PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 2, Pages 429–432
S 0002-9939(00)05748-8
Article electronically published on August 28, 2000

PSEUDO-ADVECTION METHOD FOR THE TWO-DIMENSIONAL STATIONARY EULER EQUATIONS

TAKAHIRO NISHIYAMA

(Communicated by David S. Tartakoff)

Abstract. The existence of generalized solutions to the two-dimensional stationary Euler equations with nonvanishing vorticity is proved by a new method completely different from the usual variational approaches.

The two-dimensional steady flow of an inviscid incompressible fluid under no external force effect is described by the stationary Euler equations

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p\,, \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2}$$

where $\boldsymbol{u} = \boldsymbol{u}(x,y)$ and $p = p(x,y)$ represent the velocity and pressure, respectively, of the fluid filling the domain Ω ($\subset \mathbf{R}^2$). Let J be the 2×2 matrix denoting $\pi/2$ counterclockwise rotation. Then $(J\nabla)\cdot\boldsymbol{u}$ $(= -\nabla\cdot(J\boldsymbol{u}))$ gives the vorticity. If Ω has the boundary Γ, then

$$\boldsymbol{u}\cdot\boldsymbol{n}\,|_\Gamma = 0 \tag{3}$$

is imposed, where $\boldsymbol{n}$ is the unit outward normal vector on Γ.

Turkington [6], [7] proved the existence of solutions to the problem (1)–(3) with nonvanishing vorticity by the variational method related to the kinetic energy. After that, variational approaches to steady vortical flows were also adopted in some other papers, for example, [1], [2], [4].

In order to obtain solutions to (1)–(3), Vallis et al. [8] proposed the nonstationary Euler equations with 'pseudo-advection' of vorticity (see also [3]). Their equations are written in the form

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \alpha\omega J\boldsymbol{v}_t = -\nabla q, \tag{4}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{5}$$

with $\boldsymbol{v}\cdot\boldsymbol{n}\,|_\Gamma = 0$ and an initial condition, where α is a nonzero constant, $(\boldsymbol{v}, q)$: $\Omega\times\{t>0\}\to\mathbf{R}^2\times\mathbf{R}$ and $\omega = (J\nabla)\cdot\boldsymbol{v}$. They asserted that, since the energy changes monotonically in t and the enstrophy is conserved, $\boldsymbol{v}$ converges to a solution to (1)–(3), as $t\to\infty$, which has the initially given enstrophy. This approach is completely different from the variational ones. However, it is not correct in a strict

Received by the editors April 15, 1999.
2000 *Mathematics Subject Classification.* Primary 35Q30, 76B03.
Key words and phrases. Two-dimensional stationary Euler equations, vorticity, Galerkin method, pseudo-advection.

sense until the existence of a solution to (4) and (5) is proved globally in time. In fact, its proof seems difficult because the nonlinearity of the term $\alpha\omega J\boldsymbol{v}_t$ in (4) is too strong.

The aim of this paper is to combine the idea of Vallis et al. with the Galerkin method and prove the existence of generalized solutions to (1)–(3) with nonvanishing vorticity for an arbitrary Ω. By letting the number of basis functions in the Galerkin method go to infinity at the same time as $t \to \infty$, we can evade the difficulty coming from the strong nonlinearity of $\alpha\omega J\boldsymbol{v}_t$ in (4). The author has referred to [5] to apply the Galerkin method.

We should note that the generalized solvability of (1)–(3) was derived from the solvability of the barotropic quasi-geostrophic equation with external flow and dissipation in [9]. However, our method is also completely different from this.

Let us introduce our notation. For scalar functions f, g and vector ones $\boldsymbol{f}$, $\boldsymbol{g}$ on Ω, we define $\langle f,g\rangle = \int_\Omega fg\,dxdy$ and $\langle \boldsymbol{f},\boldsymbol{g}\rangle = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{g}\,dxdy$. If $f=g$ and $\boldsymbol{f}=\boldsymbol{g}$, then we write their square roots as $\|f\|$ and $\|\boldsymbol{f}\|$, respectively. Let

$$\boldsymbol{X}_0 = \{\boldsymbol{f} : \Omega \to \mathbf{R}^2 \ \mid\ \|\boldsymbol{f}\| < \infty, \quad \nabla\cdot\boldsymbol{f} = 0, \quad \boldsymbol{f}\cdot\boldsymbol{n}|_\Gamma = 0\}$$

with the scalar product $\langle\cdot,\cdot\rangle$. It is a closed subspace of $L^2(\Omega)\times L^2(\Omega)$. We also use

$$\boldsymbol{X}_1 = \boldsymbol{X}_0 \cap \{\boldsymbol{f} \mid \|\boldsymbol{f}_x\| + \|\boldsymbol{f}_y\| < \infty\}$$

with the scalar product defined by $\langle \boldsymbol{f},\boldsymbol{g}\rangle_1 = \langle \boldsymbol{f},\boldsymbol{g}\rangle + \langle \boldsymbol{f}_x,\boldsymbol{g}_x\rangle + \langle \boldsymbol{f}_y,\boldsymbol{g}_y\rangle$.

Definition. If $\boldsymbol{u} \in \boldsymbol{X}_1$ satisfies

$$\langle (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{\Phi}\rangle = 0 \tag{6}$$

for any $\boldsymbol{\Phi} \in \boldsymbol{X}_1$, then we say that $\boldsymbol{u}$ is a *generalized solution* to (1)–(3).

The following is our main theorem.

Theorem. *Let Ω $(\subset \mathbf{R}^2)$ be an arbitrary, bounded and simply connected domain with the sufficiently smooth boundary Γ and let $\boldsymbol{v}_0$ be an arbitrary function in $\boldsymbol{X}_1$ which does not satisfy (6). Then (1)–(3) has at least two generalized solutions $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{X}_1$ such that $\|(J\nabla)\cdot\boldsymbol{u}_1\| = \|(J\nabla)\cdot\boldsymbol{u}_2\| = \|(J\nabla)\cdot\boldsymbol{v}_0\|$ and $\|\boldsymbol{u}_1\| > \|\boldsymbol{v}_0\| > \|\boldsymbol{u}_2\|$.*

Proof. Let $\{\psi^{(k)} \in C^2(\Omega)\cap C^1(\bar{\Omega})\}_{k=1}^\infty$ be eigenfunctions of $-\Delta\psi^{(k)} = \mu_k\psi^{(k)}$ $(0 < \mu_1 < \mu_2 \le \mu_3 \le \ldots \to \infty)$ in Ω with $\psi^{(k)}|_\Gamma = 0$ and set $\boldsymbol{w}^{(k)} = -J(\nabla\psi^{(k)})/\|\nabla\psi^{(k)}\|$. Then $\{\boldsymbol{w}^{(k)}\}_{k=1}^\infty$ is a complete orthonormal system in $\boldsymbol{X}_0$. We consider the equations for $g_k^n(t)$ $(k = 1, 2, \ldots, n)$:

$$\boldsymbol{v}^{(n)} = \sum_{j=1}^n g_j^n(t)\boldsymbol{w}^{(j)}(x,y), \tag{7}$$

$$\langle (1+\alpha\omega^{(n)}J)\boldsymbol{v}_t^{(n)}, \boldsymbol{w}^{(k)}\rangle = -\langle (\boldsymbol{v}^{(n)}\cdot\nabla)\boldsymbol{v}^{(n)}, \boldsymbol{w}^{(k)}\rangle, \tag{8}$$

$$g_k^n(0) = \langle \boldsymbol{v}_0, \boldsymbol{w}^{(k)}\rangle. \tag{9}$$

Here α is a fixed positive or negative number and $\omega^{(n)} = (J\nabla)\cdot\boldsymbol{v}^{(n)}$. Since the $n\times n$ matrix whose (k,j)-component is defined by $\langle \alpha\omega^{(n)}J\boldsymbol{w}^{(j)}, \boldsymbol{w}^{(k)}\rangle$ is anti-symmetric and all real parts of its eigenvalues are equal to zero, the system (7)–(9) is uniquely solvable on some time interval.

Applying Green's formula to (8), we obtain

$$\langle \omega_t^{(n)} + (\boldsymbol{v}^{(n)} + \alpha \boldsymbol{v}_t^{(n)}) \cdot \nabla \omega^{(n)}, \psi^{(k)} \rangle = 0. \tag{10}$$

Note that

$$\omega^{(n)} = -\sum_{j=1}^{n} g_j^n(t) \Delta \psi^{(j)} / \|\nabla \psi^{(j)}\| = \sum_{j=1}^{n} \sqrt{\mu_j} g_j^n(t) \psi^{(j)} / \|\psi^{(j)}\|.$$

Then, by summing up the products of (10) and $\sqrt{\mu_k} g_k^n(t)/\|\psi^{(k)}\|$ from $k = 1$ to n, we have $(d/dt)\|\omega^{(n)}\|^2 = 0$, or

$$\|\omega^{(n)}\| = \|(J\nabla) \cdot \boldsymbol{v}_0^{(n)}\| \leq \|(J\nabla) \cdot \boldsymbol{v}_0\|. \tag{11}$$

Here $\boldsymbol{v}_0^{(n)} = \sum_{j=1}^n g_j^n(0) \boldsymbol{w}^{(j)}$. In virtue of $\|\omega^{(n)}\|^2 = \sum_{j=1}^n \mu_j (g_j^n(t))^2 \geq \mu_1 \|\boldsymbol{v}^{(n)}\|^2$ and the well-known inequality $\|\nabla \psi_x^{(j)}\| + \|\nabla \psi_y^{(j)}\| \leq \text{const.}\|\Delta \psi^{(j)}\|$, the estimate (11) implies that $\boldsymbol{v}^{(n)}$ is bounded in $\boldsymbol{X}_1$ uniformly in t and n, and (7)–(9) is solvable globally in time. Furthermore, there exists a subsequence $\{n_\nu \in \mathbf{N} \mid n_1 < n_2 < \ldots < n_\nu < \ldots\}$ such that $\boldsymbol{v}^{(n_\nu)}$ converges strongly in $\boldsymbol{X}_0$ and weakly in $\boldsymbol{X}_1$ as $\nu \to \infty$ for any rational t.

Multiplying (8) by $g_k^n(t)$ and $(d/dt) g_k^n(t)$, we have, respectively,

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{v}^{(n)}\|^2 = -\alpha \langle \omega^{(n)} J \boldsymbol{v}_t^{(n)}, \boldsymbol{v}^{(n)} \rangle,$$

$$\|\boldsymbol{v}_t^{(n)}\|^2 = -\langle (\boldsymbol{v}^{(n)} \cdot \nabla) \boldsymbol{v}^{(n)}, \boldsymbol{v}_t^{(n)} \rangle.$$

Since $(\boldsymbol{v}^{(n)} \cdot \nabla) \boldsymbol{v}^{(n)} = \omega^{(n)} J \boldsymbol{v}^{(n)} + \nabla |\boldsymbol{v}^{(n)}|^2/2$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{v}^{(n)}\|^2 = -\alpha \|\boldsymbol{v}_t^{(n)}\|^2. \tag{12}$$

This means that $\|\boldsymbol{v}^{(n)}\|$ for $\alpha < 0$ ($\alpha > 0$) is monotonically increasing (decreasing) in t. First, let us consider the case $\alpha < 0$. Since

$$0 \leq -2\alpha \int_0^t \|\boldsymbol{v}_t^{(n)}\|^2 dt = \|\boldsymbol{v}^{(n)}\|^2 - \|\boldsymbol{v}_0^{(n)}\|^2 \leq \frac{1}{\mu_1} \|(J\nabla) \cdot \boldsymbol{v}_0\|^2 \tag{13}$$

follows from (11) and (12), $\|\boldsymbol{v}_t^{(n)}\|^2$ is integrable over $(0, \infty)$. Therefore, there exists a sequence $\{t_\nu \mid 0 < t_1 < t_2 < \ldots < t_\nu < \ldots \to \infty\}$ such that $\|\boldsymbol{v}_t^{(n_\nu)}(\cdot, \cdot, t_\nu)\| \to 0$ as $\nu \to \infty$. Then, setting $n = n_\nu$, $t = t_\nu$ in (8) and using (11), the continuity of $\boldsymbol{w}^{(k)}$ on $\bar{\Omega}$ and the convergence of $\boldsymbol{v}^{(n_\nu)}$, we have $\langle (\boldsymbol{u} \cdot \nabla) \boldsymbol{u}, \boldsymbol{w}^{(k)} \rangle = 0$ for $\boldsymbol{u} = \lim_{\nu \to \infty} \boldsymbol{v}^{(n_\nu)}(x, y, t_\nu)$ and any $k \in \mathbf{N}$. Hence $\boldsymbol{u}$ is a generalized solution to (1)–(3) because $\boldsymbol{X}_1$ is dense in $\boldsymbol{X}_0$ spanned by $\{\boldsymbol{w}^{(k)}\}_{k=1}^\infty$. It is clear from (11) that $\|(J\nabla) \cdot \boldsymbol{u}\| = \|(J\nabla) \cdot \boldsymbol{v}_0\|$.

We derive $\|\boldsymbol{u}\| \geq \|\boldsymbol{v}_0\|$ from (13). Now, assume that $\|\boldsymbol{u}\| = \|\boldsymbol{v}_0\|$. Then we have $\lim_{\nu \to \infty} \int_0^{t_\nu} \|\boldsymbol{v}_t^{(n_\nu)}\|^2 dt = 0$, which leads to $\lim_{\nu \to \infty} \int_0^T \|\boldsymbol{v}_t^{(n_\nu)}\|^2 dt = 0$ for any finite $T > 0$. Since

$$\|\boldsymbol{v}^{(n_\nu)} - \boldsymbol{v}_0^{(n_\nu)}\| \leq \int_0^t \|\boldsymbol{v}_t^{(n_\nu)}\| dt \leq \left(t \int_0^t \|\boldsymbol{v}_t^{(n_\nu)}\|^2 dt \right)^{1/2} \to 0 \quad \text{as } \nu \to \infty,$$

the function defined by $\lim_{\nu \to \infty} \boldsymbol{v}^{(n_\nu)}(x, y, t)$ is equal to $\boldsymbol{v}_0$ for not only rational but also all $t \in [0, T]$. Integrating (8) over $(0, T)$ and letting $n = n_\nu \to \infty$, we deduce $\langle (\boldsymbol{v}_0 \cdot \nabla) \boldsymbol{v}_0, \boldsymbol{w}^{(k)} \rangle = 0$. This conflicts with the assumption that $\boldsymbol{v}_0$ does not satisfy (6). Therefore, $\|\boldsymbol{u}\| > \|\boldsymbol{v}_0\|$ holds.

The case $\alpha > 0$ is discussed in the same way and the existence of another generalized solution to (1)–(3) is proved. Indeed, instead of (13),

$$\|\boldsymbol{v}^{(n)}\|^2 + 2\alpha \int_0^t \|\boldsymbol{v}_t^{(n)}\|^2 dt = \|\boldsymbol{v}_0^{(n)}\|^2 \le \|\boldsymbol{v}_0\|^2 \tag{14}$$

is deduced from (12). In this case, we have $\|\boldsymbol{u}\| < \|\boldsymbol{v}_0\|$. □

The theorem implies the following. If there exists a function $\widetilde{\boldsymbol{u}} \in \boldsymbol{X}_1$ such that $\|(J\nabla)\cdot\widetilde{\boldsymbol{u}}\| = \|(J\nabla)\cdot\boldsymbol{v}_0\|$ and $\|\widetilde{\boldsymbol{u}}\| > \|\boldsymbol{u}_1\|$ (resp. $\|\widetilde{\boldsymbol{u}}\| < \|\boldsymbol{u}_2\|$), then we can obtain another generalized solution $\boldsymbol{u}_3$ such that $\|(J\nabla)\cdot\boldsymbol{u}_3\| = \|(J\nabla)\cdot\boldsymbol{v}_0\|$ and $\|\boldsymbol{u}_3\| > \|\boldsymbol{u}_1\|$ (resp. $\|\boldsymbol{u}_3\| < \|\boldsymbol{u}_2\|$) by replacing $\boldsymbol{v}_0$ in the theorem by $\widetilde{\boldsymbol{u}}$. We can repeat this process until we cannot find such a function $\widetilde{\boldsymbol{u}}$.

Remark 1. If Ω is bounded and multiply connected, then we can prove that there exists at least a generalized solution to (1)–(3) as follows. Assume that Γ consists of the simple closed curves $\Gamma_1, \Gamma_2, \dots, \Gamma_{N+1}$ ($N \ge 1$) and $\phi^{(i)} \in C^2(\bar{\Omega})$ ($i = 1, 2, \dots, N$) are the harmonic functions satisfying $\phi^{(i)}|_{\Gamma_i} = 1$ and $\phi^{(i)}|_{\Gamma - \Gamma_i} = 0$. Then, by adding the N functions $-J(\nabla\phi^{(i)})/\|\nabla\phi^{(i)}\|$ to the above $\{\boldsymbol{w}^{(k)}\}_{k=1}^{\infty}$, we have a complete orthonormal system in $\boldsymbol{X}_0$. Therefore, the same argument as in the proof of the theorem is possible if $\alpha > 0$.

Remark 2. We can prove the existence of at least a generalized solution also for an unbounded Ω by assuming $\alpha > 0$. Indeed, the set of all smooth functions that vanish near infinity is dense in $\boldsymbol{X}_1$ and the estimates (11) and (14) do not depend on the size of Ω.

The name 'pseudo-advection method' in the title is from (10). This method is also applicable to the axisymmetric stationary Euler equations, which will be discussed in another paper.

References

[1] T. V. Badiani, *Existence of steady symmetric vortex pairs on a planar domain with an obstacle*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 365–384. MR **98m:**76029

[2] G. R. Burton, *Steady symmetric vortex pairs and rearrangements*, Proc. Roy. Soc. Edinburgh **108A** (1988), 269–290. MR **89f:**35178

[3] G. F. Carnevale and G. K. Vallis, *Pseudo-advective relaxation to stable states of inviscid two-dimensional fluids*, J. Fluid Mech. **213** (1990), 549–571. MR **91c:**76058

[4] A. R. Elcrat and K. G. Miller, *Rearrangements in steady vortex flows with circulation*, Proc. Amer. Math. Soc. **111** (1991), 1051–1055. MR **91g:**35205

[5] R. Temam, *On the Euler equations of incompressible perfect fluids*, J. Funct. Anal. **20** (1975), 32–43. MR **55:**3573

[6] B. Turkington, *On steady vortex flow in two dimensions.* I, Comm. Partial Differential Equations **8** (1983), 999–1030. MR **85g:**35110

[7] ———, *On steady vortex flow in two dimensions.* II, Comm. Partial Differential Equations **8** (1983), 1031–1071. MR **85g:**35110

[8] G. K. Vallis, G. F. Carnevale, and W. R. Young, *Extremal energy properties and construction of stable solutions of the Euler equations*, J. Fluid Mech. **207** (1989), 133–152. MR **90m:**76026

[9] G. Wolansky, *Existence, uniqueness, and stability of stationary barotropic flow with forcing and dissipation*, Comm. Pure Appl. Math. **41** (1988), 19–46. MR **88k:**35167

Department of Mathematics, Keio University, Yokohama 223–8522, Japan
E-mail address: `nisiyama@math.keio.ac.jp`