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A NOTE ON THE EXISTENCE OF A LARGEST TOPOLOGICAL FACTOR WITH ZERO ENTROPY

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ABSTRACT. Given a topological system T on a σ-compact Hausdorff space and its factor S we show the existence of a largest topological factor $\hat{S}$ containing S such that for each $\hat{S}$-invariant measure μ, $h_\mu(\hat{S}|S) = 0$. When a relative variational principle holds, $h(\hat{S}) = h(S)$.

INTRODUCTION

In [1], F. Blanchard introduced the notion of entropy pairs as a tool to localize entropy in topological systems on compact Hausdorff spaces. This notion turned out to be very fruitful and caused a series of articles ([2], [11], [12]) also introducing a related concept in the measure-theoretic setup ([3], [4], [10]). In particular, in [2] F. Blanchard and Y. Lacroix show the existence of a largest factor with zero entropy (so-called topological Pinsker factor).

In this note we propose a different approach, via invariant measures. This will allow us to define (in Section 1) a certain largest factor for topological systems on arbitrary σ-compact Hausdorff spaces. When additionally the variational principle holds, this factor turns out to be the topological Pinsker factor.

The first approach to the relativization of the notion of the topological Pinsker factor was presented by E. Glasner and B. Weiss in [12] as a generalization of the result of [2]. Namely, between a compact system X and its factor Y, a largest topological factor whose fibers do not contain entropy pairs is proved to exist. It turns out that this factor has the same topological entropy as Y and it is called (in [12]) the relative topological Pinsker factor of X with respect to Y. In this note it will be called the *relative topological Pinsker*$_1$ *factor*. In Section 2 we relativize (with respect to a factor Y) the result of Section 1. A largest topological factor Z (between X and Y) for which the measure-theoretic relative entropy $h_\nu(Z|Y)$ equals zero for any invariant measure ν on Z turns out to exist. It follows that $h(Z) = h(Y)$ whenever X is compact. Moreover, the relative topological Pinsker$_1$ factor is a factor (in general, a proper factor) of Z. We call Z the *relative topological Pinsker*$_2$ *factor* of X with respect to Y. It should be emphasized that in general in the family $\{Y_\alpha\}_{\alpha\in\Lambda}$ of all topological factors between X and Y satisfying $h(Y_\alpha) = h(Y)$ there

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is no largest element. A relevant example was pointed out to us by T. Downarowicz (Example 3 below).

When the underlying space is compact metric the relative variational principle holds ([13]): for each invariant measure ν on Y

$$\sup h_\mu(X|Y) = \int_Y h(X|\pi^{-1}(y))\,\nu(y),$$

where the supremum is taken over all invariant measures μ on X whose image via the factor map $\pi : X \to Y$ equals ν. We can then describe the relative Pinsker$_2$ factor in terms of a behaviour of Bowen's entropies $h(X|\pi^{-1}(y))$ on fibers (see Lemma 4 below). However, even in the non-metrizable case, recently, T. Downarowicz and J. Serafin introduced in [8] the notion of the relative topological entropy $h(X|Y)$ of a system X with respect to its factor Y and they proved a relative variational principle. It turns out that a largest factor $\widetilde{Y}$ between X and Y satisfying $h(\widetilde{Y}|Y) = 0$ exists and moreover it is equal to the relative topological Pinsker$_2$ factor.

We assume that the reader is familiar with basic properties of measure-theoretic entropy, topological entropy, Bowen's entropy (see e.g. [16]). For the reader's convenience we recall the definitions of entropy pairs and measure-theoretic entropy pairs. Given a compact Hausdorff space X and its homeomorphism T a pair $(x_1, x_2) \in X \times X$, $x_1 \neq x_2$ is called an *entropy pair* if for every open cover $\mathcal{U} = \{U_1, U_2\}$ of X such that U_i is non-dense and $x_i \in \operatorname{Int}(U_i^c)$, $i = 1, 2$, the topological entropy $h(T, \mathcal{U})$ is positive. Similarly, given a T-invariant measure m on X, a pair $(x_1, x_2) \in X \times X$, $x_1 \neq x_2$ is called an *m-entropy pair* if for every Borel partition $\mathcal{F} = \{F_1, F_2\}$ of X with $x_i \in \operatorname{Int} F_i$, $i = 1, 2$, the measure-theoretic entropy $h_m(T, \mathcal{F})$ is positive. For basic properties of entropy pairs and measure-theoretic entropy pairs we refer to [2], [3], [4], [10], [11] and [12].

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1. The existence of a topological Pinsker factor

Let X be a topological Hausdorff space. Throughout $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of X. If $\Phi \subset C(X)$ is a family of continuous functions on X, then by $\operatorname{Alg}(\Phi)$ we denote the sub-algebra of $C(X)$ generated by Φ. Let $\{R_i\}_{i\in I}$ be a family of closed (as subsets of $X \times X$) equivalence relations on X. Put $R = \bigcap_{i\in I} R_i$. Denote by π_i (π) the canonical map $\pi_i : X \to X/R_i$ ($\pi : X \to X/R$), $i \in I$. Fix $K \subset X$ a compact set. Denote $K/R := \pi(K) \subset X/R$ and $K/R_i := \pi_i(K) \subset X/R_i$. Let $p_i : X/R \to X/R_i$ and $p_{i,K} : K/R \to K/R_i$ stand for the corresponding canonical maps. Then the map

$$C(K/R_i) \ni f \mapsto f \circ p_{i,K} \in C(K/R)$$

allows us to embed the space $C(K/R_i)$ of continuous maps on K/R_i as a subspace $\widetilde{C}(K/R_i)$ of $C(K/R)$.

Lemma 1. $C(K/R) = \overline{\operatorname{Alg}(\bigcup_{i\in I} \widetilde{C}(K/R_i))}$, *where the closure is in the topology of uniform convergence in* $C(K/R)$.

Proof. Clearly $\operatorname{Alg}(\bigcup_{i\in I} \widetilde{C}(K/R_i)) \subset C(K/R)$. Now take $x, y \in K$ and suppose that $[x]_R \neq [y]_R$. Then there exists i such that $[x]_{R_i} \neq [y]_{R_i}$ and therefore there exists $f \in C(K/R_i)$ such that $f([x]_{R_i}) \neq f([y]_{R_i})$. It follows that the function

$f \circ p_{i,K} \in \tilde{C}(K/R_i)$ separates $[x]_R$ and $[y]_R$. By the Stone-Weierstrass theorem $\mathrm{Alg}(\bigcup_{i\in I} \tilde{C}(K/R_i))$ is dense in $C(K/R)$. □

Let m be a probability measure on $\mathcal{B}(X/R)$. By $m|$ denote its restriction to K/R. Put $\tilde{\mathcal{B}}(K/R_i) = p_{i,K}^{-1}(\mathcal{B}(K/R_i))$ and let the join $\bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i)$ denote the smallest σ-algebra of $\mathcal{B}(K/R)$ containing all $\tilde{\mathcal{B}}(K/R_i)$, $i \in I$.

Lemma 2. *$\mathcal{B}(K/R) = \bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i)$ mod $m|$.*

Proof. The result is true if $m|$ is the zero measure. Now, take m so that $m|$ is non-zero. For a fixed $i_0 \in I$ we have

$$\tilde{C}(K/R_{i_0}) \subset L^2(\tilde{\mathcal{B}}(K/R_{i_0}), m|) \subset L^2(\bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i), m|).$$

Hence, $\mathrm{Alg}(\bigcup_{i\in I} \tilde{C}(K/R_i)) \subset L^2(\bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i), m|)$ and therefore, by Lemma 1

$$C(K/R) \subset L^2(\bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i), m|).$$

However, $C(K/R)$ is dense in $L^2(\mathcal{B}(K/R), m|)$ in $L^2(m|)$ topology, so

$$L^2(\mathcal{B}(K/R), m|) = L^2(\bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i), m|)$$

and the result follows. □

Denote $\tilde{\mathcal{B}}(X/R_i) = p_i^{-1}(\mathcal{B}(X/R_i))$.

Lemma 3. *If X is σ-compact, then $\mathcal{B}(X/R) = \bigvee_{i\in I} \tilde{\mathcal{B}}(X/R_i)$ mod m.*

Proof. Let $\{K_n\}_{n\geq 1}$ be a family of compact sets satisfying $K_1 \subset K_2 \subset \dots$ and $\bigcup_{n\geq 1} K_n = X$. We have $K_n/R \subset X/R$ and let $\mathcal{B}(K_n/R)$ denote the family of Borel sets of X/R contained in K_n/R. Consider $\mathcal{B}(X/R)$ as a metric space with the metric induced by m. Then $\bigcup_{n=1}^{\infty} \mathcal{B}(K_n/R)$ is dense in $\mathcal{B}(X/R)$. Indeed, if $A \in \mathcal{B}(X/R)$, then

$$A = \bigcup_{n=1}^{\infty} (\pi_R^{-1}(A) \cap K_n)/R$$

and $(\pi_R^{-1}(A) \cap K_n)/R \in \mathcal{B}(K_n/R)$. It follows from Lemma 2 that the set $\bigcup_{n\geq 1} \bigvee_{i\in I} \tilde{\mathcal{B}}(K_n/R_i)$ is dense in $\mathcal{B}(X/R)$. However

$$\bigcup_{n\geq 1} \bigvee_{i\in I} \tilde{\mathcal{B}}(K_n/R_i) \subset \bigvee_{i\in I} \tilde{\mathcal{B}}(X/R_i) \subset \mathcal{B}(X/R),$$

so the result follows. □

Given a homeomorphism of a topological Hausdorff space X we denote by $M(X,T)$ the set of Borel probability measures on X. Given $m \in M(X,T)$ and a T-invariant sub-σ-algebra $\mathcal{A} \subset \mathcal{B}(X)$ (i.e. a measure-theoretic factor) the measure-theoretic entropy of the corresponding factor will be denoted by $h_m(T, \mathcal{A})$. Consider now the family $\{R_i\}$ of closed (as subsets of $X \times X$) $T \times T$-invariant equivalence relations on X such that

$$(\forall m_i \in M(X/R_i, T)) \;\; h_{m_i}(T, \mathcal{B}(X/R_i)) = 0.$$

If such a family contains a smallest element, then the corresponding largest factor, by some abuse of vocabulary, will be called the topological Pinsker factor of T. Let us also recall that in the measure-theoretic setup any join of factors with zero entropy has again zero entropy (since the measure-theoretic Pinsker factor always exists and it will contain any "coordinate" of the join). Hence, the remarks above show the following.

Proposition 1. *If T is a homeomorphism of a σ-compact Hausdorff space, then the topological Pinsker factor exists.* □

Assume that $T : X \to X$ is a homeomorphism of such a Hausdorff space that the entropy theory can be considered (see [5] and [15]). Let $\{R_i\}_{i\in I}$ be the family of all ICERs such that the topological entropy $h(T, X/R_i)$ equals zero. The next proposition shows that in this case the topological Pinsker factor coincides with a largest topological zero entropy.

Proposition 2. *If X is σ-compact and if the variational principle holds for all factors of T, then there exists a largest topological factor with zero entropy.*

Proof. All we need to show is that $h(T, X/R) = 0$, where $R = \bigcap_i R_i$. Take any Borel measure $m \in M(X/R, T)$. Then by the variational principle $h_m(T, \tilde{\mathcal{B}}(X/R_i)) = 0$ for each $i \in I$. Now X/R is σ-compact, so

$$h_m(T, \mathcal{B}(X/R)) = h_m(T, \bigvee \tilde{\mathcal{B}}(X/R_i)) = 0.$$

Using the variational principle once more we obtain that $h(T, X/R) = 0$. □

It is well-known that the variational principle holds in the compact case (e.g. [14] or [16] §8.2 in the metrizable case). Hence

Corollary 1 ([2]). *If $T : X \to X$ is a homeomorphism of a compact Hausdorff space, then the topological Pinsker factor exists.* □

2. The relative Pinsker factor

Assume that $T : X \to X, S : Y \to Y$ are homeomorphims of topological Hausdorff spaces and let $\pi : X \to Y$ be a continuous surjection such that $\pi T = S\pi$. The method from the previous section allows us to define the relative Pinsker factor $\hat{S}$ whenever the underlying space X is σ-compact. Indeed, we take the intersection of all ICERs R_i for which:

(i) R_i is contained in the preimage (via $\pi \times \pi$)) of the diagonal on Y

and

(ii) for each $m \in M(X/R_i, T)$ the relative entropy $h_m((T, X/R_i)|S) = 0$.

Since the relative measure-theoretic Pinsker factor always exists, the existence of the relative topological Pinsker factor follows.

In what follows we consider only the case when X is compact Hausdorff. We say that T is a *zero entropy extension* of S if $h(T) = h(S)$. We will consider other properties of extensions (possibly) stronger than the property of zero entropy. We say that a property P is a *Pinsker property* if:

1) for each $T : X \to X$, for arbitrary ICERs W, R_i $(i \in I)$ with $W \supset R_i$ for which $(T, X/R_i)$ is a P-extension of $(T, X/W)$ for each $i \in I$, we have

$$(T, X/\textstyle\bigcap_{i\in I} R_i) \text{ is a } P\text{-extension of } (T, X/W);$$

2) every extension with the P property is a zero entropy extension.

It is clear that for each Pinsker property we can define the corresponding relative Pinsker factor. Let us now define the following three properties:

P_1 (null entropy extension from [12]): an extension T of S has P_1 property if no fiber $\pi^{-1}(y)$ contains an entropy pair;

P_2: an extension T of S has P_2 property if for each $m \in M(X,T)$, $h_m(T) = h_{\pi_*(m)}(S)$ (i.e. $h_m(T|S) = 0$), where $\pi_*(m)$ denotes the image of m via π;

P_3: an extension T of S has P_3 property if it is a zero entropy extension.

The relative Pinsker factor defined by P_i property ($i = 1,2$) we will call the *relative Pinsker$_i$ factor*.

Lemma 4. *Assume that X is a compact metric space. An extension T of S has P_2 property if and only if for each $\nu \in M(Y,S)$, Bowen's entropy $h(T, \pi^{-1}(y)) = 0$ for ν-a.a. $y \in Y$.*

Proof. Fix $\nu \in M(Y,S)$. For each $m \in M(X,T)$ with $\nu = \pi_*(m)$ clearly, $h_m(T) \geq h_\nu(S)$. The relative variational principle says (see [13]) that

$$\sup_{\{m \in M(X,T)\colon \pi_*(m)=\nu\}} h_m(T) = h_\nu(S) + \int_Y h(T, \pi^{-1}(y))\, d\nu(y),$$

so in fact $h_m(T) = h_\nu(S)$ for each m satisfying

$$\pi_*(m) = \nu \quad \text{iff} \quad \int_Y h(T, \pi^{-1}(y))\, d\nu(y) = 0$$

and the result easily follows. □

Remark 1. It follows however from a recent paper [8] that in fact $h(T, \pi^{-1}(y)) = 0$ for ν-a.a. $y \in Y$ for all $\nu \in M(Y,S)$ if and only if $h(T, \pi^{-1}(y)) = 0$ for all $y \in Y$. In particular, it follows that in the compact metrizable case an extension T of S has P_2 property iff $h(T, \pi^{-1}(y)) = 0$ for all $y \in Y$.

In order to see a relationship between P_1 and P_2 we will need the following.

Lemma 5. *Let $m \in M(X,T)$ and assume that $h_m(T) > h_\nu(S)$, where $\nu = \pi_*(m)$. Then for ν-a.a. $y \in Y$, $\pi^{-1}(y)$ contains an m-entropy pair.*

Proof. Suppose that $h_m(T) > h_\nu(S)$. Then there exists a measure-theoretic (with respect to m) factor $\mathcal{A} \subset \mathcal{B}(X)$ such that $\mathcal{A} \supset \pi^{-1}(\mathcal{B}(Y))$, $h_m(T, \mathcal{A}) = h_\nu(S)$ and T is relatively K over $\mathcal{A}$. The measurable partition p of X defining $\mathcal{A}$ is finer than the partition into fibers $\{\pi^{-1}(y)\colon y \in Y\}$. Moreover two points in the fiber are in the same atom of p iff they cannot be distinguished by sets belonging to $\mathcal{A}$. Now, fix two points $x, x' \in X$ so that:

(i) $x, x' \in \pi^{-1}(y)$;

(ii) x, x' are in the topological support of m;

(iii) x, x' cannot be distinguished by sets belonging to $\mathcal{A}$.

Assume that we have a partition (Q, Q^c) of X with $x \in \mathrm{Int}(Q)$ and $x' \in \mathrm{Int}(Q^c)$. It follows that $m(Q), m(Q^c) > 0$ and that $Q \notin \mathcal{A}$ (because x and x' are distinguished by Q and Q^c). Therefore the factor $\mathcal{A}'$ generated by $\mathcal{A}$ and Q is strictly bigger than $\mathcal{A}$. By the relative K-property, $h_m(\mathcal{A}') > h_m(\mathcal{A})$ which implies that $h_m(T, (Q, Q^c)) > 0$. Hence (x, x') is an m-entropy pair. □

Proposition 3. *If X is a compact Hausdorff space, then $P_1 \Rightarrow P_2 \Rightarrow P_3$.*

Proof. $P_1 \Rightarrow P_2$ follows directly from Lemma 5 and the fact that each m-entropy pair is an entropy pair.

$P_2 \Rightarrow P_3$ follows from the usual variational principle. □

Clearly all these notions coincide if Y is the trivial one-point factor. We will show that in general no converse implication holds.

Example 1. In any u.p.e. system T (see [1]) for any factor S with finite fibers we have that the corresponding extension has P_2 property and does not have P_1 property. We easily realize such a situation in any full shift.

Example 2. Assume that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ with both subsets closed and T-invariant. We assume that $h(T|_{X_1}) \geq h(T|_{X_2}) > 0$. Define R by $(x, y) \in R$ for each $x, y \in X_2$ (and only $(x, x) \in R$ whenever $x \in X_1$). Clearly (T, X) is a P_3 extension of $(T, X/R)$ and it is not a P_2-extension.

E. Glasner and B. Weiss in [12] show that P_1 is a Pinsker property. We have already observed (in fact, in a more general context) that

Proposition 4. *If X is a compact Hausdorff space, then P_2 is a Pinsker property.* □

The following example, due to T. Downarowicz, shows that P_3 is not a Pinsker property.

Example 3. In order to show that P_3 is not a Pinsker property it is enough to construct a homeomorphism T of a compact metric space X with the following properties:

(i) it admits two factors (S_i, Y_i) via factor maps $\pi_i : X \to Y_i$, $i = 1, 2$;

(ii) both (S_1, Y_1) and (S_2, Y_2) factor onto the same system (U, Z) via ρ_1 and ρ_2, respectively, so that $\rho_1\pi_1 = \rho_2\pi_2$;

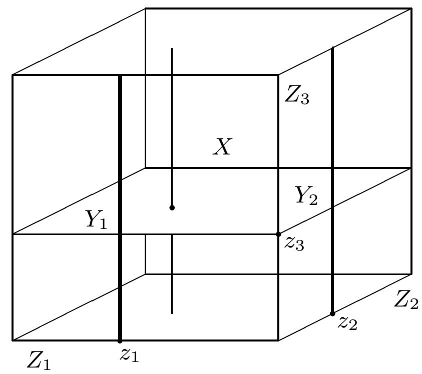
(iii) $R_{\pi_1} \cap R_{\pi_2} = \Delta_X$ (here Δ_X denotes the diagonal in $X \times X$ and $R_{\pi_i} = \{(y_i, y_i') \in Y_i \times Y_i \colon \pi_i(y_i) = \pi_i(y_i')\}$, $i = 1, 2$);

(iv) $h(S_1) = h(S_2) = h(U)$, while $h(T) = 2h(U)$.

We pass to a construction of such a T. First let U_i be a homeomorphism of a compact metric space Z_i and let z_i be a fixed point of U_i, $i = 1, 2, 3$. We assume that $h = h(U_1) = h(U_2) = h(U_3) > 0$. Put

$$X = \{(a, b, z_3) :\ a \in Z_1,\, b \in Z_2\} \cup \{(z_1, z_2, c) :\ c \in Z_3\}.$$

Clearly, X is a closed subset of $Z_1 \times Z_2 \times Z_3$. Moreover, it is invariant under the product action $T = U_1 \times U_2 \times U_3$. We then define $Y_1 = Z_1 \times Z_3$ and $Y_2 = Z_2 \times Z_3$ and on both spaces we consider the corresponding product actions. We put $\pi_1 : X \to Y_1$ as $\pi_1(a, b, c) = (a, c)$. Similarly, $\pi_2 : X \to Y_2$ is the projection onto the second and third coordinate: $\pi_2(a, b, c) = (b, c)$. Finally, both ρ_1 and ρ_2 are defined (on different spaces) as the projection onto the second coordinate (which in both cases coincides with the original third coordinate): $\rho_1(a, c) = c$, $\rho_2(b, c) = c$ (see the picture).



Now note that

$$Y_1 = \pi_1(X) = \{(a, z_3)\colon a \in Z_1\} \cup \{(z_1, c)\colon c \in Z_3\},$$

so it is a union of two S_1-invariant closed subsets on each of which the entropy equals h and moreover these two subsets have only a fixed point (z_1, z_3) in common. Thus $h(S_1) = h$. A similar argument shows $h(S_2) = h$. Their common factor $Y_3 = \rho_1(Y_1) = \rho_2(Y_2)$ coincides with Z_3, so its topological entropy is h, as well. Finally, $h(T) = h(U_1 \times U_2) = 2h$ by the arguments above. Observe that the equivalence classes of the ICER R_{π_1} corresponding to π_1 are of the form either $\{a\} \times Z_2 \times \{z_3\}$, $a \in Z_3$ or $\{(z_1, z_2, c)\}$, $c \in Z_3$, while the equivalence classes of the ICER corresponding to π_2 are of the form either $Z_1 \times \{b\} \times \{z_3\}$, $b \in Z_2$ or $\{(z_1, z_2, c)\}$, $c \in Z_3$. Thus, clearly $R_{\pi_1} \cap R_{\pi_2} = \Delta_X$.

Remark 2. T. Downarowicz ([6]) has informed us that the above example can be modified, using Downarowicz-Lacroix' version ([7]) of Furstenberg-Weiss' theorem ([9]), so that (T, X) (hence also all remaining systems) becomes minimal.

There is one natural case when the notions P_2 and P_3 are equivalent (as before, we consider only compact Hausdorff spaces). We say that (S, Y) has *constant entropy function* (c.e.f. for short) if $h_\nu(S) = h_\mu(S)$ for every $\nu, \mu \in M(Y, S)$. Uniquely ergodic systems or, more generally, distal extensions of uniquely ergodic systems serve as examples of systems with c.e.f. property.

Proposition 5. *If S has c.e.f. property, then each P_3 extension is also a P_2 extension.*

Proof. Let T be a zero entropy extension of S. Hence for each $m \in M(X, T)$

$$h(T) \geq h_m(T) \geq h_{\pi_*(m)}(S) = h(S),$$

where the equality follows from the c.e.f. property of S and the variational principle. Therefore $h_m(T) = h_{\pi_*(m)}(S)$ and the result follows. □

It follows that if a topological factor S of T has c.e.f. property, then the relative Pinsker$_2$ topological factor of T is the largest zero entropy extension of S.

In general, the relative Pinsker$_2$ topological factor is strictly larger than the relative Pinsker$_1$ topological factor. Indeed, in Example 1, the relative Pinsker$_2$ topological factor equals T while the relative Pinsker$_1$ topological factor equals S.

Assume that S is a topological factor of T and $h(S) = 0$; then the relative Pinsker$_2$ factor of T equals the Pinsker factor of T (and therefore it is again the largest zero entropy extension of S). Indeed, we repeat the argument from the proof of Proposition 5.

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