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PARAMETER DEPENDENCE OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS IN SPACES OF REAL ANALYTIC FUNCTIONS

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Dedicated to V. P. Zaharjuta on the occasion of his 60th birthday

Abstract. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $\mathrm{A}(\Omega)$ denote the class of real analytic functions on Ω. It is proved that for every surjective linear partial differential operator $P(D,x) : \mathrm{A}(\Omega) \to \mathrm{A}(\Omega)$ and every family $(f_\lambda) \subseteq \mathrm{A}(\Omega)$ depending holomorphically on $\lambda \in \mathbb{C}^m$ there is a solution family $(u_\lambda) \subseteq \mathrm{A}(\Omega)$ depending on λ in the same way such that

$$P(D,x)u_\lambda = f_\lambda, \qquad \text{for } \lambda \in \mathbb{C}^m.$$

The result is a consequence of a characterization of Fréchet spaces E such that the class of "weakly" real analytic E-valued functions coincides with the analogous class defined via Taylor series. An example shows that the analogous assertions need not be valid if $\mathbb{C}^m$ is replaced by another set.

1. Introduction

In the paper [4], the authors proved that for any surjective linear continuous map (operator) $T : \mathrm{A}(\mathbb{R}) \to \mathrm{A}(\mathbb{R})$ and any family $(f_\lambda) \subseteq \mathrm{A}(\mathbb{R})$ depending holomorphically on $\lambda \in U$ there is a parametrized family $(u_\lambda) \subseteq \mathrm{A}(\mathbb{R})$ depending holomorphically on $\lambda \in U$ and

$$Tu_\lambda = f_\lambda \qquad \text{for } \lambda \in U,$$

whenever U is a Stein manifold satisfying the *strong Liouville property*, i.e., every bounded plurisubharmonic function on U is constant. The aim of this paper is to generalize the above result to arbitrary open sets $\Omega \subseteq \mathbb{R}^n$ instead of $\Omega = \mathbb{R}$ (see Corollary 7 below). The proof is similar to that in [4] so we concentrate below on the differences and the paper can be treated as a complement to [4]. We also give in Section 3 some other positive and negative results on a parameter dependence. For other results on a parameter dependence of solutions of functional equations see [10], [11], [24], [25], [26] and [34].

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Let us recall [19] (comp. [4]) that a map $f : \Omega \to E$ with values in a locally convex sequentially complete space E is called *real analytic* if for every $u \in E'$, $u \circ f \in \mathrm{A}(\Omega)$. Analogously, f is called *topologically real analytic* or *bornologically real analytic* if for every $t \in \Omega$, $f(x) = \sum_{j=0}^{\infty} a_j(x-t)^j$ for all x belonging to some neighbourhood of t where the series converges in E or in E_B, resp., where B is a bounded absolutely convex closed set not depending on t. By E_B we denote the Banach space $\mathrm{lin}B$ equipped with the gauge functional of B as its norm. We denote by $\mathrm{A}(\Omega, E)$, $\mathrm{A}_t(\Omega, E)$ and $\mathrm{A}_b(\Omega, E)$, respectively, the classes of all real analytic, topologically real analytic and bornologically real analytic functions $f : \Omega \to E$. It is known [4] that $\mathrm{A}(\Omega, E) = \mathrm{A}_t(\Omega, E)$ for any complete LB-space E as well as $\mathrm{A}_t(\Omega, E) = \mathrm{A}_b(\Omega, E)$ for any Fréchet space E (comp. [1], [3]). The book [20] (and an earlier paper [19]) contains some applications of vector valued real analytic functions.

Our crucial step in the proof of the main result is a characterization of those Fréchet spaces E such that $\mathrm{A}(\Omega, E) = \mathrm{A}_t(\Omega, E)$ (Theorem 3). We also characterize when $\mathrm{A}_t(\Omega, E) = \mathrm{A}_b(\Omega, E)$ for complete LB-spaces E (Theorem 5). These proofs differ slightly from the case $\Omega = \mathbb{R}$ presented in [4] although the characterization is given by the same conditions. We explain the differences below.

We denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$. For every open set U in $\mathbb{C}^n$ or in an arbitrary Stein manifold we consider the space $H(U)$ of all holomorphic functions on U equipped with the compact open topology. Analogously, $H^\infty(U) \subseteq H(U)$ consists of bounded functions and it is a Banach space with the uniform norm. If $K \subseteq \mathbb{C}^n$ is a compact set, we denote by $H(K)$ the space of holomorphic germs equipped with its natural topology of inductive limit $\mathrm{ind}_{n\in\mathbb{N}} H(U_n)$, where (U_n) is a decreasing fundamental sequence of open neighbourhoods of K. Finally, on $\mathrm{A}(\Omega)$ we consider two topologies known to be equal [28, Prop. 1.9, Th. 1.2]: $\mathrm{ind}_U H(U)$, where U runs over all $\mathbb{C}^n$-neighbourhoods of Ω and $\mathrm{proj}\,_K H(K)$, where K runs over all compact subsets of Ω.

Theorem 1 (Martineau [27, 28], comp. [4])**.** *The space $A(\Omega)$ is an ultrabornological projective limit of nuclear LB-spaces and its dual is a complete nuclear LF-space.*

Let X, Y be locally convex spaces. We denote by X'_β the strong dual of X. By $L(X, Y)$ and $LB(X, Y)$ we denote the space of all operators $T : X \to Y$ (i.e., linear continuous maps) and all bounded operators $T : X \to Y$ (i.e., T maps some 0-neighbourhood into a bounded set). A Fréchet space X is called a *quojection* whenever every quotient of X with a continuous norm is a Banach space. By $A \subset\subset B$ we denote that A is relatively compact and its closure is contained in B. In some places we use the theory of the functor Proj^1 as explained in [40] or [39] (comp. also [4, Sec. 7]). For other non-explained notions from functional analysis see [2], [16], [18] and [30], [31]. Similarly, [15] is our reference book from complex analysis, and [17] for real analytic functions.

2. The main results

The following natural extension of [4, Th. 16] (which can be proved similarly) is needed in the sequel.

Theorem 2. *Let E be a sequentially complete locally convex space. The spaces $A(\Omega, E)$ and $A(\Omega)\varepsilon E = L(A(\Omega)'_\beta, E)$ are algebraically isomorphic in a canonical way. Moreover, this isomorphism maps $A_b(\Omega, E)$ onto $LB(A(\Omega)'_\beta, E)$.*

A Fréchet space E with a fundamental sequence of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is said to satisfy the property (DN) (comp. [30], [43] or [37]) if

$$\exists\ n\ \ \forall\ m\geq n\ \ \exists\ l\geq m, C<\infty\ \ \forall\ x\in E:\qquad \|x\|_m^2\leq C\|x\|_n\|x\|_l.$$

Every power series space of infinite type has the property (DN) and a nuclear Fréchet space has the property (DN) if and only if it is isomorphic to a subspace of the space s of rapidly decreasing sequences. We refer the reader to [30], [36], [37], [38] and [43, 2.3] for more information on this important topological invariant.

Theorem 3. *Let E be a Fréchet space. The following assertions are equivalent:*

(a) *E has the property (DN);*
(b) *there is an open set $\Omega\subseteq\mathbb{R}^n$ such that $A(\Omega,E)=A_t(\Omega,E)$;*
(c) *for every open set $\Omega\subseteq\mathbb{R}^n$ we have $A(\Omega,E)=A_t(\Omega,E)$.*

Proof. (a)⇒(c): Using exactly the same proof as in [4, Th. 18], we obtain $A(\mathbb{R}^n,E)$ $=A_t(\mathbb{R}^n,E)$. Since $\mathrm{A}_t(\Omega,E)=\mathrm{A}_b(\Omega,E)$, by Theorem 2 above, we have $\mathrm{A}(\Omega,E)=\mathrm{A}_t(\Omega,E)$ if and only if $L(\mathrm{A}(\Omega)'_\beta,E)=LB(\mathrm{A}(\Omega)'_\beta,E)$. Thus the equality depends only on the isomorphic class of the space $\mathrm{A}(\Omega)$. Let $\varphi:\mathbb{R}^n\to r\mathbb{D}^n$, $r>0$, be a real analytic diffeomorphism. The map

$$C_\varphi: A(r\mathbb{D}^n)\to A(\mathbb{R}^n),\qquad C_\varphi(f)=f\circ\varphi$$

is an isomorphism. Accordingly, $A(r\mathbb{D}^n,E)=A_t(r\mathbb{D}^n,E)$ for every $r>0$.

Let Ω be arbitrary, $f\in\mathrm{A}(\Omega,E)$, $x\in\Omega$. For suitable $r>0$, $g:r\mathbb{D}^n\to E$, $g(y):=f(y-x)$ belongs to $A(r\mathbb{D}^n,E)=A_t(r\mathbb{D}^n,E)$. Hence f develops into the Taylor series around x for any $x\in\Omega$; this completes the proof.

(c)⇒(b): Obvious.

(b)⇒(a): As in the proof of [4, Th. 18], it suffices to show that $\mathrm{A}(\Omega)'_\beta$ has a quotient isomorphic to $H(\mathbb{D}^n)$ (since, by [35, Th. 2.1], E has (DN) if and only if $L(H(\mathbb{D}^n),E)=LB(H(\mathbb{D}^n),E)$). As in [4, Prop. 5] we show that $A(\mathbb{R}^n)$ contains the subspace X of t-periodic functions isomorphic to the LB-space of germs $H(\overline{\mathbb{D}}^n)$. Since $A(\mathbb{R}^n)=\mathrm{proj}_{\,m\in\mathbb{N}}A([-m,m]^n)$, it is easy to see that there is $m\in\mathbb{N}$ such that the topologies of $A(\mathbb{R}^n)$ and $A([-m,m]^n)$ coincide on X. By a suitable change of variable $x\longmapsto rx+x_0$, we may assume that $[-m,m]^n\subseteq\Omega$. Therefore we have restriction maps

$$A(\mathbb{R}^n)\to\mathrm{A}(\Omega)\to A([-m,m]^n)$$

which are isomorphisms when restricted to $X\simeq H(\overline{\mathbb{D}}^n)$. By duality, $\mathrm{A}(\Omega)'_\beta$ has a quotient isomorphic to $H(\overline{\mathbb{D}}^n)'_\beta$. The latter space is isomorphic to $H(\mathbb{D}^n)$ by the Grothendieck-Köthe-Silva duality (see [12], [28], [32] or [43], comp. [4]). This completes the proof. □

Before we prove an analogue of Theorem 3 for LB-spaces we need the following lemma which substitutes [4, Lemma 22]. We start with some notation. For every $f\in H(U)$ we denote $\|f\|_U:=\sup_{z\in U}|f(z)|$. If $T\in L(F,H(U))$, F is a Fréchet space with a fundamental sequence of seminorms $(\|\cdot\|_N)_{N\in\mathbb{N}}$, we set

$$\|T\|_{K,U}:=\sup\{\|Tx\|_U:\|x\|_K\leq 1\}.$$

We recall that F is said to satisfy the property $(\overline{\overline{\Omega}})$ [35] if

$$\forall\, N\ \exists\, M \geq N\ \forall\, L \geq N, \delta \in]0,1[\ \exists\, C < \infty\ \forall\, u \in F' : \|u\|_M^* \leq C(\|u\|_N^*)^{1-\delta}(\|u\|_L^*)^{\delta}.$$

Here $\|\cdot\|_N^*$ denotes the dual norm of $\|\cdot\|_N$, i.e., $\|u\|_N^* := \sup\{|\langle u, x\rangle| : \|x\|_N \leq 1\}$ (see [35] and [29] for examples of spaces with $(\overline{\overline{\Omega}})$).

Lemma 4. *Let F be a Fréchet space with the property $(\overline{\overline{\Omega}})$. Then for any $N \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that for any $M \in \mathbb{N}$, every connected open subset U of a connected real analytic manifold and every triple of open sets $\emptyset \neq U_1 \subseteq U_2 \subset\subset U_3 \subset\subset U$ there is a constant C such that*

$$\|T\|_{K,U_2} \leq C \max(\|T\|_{N,U_1}, \|T\|_{M,U_3})$$

for every operator $T : F \to H(U)$.

Proof. By the property $(\overline{\overline{\Omega}})$, we have

$$\forall\ N\ \exists\ K\ \forall\ M\ \forall\ \theta \in]0,1[\ \exists\ C_2 : \|u\|_K^* \leq C_2\,(\|u\|_N^*)^{1-\theta}\,(\|u\|_M^*)^{\theta}$$

for every $u \in F'$. By the Hadamard Three Circle Theorem (comp. the version in [33, Satz 5.1]) there are $\tau \in (0,1)$ and C_1 such that

$$\|f\|_{U_2} \leq C_1 \|f\|_{U_1}^{1-\tau} \cdot \|f\|_{U_3}^{\tau}$$

for all $f \in H(U)$. We choose θ such that $\tau < \theta < 1$. Now, if $\|T\|_{N,U_1}, \|T\|_{M,U_3} < \infty$, then by the interpolation Lemma [30, 29.17], there is C_3 such that

$$\|T\|_{K,U_2} \leq C_3 \|T\|_{N,U_1}^{1-\tau} \cdot \|T\|_{M,U_3}^{\tau}.$$

This completes the proof. □

Theorem 5. *For every complete LB-space E the following assertions are equivalent:*

(a) *E'_β has the property $(\overline{\overline{\Omega}})$;*
(b) *there is an open connected set $\Omega \subseteq \mathbb{R}^n$ such that $A_t(\Omega, E) = A_b(\Omega, E)$;*
(c) *for every open connected set $\Omega \subseteq \mathbb{R}^n$ we have $A_t(\Omega, E) = A_b(\Omega, E)$.*

Proof. We write $E = \mathrm{ind}_{N\in\mathbb{N}} E_N$, where (E_N, p_N) is a Banach space. By (K_N) we denote a fundamental sequence of compact subsets of Ω, $K_N \subseteq \mathrm{int} K_{N+1}$ for $N \in \mathbb{N}$. Finally,

$$\begin{aligned}
\|u\|_N &:= \sup\{|\langle u, x\rangle| : p_N(x) \leq 1\} \quad \text{for } u \in E',\\
\|f\|_{(N,L)} &:= \sup\{|f(z)| :\ z \in K_N + \mathbb{D}_L^n\} \quad \text{for } f \in \mathrm{A}(\Omega),\\
\|S\|_{M,(N,L)} &:= \sup\{\|Sx\|_{(N,L)} :\ \|x\|_M \leq 1\} \quad \text{for } S \in L(E'_\beta, \mathrm{A}(\Omega)),
\end{aligned}$$

where $\mathbb{D}_L := \frac{1}{L}\mathbb{D}$.

(c)⇒(b): Obvious.

(b)⇒(a): The proof is similar to the proof of necessity in [4, Th. 21]. We define for $x \in \Omega$, $x = (x_1, \ldots, x_n)$,

$$g_j(x) := (\arctan x_1)^j.$$

Moreover,

$$A_n := \sup_{x\in K_n} |g_1(x)|, \qquad A := \sup_{x\in\Omega} |g_1(x)|;$$

clearly $\lim_{n\to\infty} A_n = A$.

We choose, for a given sequence of natural numbers $(S(N))$, another sequence $(L(N))$ such that

$$\|g_1\|_{(S(N),L(N))} \leq A_{S(N+1)}.$$

Exactly as in [4, Proof of Th. 21] we obtain for any sequence of natural numbers $(S(N))_{N\in\mathbb{N}}$, a sequence $(M(N))_{N\in\mathbb{N}}$ such that

$$\forall\ K\ \ \exists\ n\ \ \forall\ N_0, C<\infty\ \ \forall\ j\in\mathbb{N}, x\in E_{M(1)}:$$
$$p_K(x)(A_n)^j \leq C \max_{N=1,\ldots,N_0} p_{M(N)}(x)(A_{S(N)})^j$$

which implies $(\overline{\overline{\Omega}})$ for E'_β as in [4].

(a)⇒(c): As in [4] it suffices to show that every operator $T: E'_\beta \to \mathrm{A}(\Omega)$ is bounded. It is easily seen that for every N there is $L(N)$ such that

$$T: E'_\beta \to H^\infty(K_N + \mathbb{D}^n_{L(N)}) \subseteq H(K_N + \mathbb{D}^n_{L(N)}).$$

Thus there is $M(N)$ such that $\|T\|_{M(N),(N,L(N))} < \infty$. We claim that

$$\exists\ K\ \ \forall\ m\ \ \exists\ k, N_0, C<\infty:$$
$$\|T\|_{K,(m,L)} \leq C\max(\|T\|_{M(1),(1,L(1))}, \|T\|_{M(N_0),(N_0,L(N_0))})$$

and this completes the proof by the same argument as in [4, Th. 21].

We take $N = M(1)$, find K from Lemma 4, take any m, $N_0 > m$ and apply Lemma 4 for $M = M(N_0)$, $U := K_{N_0} + \mathbb{D}^n_{L(N_0)}$. We choose $U_2 := K_m + \mathbb{D}^n_k$ for suitable k, $U_1 \subseteq U_2 \cap (K_1 + \mathbb{D}^m_{L(1)})$ and $U_2 \subset\subset U_3 \subset\subset U$. Then we obtain the claim immediately from Lemma 4. □

It is worth noting that in many other results contained in [4] we can put an arbitrary open set $\Omega \subseteq \mathbb{R}^m$ instead of $\mathbb{R}$ (for instance, in Theorems 1, 2, Propositions 4, 6, 9, 10, Corollary 11, Proposition 12, Lemma 14, Corollaries 25 and 26).

Let us assume $T: A(\Omega_1) \longrightarrow A(\Omega_2)$ is a surjective continuous linear operator. Let E be a complete locally convex space. One canonically defines a continuous linear operator $\tilde{T}: A(\Omega_1, E) \longrightarrow A(\Omega_2, E)$, using the identification given in Theorem 2: since $\mathrm{A}(\Omega, E) = \mathrm{A}(\Omega)\varepsilon E = L(E'_{co}, \mathrm{A}(\Omega))$, we put $\tilde{T}(W) := T \circ W$ for each $W \in A(\Omega_1)\varepsilon E$. Clearly $\tilde{T}$ coincides with the unique extension to the completion of the operator

$$T\otimes \mathrm{id}: A(\Omega_1)\otimes_\varepsilon E \longrightarrow A(\Omega_2)\otimes_\varepsilon E,$$
$$(T\otimes\mathrm{id})(f\otimes x) := Tf\otimes x, \quad f\in A(\Omega_1), x\in E.$$

We study conditions on E to assure that the operator $\tilde{T}$ is also surjective and we prove the main result (comp. [4, Th. 38]):

Theorem 6. *Let $\Omega_1 \subseteq \mathbb{R}^n$, $\Omega_2 \subseteq \mathbb{R}^m$ be open sets. Let $T: A(\Omega_1) \to A(\Omega_2)$ be a surjective continuous linear map. The map $T\otimes \mathrm{id}: A(\Omega_1, E) \to A(\Omega_2, E)$ is surjective if the locally convex space E satisfies one of the following conditions:*

(i) *E is a Fréchet quojection, in particular, E is a Banach space;*
(ii) *E is a Fréchet space satisfying the property (DN);*
(iii) *E is a complete LB-space such that E'_β has the property $(\overline{\overline{\Omega}})$.*

Proof. The case (i) follows directly from the remark after the proof of [4, Th. 36], since by Theorem 1, $A(\Omega_1)$ is an ultrabornological space and $\mathrm{Proj}^1 A(\Omega_1) = 0$ by [41] (comp. [8]). Let us assume (ii) or (iii) and let us take $f \in A(\Omega_2, E)$. By Theorems 3 and 5, there is a closed absolutely convex subset B of E such that $f \in \mathrm{A}(\Omega_2, E_B)$. Since E_B is a Banach space, we can apply [4, Th. 36] to find $g \in \mathrm{A}(\Omega_1, E_B)$ such that $T \otimes \mathrm{id}_{E_B}(g) = f$. Clearly $T \otimes \mathrm{id}_E(g) = f$ as well. ☐

It is worth noting that there is an extensive theory of surjectivity of convolution operators on $\mathrm{A}(\Omega)$ and related spaces (see [5], [7], [14], [22], [23]).

Corollary 7. *Let U be a Stein manifold with the strong Liouville property (for instance, $U = \mathbb{C}^n$). Let Ω_1, Ω_2 be open subsets in $\mathbb{R}^m$. Let $T = T_\mu : A(\Omega_1) \to A(\Omega_2)$ be a surjective convolution operator (or $\Omega_1 = \Omega_2$, $T = P(D, x) : A(\Omega_1) \to A(\Omega_1)$ be a surjective linear PDO). Then for every function f holomorphic on some neighbourhood of $\Omega_2 \times U$ there is a function g holomorphic on some neighbourhood of $\Omega_1 \times U$ such that*

$$Tg_z = f_z \quad \text{for } z \in U,$$

where $f_z(t) := f(t, z)$, $g_z(t) := g(t, z)$.

Proof. It is enough to know that $H(U)$ has (DN) if and only if U has the strong Liouville property [43]. ☐

3. Examples

First, we give some examples of spaces E for which an analogue of Theorem 6 does not hold.

According to the result of De Giorgi and Cattabriga from 1971 every linear partial differential operator with constant coefficients is surjective on $\mathrm{A}(\mathbb{R}^2)$ but the operator $T := \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 : \mathrm{A}(\mathbb{R}^3) \longrightarrow \mathrm{A}(\mathbb{R}^3)$ is not surjective (see [9], comp. [14] and [6, Ex. 3.3]). It is well known that $H([-N,N]^3) \simeq H([-N,N]^2)\tilde{\otimes}_\varepsilon H([-N,N])$ and thus, by [16, 16.3.2],

$$\begin{aligned}\mathrm{A}(\mathbb{R}^3) &= \mathrm{proj}_{N\in\mathbb{N}} H([-N,N]^3) \simeq \mathrm{proj}\ H([-N,N]^2)\tilde{\otimes}_\varepsilon \mathrm{proj}\ H([-N,N]) \\ &= \mathrm{A}(\mathbb{R}^2)\tilde{\otimes}_\varepsilon \mathrm{A}(\mathbb{R}).\end{aligned}$$

Clearly, $T = P(D) \otimes \mathrm{id}$, where $P(D) = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$ so we have found a case when there is no surjectivity after tensorizing.

We modify our example to get an (LB)-space as a second factor. Let F be the subspace of $\mathrm{A}(\mathbb{R}^3)$ consisting of those functions which are 2π-periodic in the third variable. It is known that the space $\mathrm{A}_{\mathrm{per}}(\mathbb{R})$ of 2π-periodic real analytic functions is isomorphic to $H(\overline{\mathbb{D}})$ and complemented in $\mathrm{A}(\mathbb{R})$ (see [4, Prop. 5]). Thus our space F is isomorphic to $\mathrm{A}(\mathbb{R}^2)\tilde{\otimes}_\varepsilon H(\overline{\mathbb{D}}) = \mathrm{A}(\mathbb{R}^2, H(\overline{\mathbb{D}}))$ and it is complemented in $\mathrm{A}(\mathbb{R}^3)$. Since the latter space is ultrabornological (see Theorem 1) the former one also has the same property. Moreover, by [16, 16.3.2], F is a projective limit of a sequence of nuclear LB-spaces. By [41], ultrabornologicity implies that $\mathrm{Proj}^1 F = 0$.

Let us assume that $P(D) : \mathrm{A}(\mathbb{R}^2) \longrightarrow \mathrm{A}(\mathbb{R}^2)$ is an arbitrary elliptic linear partial differential operator with constant coefficients; for instance, let $P(D)$ be the Laplace operator. By [9], $P(D)$ is surjective. Let us assume that $S := P(D) \otimes \mathrm{id}_{H(\overline{\mathbb{D}})} : \mathrm{A}(\mathbb{R}^2, H(\overline{\mathbb{D}})) \longrightarrow \mathrm{A}(\mathbb{R}^2, H(\overline{\mathbb{D}}))$ is surjective as well. Since Proj^1 vanishes for the domain of S then it also vanishes for its kernel (use [40, Th. 5.4] or [39]). By [40, Th. 3.4] (comp. [39]), the kernel of S is ultrabornological.

Since $P(D)$ is elliptic its kernel coincides for various spaces; in particular, by [42, 2.2, Satz 9], $\ker P(D)$ is isomorphic to the power series space of infinite type $\Lambda_\infty(j)$. On the other hand, $H(\overline{\mathbb{D}})$ is isomorphic to the strong dual of $\Lambda_1(j)$. Clearly, $\ker S \simeq \ker P(D)\tilde{\otimes}_\varepsilon H(\overline{\mathbb{D}}) \simeq L_b(\Lambda_1(j), \Lambda_\infty(j))$, the latter space equipped with the topology of uniform convergence on bounded subsets. Combining [21, Th. 1.1 and 2.1] with [38, Cor. 4.4] we obtain that $\ker S$ is not bornological (or equivalently, by completeness, ultrabornological); a contradiction. We have proved that S is not surjective. See [6, Ex. 3] for a similar argument.

So we have proved the following result.

Theorem 8. *For every elliptic (surjective) linear partial differential operator with constant coefficients $P(D) : A(\mathbb{R}^2) \longrightarrow A(\mathbb{R}^2)$ the map*

$$P(D) \otimes \mathrm{id} : A(\mathbb{R}^2, H(\overline{\mathbb{D}})) \longrightarrow A(\mathbb{R}^2, H(\overline{\mathbb{D}}))$$

is not surjective.

Such an example cannot be constructed for Fréchet spaces instead of $H(\overline{\mathbb{D}})$.

Proposition 9. *Let E be either a Fréchet space or a strong dual of a Fréchet space F with the property (DN) and let $\Omega \subseteq \mathbb{R}^n$ be an open subset. Then for every linear elliptic partial differential operator $P(D)$ with constant coefficients the map $P(D) \otimes \mathrm{id} : A(\Omega, E) \longrightarrow A(\Omega, E)$ is surjective.*

Proof. By the results of Grothendieck [13] (the Fréchet case) and of Vogt [34] (the LB-case),

$$P(D) \otimes \mathrm{id} : C^\infty(\Omega, E) \longrightarrow C^\infty(\Omega, E)$$

is surjective. Thus for any $g \in \mathrm{A}(\Omega, E)$ there is $f \in C^\infty(\Omega, E)$ such that $P(D) \otimes \mathrm{id} f = g$. Since $P(D)(u \circ f) = u \circ (P(D) \otimes \mathrm{id}(f))$ for each $u \in E'$, we obtain that $u \circ f$ is analytic for each $u \in E'$ by the ellipticity of $P(D)$. Hence $f \in \mathrm{A}(\Omega, E)$. □

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