 and C_2 are positive constants, provided that $\delta \in (0, \lambda/C_1)$ and $\|\nabla u\|_p = \rho$ is sufficiently small. Next, clearly $\Phi(0) = 0$ and we will show that the maximal solution $\overline{u}_\lambda$ of $(\mathrm{P})_\lambda$ satisfies $\Phi(\overline{u}_\lambda) < 0$ for sufficiently small $\lambda > 0$. Since $\lambda\|\nabla \overline{u}_\lambda\|_p^p = \int_\Omega \overline{u}_\lambda f(\overline{u}_\lambda)dx$, $\Phi(\overline{u}_\lambda)$ can be expressed as

$$\Phi(\overline{u}_\lambda) = \left(\frac{1}{p} - \frac{1}{q} \right) \|\overline{u}_\lambda\|_q^q - \left(\frac{1}{p} - \frac{1}{q+r} \right) \|\overline{u}_\lambda\|_{q+r}^{q+r}$$

with use of $\overline{F}(\overline{u}_\lambda) = F(\overline{u}_\lambda)$, due to Proposition 2.1. Noting that $\|\overline{u}_\lambda\|_\tau^\tau = |\mathcal{O}_\lambda| + \int_{\Omega\setminus\mathcal{O}_\lambda} |\overline{u}_\lambda|^\tau$ for any $\tau \geq 1$, we have

$$\begin{aligned}\Phi(\overline{u}_\lambda) &= \int_{\Omega\setminus\mathcal{O}_\lambda} \left\{ \left(\frac{1}{p} - \frac{1}{q}\right)|\overline{u}_\lambda|^q - \left(\frac{1}{p} - \frac{1}{q+r}\right)|\overline{u}_\lambda|^{q+r} \right\} dx - \left(\frac{1}{q} - \frac{1}{q+r}\right)|\mathcal{O}_\lambda| \\ &\leq C|\Omega \setminus \mathcal{O}_\lambda| - \left(\frac{1}{q} - \frac{1}{q+r}\right)|\mathcal{O}_\lambda| \\ &= C|\Omega| - \left(C + \frac{1}{q} - \frac{1}{q+r}\right)|\mathcal{O}_\lambda|.\end{aligned}$$

Take $\varepsilon > 0$ so sufficiently small that

$$|\Omega_\varepsilon| < \frac{\frac{1}{q} - \frac{1}{q+r}}{C + \frac{1}{q} - \frac{1}{q+r}}|\Omega|.$$

We see from (iii) of Theorem 1.2 that there exists $\Lambda \in (0, \overline{\lambda}]$ such that if $\lambda < \Lambda$, then $|\mathcal{O}_\lambda| > |\Omega \setminus \Omega_\varepsilon|$. Thus, if $\lambda \in (0, \Lambda)$, then

$$\begin{aligned}\Phi(\overline{u}_\lambda) &< C|\Omega| - \left(C + \frac{1}{q} - \frac{1}{q+r}\right)|\Omega \setminus \Omega_\varepsilon| \\ &= \left(C + \frac{1}{q} - \frac{1}{q+r}\right)|\Omega_\varepsilon| - \left(\frac{1}{q} - \frac{1}{q+r}\right)|\Omega| \\ &< 0.\end{aligned}$$

Therefore, all conditions for the Mountain Pass Theorem hold; so we obtain a solution u_λ of $(\mathrm{P})_\lambda$, which is distinct from $\overline{u}_\lambda$ and satisfies $\Phi(u_\lambda) > 0$. □

Remark 2.2. In connection with multiplicity of solutions, we have known a number of results on the linear diffusion case. Rabinowitz [16] has studied the case $p = 2 < q$ (for example, equations like $\lambda\Delta u + u^2(1-u) = 0$) by combining critical point theory and the Leray-Schauder degree theory, and proved there exists $\Lambda > 0$ such that if $\lambda > \Lambda$, then $(\mathrm{P})_\lambda$ has no solution and if $\lambda < \Lambda$, then $(\mathrm{P})_\lambda$ has at least two distinct solutions (see also Ambrosetti and Rabinowitz [1], and Rabinowitz [17]). Particularly, when Ω is a ball, Ouyang and Shi [14] have obtained a precise global bifurcation diagram and concluded that there exist *exactly* two solutions for small λ by using a bifurcation theorem of Crandall and Rabinowitz.

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