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# A NOTE ON TRIANGULAR DERIVATIONS OF $\mathbf{k}[X_1, X_2, X_3, X_4]$

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Abstract. For a field $\mathbf{k}$ of characteristic zero, and for each integer $n \geq 4$, we construct a triangular derivation of $\mathbf{k}[X_1, X_2, X_3, X_4]$ whose ring of constants, though finitely generated over $\mathbf{k}$, cannot be generated by fewer than $n$ elements.

## 1. Introduction

Let $\mathbf{k}$ be a field of characteristic zero. If $R$ is a finitely generated $\mathbf{k}$-algebra, we write $\#(R) = s$ to indicate that $R$ can be generated by $s$ elements but not by $s-1$. The purpose of this note is to show:

**Theorem.** *Given any integer $n \geq 3$, there exists a triangular derivation $\Delta$ of the polynomial ring $\mathbf{k}[X_1, X_2, X_3, X_4]$ whose kernel satisfies $n \leq \#(\ker \Delta) \leq n+1$.*

Equivalently, the theorem asserts that, given $n \geq 3$, there exists a triangular action of $G_a = (\mathbf{k}, +)$ on $\mathbb{A}^4$ whose ring of invariants satisfies $n \leq \#\mathcal{O}(\mathbb{A}^4)^{G_a} \leq n+1$. The theorem is proved by constructing $\Delta$ explicitly for $n \geq 4$ (for $n = 3$, just use a partial derivative).

In contrast to our present result, the well-known theorem of Miyanishi [2] states that, for any locally nilpotent $\mathbf{k}$-derivation $D$ of $\mathbf{k}[X_1, X_2, X_3]$, $\#(\ker D) = 2$. At the other extreme, the authors recently found a triangular derivation of the ring $\mathbf{k}[X_1, X_2, X_3, X_4, X_5]$ whose kernel is not finitely generated as a $\mathbf{k}$-algebra [1]. It is not known whether such kernels in dimension four are always finitely generated, even for triangular derivations.

## 2. Preliminaries

A *triangular* derivation of $\mathbf{k}[X_1, \ldots, X_n]$ is a $\mathbf{k}$-derivation $\Delta : \mathbf{k}[X_1, \ldots, X_n] \to \mathbf{k}[X_1, \ldots, X_n]$ satisfying $\Delta(X_i) \in \mathbf{k}[X_1, \ldots, X_{i-1}]$ for all $i = 1, \ldots, n$.

An element of a submonoid $\Gamma$ of $(\mathbb{N}, +)$ is *primitive* if it is positive and cannot be written as the sum of two positive elements of $\Gamma$. It is easy to see that the set of primitive elements in $\Gamma$ is a finite set which generates $\Gamma$ and which is contained in every generating set.

The *support* of an element $f = \sum_{i=0}^{\infty} a_i X^i$ of the power series ring $\mathbf{k}[[X]]$ is $\mathrm{Supp}(f) = \{i \in \mathbb{N} | a_i \neq 0\}$. Given a submonoid $\Gamma$ of $(\mathbb{N}, +)$, the elements $f$ of

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$\mathbf{k}[[X]]$ satisfying $\text{Supp}(f) \subseteq \Gamma$ form a subalgebra of $\mathbf{k}[[X]]$ which we denote $\mathbf{k}[[\Gamma]]$. We observe:

(1) If $g_1, \dots, g_r \in \mathbf{k}[[\Gamma]]$ and $P \in \mathbf{k}[[T_1, \dots, T_r]]$ satisfy $\text{ord}(g_i) \geq 1$ for all $i$ and $\text{ord}(P) \geq 2$, then no primitive element of $\Gamma$ belongs to the support of $P(g_1, \dots, g_r)$.

Indeed, let $\gamma \in \text{Supp}\, P(g_1, \dots, g_r)$; then $\gamma$ must be in the support of some monomial $g_1^{i_1} \cdots g_r^{i_r}$ with $i_1 + \cdots + i_r \geq 2$, so $\gamma$ is the sum of $i_1 + \cdots + i_r$ elements of $\bigcup_{i=1}^r \text{Supp}(g_i) \subseteq \Gamma \setminus \{0\}$ and hence is not primitive.

**Lemma 1.** *Let $\Gamma$ be a submonoid of $(\mathbb{N}, +)$, let $e_1 < \cdots < e_h$ be the primitive elements of $\Gamma$, let $R = \mathbf{k}[X^{e_1}, \dots, X^{e_h}]$ and let $T$ be an indeterminate over $R$. Then:*

$$\#(R) = h \quad \textit{and} \quad \#(R[T]) = h + 1.$$

*Proof.* Given $f \in R[T]$, let $f(0) \in \mathbf{k}[X]$ be the result of evaluating $f$ at $T = 0$, and let $\text{ord}(f) \in \mathbb{N} \cup \{\infty\}$ be the $X$-order of $f(0)$, i.e., the largest $s \geq 0$ such that $X^s$ divides $f(0)$ in $\mathbf{k}[X]$. Note that $\#(R) = h$ is a consequence of $\#(R[T]) = h + 1$, so it suffices to prove the latter.

Assume that $\#(R[T]) \neq h + 1$; then $R[T]$ can be generated by $h$ elements, say $R[T] = \mathbf{k}[f_1, \dots, f_h]$. We begin by showing that, replacing if necessary the generating set $\{f_1, \dots, f_h\}$ by another one with the same cardinality $h$, we may arrange that $\text{ord}(f_j) = e_j$ for all $j = 1, \dots, h$. To see this, consider an integer $i$ satisfying $1 \leq i \leq h$ and

$$\text{ord}(f_j) = e_j, \qquad \text{for all } j < i \tag{2}$$

(this certainly holds for $i = 1$). Observe that every element of $\Gamma$ strictly less than $e_i$ belongs to the monoid generated by $\{e_1, \dots, e_{i-1}\}$; hence, replacing each $f_j$ (with $j \geq i$) by $f_j$ plus a suitable polynomial in $(f_1, \dots, f_{i-1})$, we may arrange that $\text{ord}(f_j) \geq e_i$ for all $j \geq i$. After relabelling, we obtain that $f_i, \dots, f_h$ satisfy

$$e_i \leq \text{ord}(f_i) \leq \text{ord}(f_{i+1}) \leq \cdots \leq \text{ord}(f_h).$$

Since $X^{e_i} \in R[T]$, we may write

$$X^{e_i} = \lambda_1 f_1 + \cdots + \lambda_h f_h + P(f_1, \dots, f_h),$$

where $\lambda_j \in \mathbf{k}$, $P \in \mathbf{k}[T_1, \dots, T_h]$ (the $T_j$ are indeterminates) and where every monomial occuring in $P(T_1, \dots, T_h)$ has degree at least two. Now $P(f_1, \dots, f_h)\Big|_{T=0} = P(f_1(0), \dots, f_n(0)) = \sum_j \mu_j X^{\gamma_j}$, where $\mu_j \in \mathbf{k}$ and $\gamma_j \in \Gamma$, but none of these $\gamma_j$ can be a primitive element of $\Gamma$ by (1). It follows that $\lambda_j = 0$ for all $j < i$; also, $\text{ord}(f_i) = e_i$, so we arranged that (2) holds for a larger value of $i$. Thus we may arrange that

$$\text{ord}(f_j) = e_j \quad \text{for all } j = 1, \dots, h.$$

Since $T \in R[T]$, we may write

$$T = \lambda'_1 f_1 + \cdots + \lambda'_h f_h + P'(f_1, \dots, f_h), \tag{3}$$

where $\lambda'_j \in \mathbf{k}$, $P' \in \mathbf{k}[T_1, \dots, T_h]$, and where every monomial occurring in $P'(T_1, \dots, T_h)$ has degree at least two. Evaluating (3) at $T = 0$ shows that $\lambda'_j = 0$ for all $j$ (as before, $P'(f_1(0), \dots, f_h(0))$ can't produce a term $X^{e_j}$, by (1)). On the other hand, each $f_j$ evaluated at $X = 0$ is an element of $T\mathbf{k}[T]$. Thus, evaluating the equation $T = P'(f_1, \dots, f_h)$ at $X = 0$ yields $T = T^2 Q(T)$ for some $Q(T) \in \mathbf{k}[T]$. This is a contradiction, so $\#(R[T]) = h + 1$ cannot be false. □

**Lemma 2.** *Let $h, p, q$ be positive integers. If $\gcd(p, q) = 1$, then the ideal*

$$(T_0^q - T_1^p, T_1^q - T_2^p, \dots, T_{h-1}^q - T_h^p)$$

*of $\mathbf{k}[T_0, \dots, T_h]$ is prime.*

*Proof.* Consider the ideals $\mathfrak{p} = (T_0^q - T_1^p, \dots, T_{h-1}^q - T_h^p)$ of $\mathbf{k}[T_0, \dots, T_h]$ and $\mathfrak{p}' = (T_0^q - T_1^p, \dots, T_{h-2}^q - T_{h-1}^p)$ of $\mathbf{k}[T_0, \dots, T_{h-1}]$. By induction, we may assume that $\mathfrak{p}'$ is prime. This allows us to identify $R' = \mathbf{k}[T_0, \dots, T_{h-1}]/\mathfrak{p}'$ with $\mathbf{k}[X^{e_0}, \dots, X^{e_{h-1}}]$, where $X$ is an indeterminate and $e_j = p^{h-j}q^j$. Let $K'$ be the field of fractions of $R'$ and note that $K' = \mathbf{k}(X^p)$. Since $\mathbf{k}[T_0, \dots, T_h]/\mathfrak{p} \cong R'[T_h]/(T_h^p - \theta^q)$, where $\theta = T_{h-1} + \mathfrak{p}' \in R'$, it suffices to show that $T_h^p - \theta^q$ is an irreducible element of $K'[T_h]$; for this, it's enough to verify that $(\theta^q)^{i/p} \notin K'$ for all $i = 1, \dots, p-1$. But $\theta = X^{e_{h-1}}$, so $(\theta^q)^{i/p} = X^{iq^h} \notin \mathbf{k}(X^p)$ for all $i = 1, \dots, p-1$. □

The following is a well-known fact about extracting roots in a power series ring.

**Lemma 3.** *Let $q$ be a positive integer, $R$ a domain containing $\mathbb{Q}$, $W$ an indeterminate over $R$ and $\sigma$ an element of $R[[W]]$ with constant term equal to $1$ (i.e., $\sigma = 1 + s_1 W + s_2 W^2 + \cdots$ where $s_i \in R$). Then there exists a unique $\rho \in R[[W]]$ satisfying $\rho^q = \sigma$ and having constant term equal to $1$.*

**Lemma 4.** *Let $h \geq 2$ be an integer and $p, q$ prime numbers such that $p^2 < q$. Then there exist $f_0, \dots, f_h \in \mathbf{k}[W, X]$ satisfying:*

(i) $f_j(0, X) = X^{p^{h-j}q^j}$ *for all $j$ such that $0 \leq j \leq h$;*
(ii) $f_{j+1} \equiv \frac{f_{j-1}^q - f_j^p}{W} \pmod{W^{h-j}}$ *for all $j$ such that $0 < j < h$;*
(iii) $f_{h-1}^q - f_h^p = 0$.

*Proof.* Define $f_h = X^{q^h}$ and $f_{h-1} = X^{pq^{h-1}}$. Suppose that $f_h, \dots, f_i \in \mathbf{k}[W, X]$ have been defined (where $0 < i < h$) and satisfy (i)–(iii) and

$$X^{p^{h-j}q^j} \mid f_j \quad (i \leq j \leq h). \tag{4}$$

Note that the assumption $p^2 < q$ implies that $f_i^p + W f_{i+1}$ is divisible by $X^{p^{h-i+1}q^i}$; define

$$\sigma = \frac{f_i^p + W f_{i+1}}{X^{p^{h-i+1}q^i}} \in \mathbf{k}[W, X] \subset \mathbf{k}[X][[W]]$$

and note that $\sigma$ has the form $\sigma = 1 + s_1 W + s_2 W^2 + \cdots$ (with $s_j \in \mathbf{k}[X]$). By Lemma 3, we may consider $\rho = 1 + r_1 W + r_2 W^2 + \cdots \in \mathbf{k}[X][[W]]$ (with $r_j \in \mathbf{k}[X]$) such that $\rho^q = \sigma$. Then $\tilde{f}_{i-1} := X^{p^{h-i+1}q^{i-1}} \rho \in \mathbf{k}[X][[W]]$ satisfies

$$\frac{\tilde{f}_{i-1}^q - f_i^p}{W} = f_{i+1} \qquad \text{and} \qquad X^{p^{h-i+1}q^{i-1}} \mid \tilde{f}_{i-1}$$

so, if $f_{i-1} \in \mathbf{k}[W, X]$ is a suitable truncation of $\tilde{f}_{i-1}$, then $f_h, \dots, f_{i-1}$ satisfy (i)–(iii) and (4). So we are done by induction. □

## 3. The examples

Given an integer $h \geq 2$, we construct a triangular derivation $\Delta : \mathbf{k}[W, X, Y, Z] \to \mathbf{k}[W, X, Y, Z]$ whose kernel satisfies $h + 2 \leq \#(\ker \Delta) \leq h + 3$.

Choose prime numbers $p, q$ satisfying $p^2 < q$; consider $f_0, \dots, f_h \in \mathbf{k}[W, X]$ as in Lemma 4 and define $F_0 = f_0 + YW^{h+1}$, $F_1 = f_1 + ZW^h$ and

$$F_{i+1} = \frac{F_{i-1}^q - F_i^p}{W} \quad (1 \leq i \leq h).$$

Let $A = \mathbf{k}[W, F_0, \dots, F_{h+1}]$. We have to prove the following two claims:

$$h + 2 \leq \#(A) \leq h + 3; \tag{5}$$

$A$ is the kernel of some triangular derivation

$$\Delta : \mathbf{k}[W, X, Y, Z] \to \mathbf{k}[W, X, Y, Z]. \tag{6}$$

We begin by showing that

$$F_j = f_j + b_j W^{h+1-j} \qquad (0 \leq j \leq h + 1), \tag{7}$$

where $b_j \in \mathbf{k}[W, X, Y, Z]$ and $b_j(0, X, Y, Z) \notin \mathbf{k}[X]$, and where we define $f_{h+1} = 0$. We proceed by induction and note that the assertion is clear for $j \leq 1$. Assume that (7) holds for $0 \leq j \leq i$, for some $i$ such that $1 \leq i \leq h$. Then $F_i = f_i + b_i W^{h+1-i}$ and $F_{i-1} = f_{i-1} + b_{i-1} W^{h+2-i}$, so

$$\begin{aligned} F_{i-1}^q - F_i^p &= (f_{i-1} + b_{i-1} W^{h+2-i})^q - (f_i + b_i W^{h+1-i})^p \\ &= (f_{i-1}^q - f_i^p) - p f_i^{p-1} b_i W^{h+1-i} + \varepsilon_1 W^{h+2-i} \end{aligned}$$

for some $\varepsilon_1 \in \mathbf{k}[W, X, Y, Z]$. Write $\frac{f_{i-1}^q - f_i^p}{W} = f_{i+1} + \varepsilon_2 W^{h-i}$, with $\varepsilon_2 \in \mathbf{k}[W, X]$; then dividing $(F_{i-1}^q - F_i^p)$ by $W$ gives

$$\begin{aligned} F_{i+1} &= (f_{i+1} + \varepsilon_2 W^{h-i}) - p f_i^{p-1} b_i W^{h-i} + \varepsilon_1 W^{h+1-i} \\ &= f_{i+1} + (\varepsilon_2 - p f_i^{p-1} b_i + \varepsilon_1 W) W^{h-i}, \end{aligned}$$

which proves (7). (In particular, the $F_j$'s are polynomials.)

Let $\pi : \mathbf{k}[W, X, Y, Z] \to \mathbf{k}[X, Y, Z]$ be the surjective $\mathbf{k}$-homomorphism defined by

$$W \mapsto 0,\ X \mapsto X,\ Y \mapsto Y,\ Z \mapsto Z.$$

Then (7) implies that $\pi(A) = \mathbf{k}[X^{e_0}, \dots, X^{e_h}, \tau]$, where $e_i = p^{h-i} q^i$ and where $\tau$ is transcendental over $\mathbf{k}(X)$. Let $R = \mathbf{k}[X^{e_0}, \dots, X^{e_h}]$; since $R[\tau]$ is a homomorphic image of $A$, $\#(A) \geq \#(R[\tau])$; since $\#(R[\tau]) = h+2$ by Lemma 1, (5) holds. Define a derivation $\Delta : \mathbf{k}[W, X, Y, Z] \to \mathbf{k}[W, X, Y, Z]$ by

$$\Delta = \begin{vmatrix} \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} & \frac{\partial}{\partial Z} \\ \frac{\partial F_0}{\partial X} & \frac{\partial F_0}{\partial Y} & \frac{\partial F_0}{\partial Z} \\ \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial Z} \end{vmatrix}. \tag{8}$$

Then $\Delta Z = -W^{h+1} \frac{\partial f_1}{\partial X}$, $\Delta Y = -W^h \frac{\partial f_0}{\partial X}$, $\Delta X = W^{2h+1}$ and $\Delta W = 0$, so $\Delta$ is a triangular derivation of $\mathbf{k}[W, X, Y, Z]$. It is clear that $\mathbf{k}[W, F_0, F_1] \subseteq \ker \Delta$, so $A \subseteq \ker \Delta$; let us now argue that $\ker \Delta \subseteq A_W$. Write $B = \mathbf{k}[W, X, Y, Z]$; since

$$B_W \supseteq A_W[X] \supseteq \mathbf{k}[W, W^{-1}, X, F_0, F_1] = \mathbf{k}[W, W^{-1}, X, Y, Z] = B_W,$$

$B_W$ is a polynomial ring over $A_W$. On the other hand, $(\ker \Delta)_W$ contains $A_W$ and is the kernel of the nonzero derivation $\Delta_W : B_W \to B_W$, so $(\ker \Delta)_W = A_W$ and we have shown that $A \subseteq \ker\Delta \subset A_W$. So, in order to prove (6), there remains only to prove

$$A \cap WB = WA. \tag{9}$$

It is easy to see that the proof of (9) reduces to that of the following: if $T_0, \ldots, T_{h+1}$ are indeterminates and $\psi \in \mathbf{k}[T_0, \ldots, T_{h+1}]$, then $\psi(F_0, \ldots, F_{h+1}) \in WB$ implies $\psi(F_0, \ldots, F_{h+1}) \in WA$. Write $\psi = \sum_{n\geq 0} \psi_n T_{h+1}^n$ with $\psi_n \in \mathbf{k}[T_0, \ldots, T_h]$. Then

$$0 = \pi(\psi(F_0, \ldots, F_{h+1})) = \sum_{n\geq 0} \psi_n(X^{e_0}, \ldots, X^{e_h})\tau^n,$$

where $\tau = \pi(F_{h+1})$ is transcendental over $\mathbf{k}(X)$, and consequently $\psi_n \in \ker\varphi$ for all $n$, where $\varphi : \mathbf{k}[T_0, \ldots, T_h] \to \mathbf{k}[X]$ is the $\mathbf{k}$-homomorphism which maps $T_i$ to $X^{e_i}$. By Lemma 2, $\ker\varphi = (T_0^q - T_1^p, \ldots, T_{h-1}^q - T_h^p)$, so $\psi_n = \sum_{j=1}^h \alpha_j(T_{j-1}^q - T_j^p)$ for some $\alpha_j \in \mathbf{k}[T_0, \ldots, T_h]$. Then

$$\begin{aligned}\psi_n(F_0, \ldots, F_h) &= \sum_{j=1}^h \alpha_j(F_0, \ldots, F_h)(F_{j-1}^q - F_j^p)\\ &= \sum_{j=1}^h \alpha_j(F_0, \ldots, F_h)WF_{j+1} \in WA.\end{aligned}$$

So (9) holds and, consequently, $\ker(\Delta) = A$. So (5) and (6) are proved.

**Example.** We exhibit a triangular derivation $\Delta$ of $\mathbf{k}[W, X, Y, Z]$ whose kernel cannot be generated by five elements over $\mathbf{k}$. Let $p = 2$, $q = 5$ and $h = 4$ and, following the proof of Lemma 4, successively define $f_4, f_3, f_2, f_1, f_0$ by:[1]

$$f_4 = X^{625}, \quad f_3 = X^{250}, \quad f_2 = X^{100} + \tfrac{1}{5}X^{225}W,$$
$$f_1 = X^{40} + \left(\tfrac{2}{25}X^{165} + \tfrac{1}{5}X^{90}\right)W + \left(-\tfrac{3}{625}X^{290} - \tfrac{8}{125}X^{215} - \tfrac{2}{25}X^{140}\right)W^2$$

and

$$\begin{aligned}f_0 = X^{16} &+ \left(\tfrac{2}{25}X^{66} + \tfrac{1}{5}X^{36} + \tfrac{4}{125}X^{141}\right)W\\ &+ \left(-\tfrac{92}{3125}X^{191} - \tfrac{23}{625}X^{116} - \tfrac{42}{15625}X^{266} + \tfrac{9}{625}X^{161} - \tfrac{8}{125}X^{86} - \tfrac{2}{25}X^{56}\right)W^2\\ &+ \Bigg(\tfrac{408}{78125}X^{241} + \tfrac{68}{15625}X^{166} + \tfrac{666}{390625}X^{316} + \tfrac{328}{15625}X^{211} + \tfrac{132}{3125}X^{136}\\ &\qquad + \tfrac{36}{625}X^{106} - \tfrac{72}{78125}X^{286} - \tfrac{28}{3125}X^{181} + \tfrac{6}{125}X^{76} + \tfrac{244}{1953125}X^{391}\Bigg)W^3.\end{aligned}$$

Define $\Delta$ as in (8) or, equivalently, by

$$\Delta W = 0, \quad \Delta X = W^9, \quad \Delta Y = -W^4\,\frac{\partial f_0}{\partial X} \quad \text{and} \quad \Delta Z = -W^5\,\frac{\partial f_1}{\partial X}.$$

Then, by (5) and (6), we have $6 \leq \#(\ker\Delta) \leq 7$.

[1] Note that the $f_j$'s are not unique.

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