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BOUNDING THE NUMBER OF CYCLES OF O.D.E.S IN $\mathbf{R}^n$

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ABSTRACT. Criteria are given under which the boundary of an oriented surface does not consist entirely of trajectories of the C^1 differential equation $\dot{x} = f(x)$ in $\mathbf{R}^n$. The special case of an annulus is further considered, and the criteria are used to deduce sufficient conditions for the differential equation to have at most one cycle. A bound on the number of cycles on surfaces of higher connectivity is given by similar conditions.

1. INTRODUCTION

The classical Bendixson-Dulac criterion (see, e.g., [6, Thm. 3.5.16]) for ruling out cycles of a planar C^1 differential equation

$$\dot{x} = f(x), \quad x \in \mathbf{R}^2 \tag{1}$$

uses the fact that any simple closed piecewise C^1 curve in $\mathbf{R}^2$ bounds a simply connected region $\mathcal{S}$. Stokes' Theorem (or Green's Theorem) in the plane is applied to the region $\mathcal{S}$ to find criteria under which the boundary curve does not consist entirely of trajectories of equation (1). In particular, this means that the boundary curve is not a cycle (*i.e.* periodic orbit, homoclinic cycle or heteroclinic cycle) of equation (1). Similar criteria can also be used to rule out cycles from the interior of $\mathcal{S}$, since any cycle in $\mathcal{S}$ will bound a simply connected subregion of $\mathcal{S}$, to which Stokes' Theorem can again be applied.

Now consider a region in $\mathbf{R}^2$ that is not simply connected: for example, an annulus $\mathcal{A}$, bounded by two simple closed piecewise C^1 curves. Then Stokes' Theorem can be applied to $\mathcal{A}$ to find criteria under which the boundary curves do not both consist entirely of trajectories of equation (1). Similar criteria can then be found under which the annulus $\mathcal{A}$ contains at most one cycle of equation (1), since any pair of cycles in $\mathcal{A}$ will bound either an annular subregion of $\mathcal{A}$ or at least one simply connected subregion of $\mathcal{A}$, to which Stokes' Theorem can again be applied. See Farkas [6, Cor. 3.5.17] for more details, and Lloyd [9] for results relating the maximum number of cycles to the connectivity of a region.

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In [2] and [3] Busenberg and van den Driessche generalize the classical Bendixson-Dulac criterion to a simple closed piecewise C^1 curve bounding a simply connected region $\mathcal{S}$ of a C^1 oriented surface (two-dimensional manifold) in $\mathbf{R}^3$. Given a vector field f on $\mathbf{R}^3$, Stokes' Theorem in $\mathbf{R}^3$ is applied to $\mathcal{S}$ to find criteria under which the positively oriented boundary curve does not consist entirely of trajectories of $\dot{x} = f(x)$. Once again similar criteria can also be used to rule out positively oriented cycles from the interior of $\mathcal{S}$. For applications of the three-dimensional result to models in epidemiology and ecology, see [15] and the references therein; and for related results, see [3, 4, 5, 7, 10, 11, 13].

In [12] Pace and Zeeman further generalize the classical result to a simple closed piecewise C^1 curve bounding a simply connected region $\mathcal{S}$ of an oriented C^1 surface in $\mathbf{R}^n$. The method is the same as that of Busenberg and van den Driessche, but re-written in the language of exterior calculus in order to apply Stokes' Theorem to surfaces in $\mathbf{R}^n$.

In this paper we consider a differential equation

$$\dot{x} = f(x), \quad \text{where } x \in \mathbf{R}^n \text{ and } f : \mathbf{R}^n \to \mathbf{R}^n \text{ is } C^1, \tag{2}$$

and a union of simple closed piecewise C^1 curves bounding a region $\mathcal{S}$ of an oriented C^1 surface in $\mathbf{R}^n$, where $\mathcal{S}$ is not necessarily simply connected. We apply Stokes' Theorem to the region $\mathcal{S}$ to find criteria under which the (oriented) boundary of $\mathcal{S}$ does not consist entirely of trajectories of equation (2). In Section 2 we prove our main theorem (Theorem 2.2) and some corollaries. In Section 3 we consider the special case when $\mathcal{S}$ is an annulus in $\mathbf{R}^n$, and present a three-dimensional example.

In Section 4 we find a bound on the number of cycles in $\mathcal{S}$, in terms of the connectivity of $\mathcal{S}$.

2. Ruling out cycles

First some definitions are given. See Boothby [1] or Spivak [14] for background material on exterior calculus on manifolds. An *oriented C^1 surface* in $\mathbf{R}^n$ is a C^1 orientable two-dimensional manifold with boundary on which a choice of orientation has been made. Throughout the paper, the *boundary* of a surface means the boundary in the sense of manifolds. The orientation chosen on $\mathcal{S}$ induces an orientation on the boundary $\partial\mathcal{S}$ of $\mathcal{S}$.

Let $g = (g_1, \dots, g_n)^T$ be a vector field on $\mathbf{R}^n$. Let $\mathbf{R}^{n*}$ denote the dual space to $\mathbf{R}^n$, and let $dx_1, \dots, dx_n$ be the basis of $\mathbf{R}^{n*}$ dual to the standard basis of $\mathbf{R}^n$. Then the 1-form *dual* to g is $\omega = g_1 dx_1 + \ldots + g_n dx_n$, and the exterior derivative $d\omega$ of ω is a 2-form on $\mathbf{R}^n$.

Remark 2.1. When $n = 3$,

$$d\omega = \left(\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3}\right) dx_2 dx_3 - \left(\frac{\partial g_3}{\partial x_1} - \frac{\partial g_1}{\partial x_3}\right) dx_3 dx_1 + \left(\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2,$$

which is the 2-form dual to $\operatorname{curl} g$ via Hodge duality. So $\int_{\mathcal{S}} d\omega = \int_{\mathcal{S}} \operatorname{curl} g \cdot h$ where h is the unit normal to $\mathcal{S}$ induced by the orientation on $\mathcal{S}$.

The statement and proof of Theorem 2.2 and its corollaries follow the same simple style as those of the classical Bendixson-Dulac theorem (as in [6, Thm. 3.5.16]) and its generalizations by Busenberg and van den Driessche [2, 3]. See especially Pace and Zeeman [12].

Theorem 2.2. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 vector field, and let $\mathcal{S}$ be a compact oriented C^1 surface with piecewise C^1 boundary $\partial\mathcal{S}$ in $\mathbf{R}^n$. If there exists a C^1 vector field $g : \mathcal{S} \to \mathbf{R}^n$, such that*

(A) $g \cdot f \geq 0 \quad (\leq 0)$ *on* $\partial\mathcal{S}$,

(B) $\displaystyle\int_{\mathcal{S}} d\omega \leq 0 \quad (\geq 0)$, *where ω is the dual 1-form to g,*

(C) *either $g \cdot f \neq 0$ on an open arc in $\partial\mathcal{S}$, or* $\displaystyle\int_{\mathcal{S}} d\omega \neq 0$,

then $\partial\mathcal{S}$ does not consist entirely of trajectories of $\dot{x} = f(x)$ traversed in the positive direction with respect to the orientation on $\mathcal{S}$.

Proof. Suppose, for contradiction, that $\partial\mathcal{S}$ does consist of trajectories of $\dot{x} = f(x)$, all traversed in the positive direction with respect to the orientation on $\mathcal{S}$. Then $\partial\mathcal{S}$ can be parameterised by a union, $\gamma(t)$, of trajectories of $\dot{x} = f(x)$, so that $\gamma' = f$. Thus

$$\begin{aligned} 0 &\leq \int_{\partial\mathcal{S}} g \cdot f \quad \text{by hypothesis (A)} \\ &= \int_{\gamma} g \cdot \gamma' \quad \text{by assumption} \\ &= \int_{\gamma} \omega \\ &= \int_{\mathcal{S}} d\omega \quad \text{by Stokes' Theorem} \\ &\leq 0 \quad \text{by hypothesis (B).} \end{aligned}$$

This contradicts hypothesis (C).

The case when inequalities (A) and (B) are both reversed follows similarly. □

The choice of parametrisation may imply that $\int_\gamma g \cdot \gamma'$ is a sum of indefinite integrals. Nevertheless the compactness and smoothness assumptions ensure that $\int_\gamma g \cdot \gamma'$ is defined.

As in [12], we can develop some rough geometric intuition for this result by thinking of $d\omega$ as a generalization of $\operatorname{curl} g$ when $n > 3$. Then hypothesis (B) states that the overall rotation of g is negative with respect to the orientation on $\mathcal{S}$, while hypothesis (A) states that trajectories of $\dot{x} = f(x)$ follow the same overall direction as g around $\partial\mathcal{S}$. Therefore at least one trajectory of f on $\partial\mathcal{S}$ is traversed in the negative direction with respect to the orientation on $\mathcal{S}$.

Note that Theorem 2.2 does not rule out the possibility that $\partial\mathcal{S}$ consists of trajectories of f, some traversed in the positive direction with respect to the orientation on $\mathcal{S}$, but at least one traversed in the negative direction with respect to the orientation on $\mathcal{S}$. In the following corollaries of Theorem 2.2 we strengthen the conclusion by considering the special cases when either $\int_{\mathcal{S}} d\omega = 0$, or $g \cdot f \equiv 0$ on $\partial\mathcal{S}$.

Corollary 2.3. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 vector field, and let $\mathcal{S}$ be a compact oriented C^1 surface with piecewise C^1 boundary $\partial\mathcal{S}$ in $\mathbf{R}^n$. If there exists a C^1 vector field $g : \mathcal{S} \to \mathbf{R}^n$, such that*

(A) $g \cdot f \geq 0 \quad (\leq 0)$ *on $\partial\mathcal{S}$, and $g \cdot f \neq 0$ on an open arc in $\partial\mathcal{S}$,*

(B) $\int_{\mathcal{S}} d\omega = 0$, *where ω is the dual 1-form to g,*

then $\partial\mathcal{S}$ does not consist of trajectories of $\dot{x} = f(x)$ that are either all traversed in the positive direction with respect to the orientation on $\mathcal{S}$, or all traversed in the negative direction with respect to the orientation on $\mathcal{S}$.

Proof. The proof of Corollary 2.3 is similar to that of Theorem 2.2. In this case, we assume that $\partial\mathcal{S}$ consists of trajectories of $\dot{x} = f(x)$, all traversed in the same direction with respect to the orientation on $\mathcal{S}$. Then $\partial\mathcal{S}$ can be parameterised by γ, where either $\gamma' = f$ on γ, or $\gamma' = -f$ on γ, and Stokes' Theorem leads to a contradiction, as in the proof of Theorem 2.2. $\square$

Corollary 2.4. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 vector field, and let $\mathcal{S}$ be a compact oriented C^1 surface with piecewise C^1 boundary $\partial\mathcal{S}$ in $\mathbf{R}^n$. If there exists a C^1 vector field $g : \mathcal{S} \to \mathbf{R}^n$, such that*

(A) $g \cdot f \equiv 0$ *on $\partial\mathcal{S}$,*

(B) $\int_{\mathcal{S}} d\omega \neq 0$, *where ω is the dual 1-form to g,*

then $\partial\mathcal{S}$ does not consist entirely of trajectories of $\dot{x} = f(x)$.

Proof. Assume that $\partial\mathcal{S}$ consists of trajectories of $\dot{x} = f(x)$, without restricting the direction in which each trajectory is traversed. Then $\partial\mathcal{S}$ can be parameterised by $\gamma(t)$, where $\gamma'(t) = \pm f(\gamma(t))$ for each t. So $g \cdot \gamma' = \pm g \cdot f = 0$ on γ. Thus Stokes' Theorem again leads to a contradiction. $\square$

3. The case of an annulus

We call $\mathcal{A} \in \mathbf{R}^n$ an *annulus* if it is homeomorphic to an annulus in $\mathbf{R}^2$. We now consider the special case of the above results when $\mathcal{S}$ is a compact C^1 oriented annulus $\mathcal{A}$ in $\mathbf{R}^n$, and $\partial\mathcal{A}$ consists of two simple closed piecewise C^1 curves. If we assume that $\partial\mathcal{A}$ consists of two periodic orbits of equation (2), then Figure 1 shows the four possible orientations of the flow around this pair of periodic orbits, relative to the orientation μ on $\mathcal{A}$.

If there exists a vector field $g : \mathbf{R}^n \to \mathbf{R}^n$ satisfying the hypotheses of Theorem 2.2, then the boundary circles of $\mathcal{A}$ are not both periodic orbits of equation (2) traversed in the positive direction with respect to the orientation on $\mathcal{A}$, and Figure 1(a) is ruled out. Note that Theorem 2.2 does not rule out Figures 1(b)–(d), since at least one of the boundary periodic orbits is traversed in the negative direction with respect to the orientation on $\mathcal{A}$.

If g satisfies the hypotheses of Corollary 2.3, then Figures 1(a) and 1(b) are both ruled out, but Figures 1(c) and 1(d) are still possible. If g satisfies the hypotheses of Corollary 2.4, then ∂S does not consist entirely of trajectories of equation (2), so Figures 1(a)–1(d) are all ruled out. Clearly, similar figures are ruled out by Theorem 2.2–Corollary 2.4 when $\partial\mathcal{A}$ consists of oriented cycles rather than periodic orbits.

In the following result, the hypotheses are stronger that those of Corollary 2.4, so that Corollary 2.4 can be applied to any subregion of $\mathcal{A}$ with piecewise C^1 boundary. We can then control the number of cycles in the interior of $\mathcal{A}$, as well as on $\partial\mathcal{A}$, so that there is at most one cycle of equation (2) in the annulus $\mathcal{A}$.

Corollary 3.1. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 vector field, and let $\mathcal{A}$ be a compact oriented C^1 annulus with simple closed piecewise C^1 boundary curves $\partial\mathcal{A}$ in $\mathbf{R}^n$. If there exists a C^1 vector field $g : \mathcal{A} \to \mathbf{R}^n$, such that*

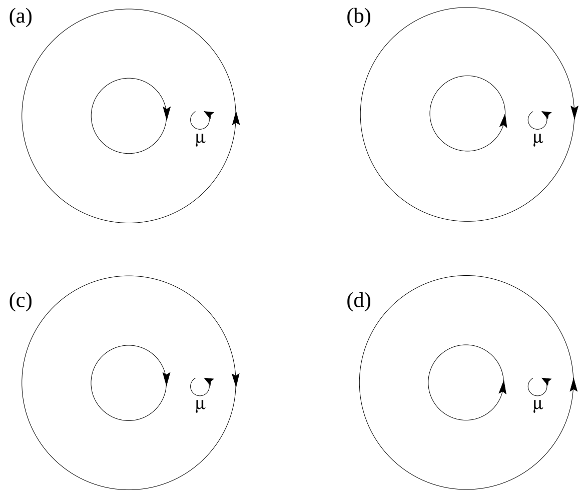

FIGURE 1. $\partial\mathcal{A}$ consists of periodic orbits, along which the flow is oriented (as shown by the arrows) (a) positively, (b) negatively, (c),(d) in opposite directions relative to the orientation μ on $\mathcal{A}$. The orientation μ is represented by a counterclockwise rotation on $\mathcal{A}$, and corresponds to a positive unit normal vector pointing out of the page, towards the reader.

(A) $g \cdot f \equiv 0$ *on* $\mathcal{A}$,

(B) *for any open region* Ω *on* $\mathcal{A}$, $\displaystyle\int_\Omega d\omega \neq 0$, *where* ω *is the dual 1-form to* g,

then $\dot{x} = f(x)$ *has at most one cycle on* $\mathcal{A}$. *Moreover, if there does exist a cycle, then it does not bound a simply connected region of* $\mathcal{A}$.

Proof. Suppose that $\dot{x} = f(x)$ has two disjoint cycles in $\mathcal{A}$. Then either these cycles bound an annular subregion of $\mathcal{A}$, or at least one of the cycles bounds a simply connected subregion of $\mathcal{A}$. In either case, call this subregion Ω. Then hypothesis (A) gives $g \cdot f \equiv 0$ on $\partial\Omega$. Now apply Corollary 2.4 to Ω. In the case when Ω is an annulus, Corollary 2.4 implies that the boundary does not consist entirely of cycles, leading to a contradiction. In the case when Ω is simply connected, Corollary 2.4 implies that $\partial\Omega$ is not a cycle. Thus, in either case, there is at most one cycle in $\mathcal{A}$ and the result follows. $\square$

As an example, we consider a variation of Example 4.5 of [8].

Example 3.2. Consider the C^1 system $\dot{x} = f(x)$, where $x \in \mathbf{R}^3$ and $f : \mathbf{R}^3 \to \mathbf{R}^3$, given by:

$$\begin{aligned}
\dot{x_1} &= x_2 w(x_3) + (a^2 - x_1^2 - x_2^2)x_1,\\
\dot{x_2} &= -x_1 w(x_3) + (a^2 - x_1^2 - x_2^2)x_2,\\
\dot{x_3} &= u(x_1, x_2) - v(x_3),
\end{aligned}$$

where $a > 0$, $w(x_3) \neq 0$ and $v'(x_3) > 0$. In addition, we assume that $k \leq u(x_1, x_2) \leq m$ and that $\exists\ p, q$ with $p < q$ such that $v(x_3) < k$ for $x_3 \leq p$, and $v(x_3) > m$ for $x_3 \geq q$.

Note that for this system the x_3-axis is invariant. From the first two equations

$$(x_1^2+x_2^2)^{\cdot} = 2(x_1\dot{x_1}+x_2\dot{x_2}) = 2(a^2-x_1^2-x_2^2)(x_1^2+x_2^2),$$

so the cylinder $\mathcal{C}$ given by $x_1^2+x_2^2=a^2$ attracts on $\mathbf{R}^3\setminus\{x_3\text{-axis}\}$. We can therefore restrict our attention to $\mathcal{C}$. Since $\dot{x_3}>0$ on $\mathcal{C}$ for $x_3\le p$, and $\dot{x_3}<0$ on $\mathcal{C}$ for $x_3\ge q$, the region $\mathcal{A}$ on $\mathcal{C}$ given by $p\le x_3\le q$ is a compact attracting annulus, and thus contains a fixed point or a periodic orbit. But since $w(x_3)\neq 0$, there is no fixed point on $\mathcal{C}$. Hence $\mathcal{A}$ contains at least one periodic orbit.

We now apply Corollary 3.1 to show that $\mathcal{A}$ contains a unique periodic orbit that encircles the x_3-axis. The orientation of $\mathcal{A}$ is chosen to have unit normal $h=\frac{1}{|a|}(x_1,x_2,0)^T$. Define g on $\mathbf{R}^3\setminus\{x_3\text{-axis}\}$ by

$$g\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\frac{-x_2}{x_1^2+x_2^2}(u(x_1,x_2)-v(x_3))\\ \frac{x_1}{x_1^2+x_2^2}(u(x_1,x_2)-v(x_3))\\ w(x_3)\end{pmatrix}.$$

Then $g\cdot f\equiv 0$ on $\mathbf{R}^3\setminus\{x_3\text{-axis}\}$, and $\operatorname{curl} g\cdot h=\frac{1}{|a|}v'(x_3)>0$ on $\mathcal{C}$. Thus, by Remark 2.1, conditions (A) and (B) of Corollary 3.1 are satisfied, and so the system has a unique periodic orbit on $\mathcal{A}$. Moreover, that periodic orbit does not bound a simply connected region of $\mathcal{A}$, and hence it encircles the x_3-axis. Note that g is singular on the x_3-axis; this singularity is necessary in using Corollary 3.1 to show that the system has at most one cycle on $\mathcal{A}$ without ruling out all cycles.

We remark that the hypotheses on $u(x_1,x_2)$ and $w(x_3)$ can be weakened. We require only that $u(x_1,x_2)$ is bounded on the circle $x_1^2+x_2^2=a^2$, and that if $w(r)=0$ for $p\le r\le q$, then $u(x_1,x_2)\neq v(r)$ on the circle. This last condition ensures that there is no fixed point on $\mathcal{A}$.

In the special case when $a=1, u(x_1,x_2)=\beta(x_1+x_2)$ and $v(x_3)=\gamma x_3$, with $\beta,\gamma>0$, this example reduces to a special case of Example 4.5 of [8] with $\alpha=1$. These choices of $u(x_1,x_2)$ and $v(x_3)$ satisfy our assumptions with the region $\mathcal{A}$ on $\mathcal{C}$ given by $|x_3|\le\sqrt{2}\frac{\beta}{\gamma}$. In [8] the existence of a unique limit cycle in an invariant toroidal region is shown by using the second additive compound.

4. Bounding the number of cycles

In this section we generalize Corollary 3.1 to find conditions that bound the number of cycles of equation (2) on surfaces of higher connectivity in $\mathbf{R}^n$. Let $\mathcal{S}$ be a compact connected oriented C^1 surface with N boundary components. Define the *number of holes* K of $\mathcal{S}$ to be the maximum number of disjoint simple closed curves, γ_i, that can lie on $\mathcal{S}$ such that no subset of $\{\gamma_i\}$ bounds an open region of $\mathcal{S}\setminus\partial\mathcal{S}$.

In the case when $N=0$, $\mathcal{S}$ is homeomorphic to a sphere with handles [1, Theorem 4.1], and K is the genus of $\mathcal{S}$. When $\mathcal{S}\in\mathbf{R}^2$, $K=N-1$, and Corollary 4.1 is similar to Theorem 1 of [9]. In the general case with $N>0$, we derive a connected oriented two-manifold $\hat{\mathcal{S}}$ from $\mathcal{S}$ by identifying each component of $\partial\mathcal{S}$ with a point. Then $\mathcal{S}$ has K holes, where $K=G+N-1$ and G is the genus of $\hat{\mathcal{S}}$.

Corollary 4.1. *Let $f:\mathbf{R}^n\to\mathbf{R}^n$ be a C^1 vector field, and let $\mathcal{S}$ be a compact connected oriented C^1 surface in $\mathbf{R}^n$ with K holes. If there exists a C^1 vector field $g:\mathcal{S}\to\mathbf{R}^n$, such that*

(A) $g\cdot f\equiv 0$ *on* $\mathcal{S}$,

(B) *for any open region Ω on $\mathcal{S}$, $\int_\Omega d\omega \neq 0$, where ω is the dual 1-form to g,*

then $\dot{x} = f(x)$ has at most K disjoint cycles on $\mathcal{S}$.

Proof. Suppose that $\dot{x} = f(x)$ has $K + 1$ disjoint cycles on $\mathcal{S}$. Let Γ denote the union of these cycles. Since $\mathcal{S}$ has K holes, there exists an open region Ω of $\mathcal{S} \setminus \partial\mathcal{S}$ such that $\partial\Omega \subset \Gamma$. Now apply Corollary 2.4 to $\Omega \cup \partial\Omega$ to show that $\partial\Omega$ does not consist entirely of trajectories of f, and hence reach a contradiction. □

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