PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 3, Pages 759–764
S 0002-9939(00)05594-5
Article electronically published on August 30, 2000

A DICHOTOMY THEOREM FOR SUBSETS OF THE POWER SET OF THE NATURAL NUMBERS

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(Communicated by Dale Alspach)

ABSTRACT. The following dichotomy is established for any pair $\mathcal{F}$, $\mathcal{G}$ of hereditary families of finite subsets of $\mathbb{N}$: Given N, an infinite subset of $\mathbb{N}$, there exists M an infinite subset of N so that either $\mathcal{G} \cap [M]^{<\infty} \subset \mathcal{F}$, or $\mathcal{F} \cap [M]^{<\infty} \subset \mathcal{G}$, where $[M]^{<\infty}$ denotes the set of all finite subsets of M.

1. INTRODUCTION

The Schreier families $\{S_\xi\}_{\xi<\omega_1}$ [1] are pointwise compact subsets of $[\mathbb{N}]^{<\infty}$, the set of all finite subsets of positive integers. These families are hereditary in the sense that they contain all subsets of their members. Their importance lies in the fact that they exhaust the complexity of all countable compact metric spaces. They also carry some nice stability properties some of which are described in the third section of our paper. Because of those properties it is often more convenient to work with Schreier families than with countable ordinal intervals.

In Banach space theory it is quite useful to know if a given hereditary subset of $[\mathbb{N}]^{<\infty}$ contains as a subset a Schreier family of certain order. This problem was considered in [10], [4], [5], [16], [15], [6]. The purpose of this note is to provide, by making use of the infinite Ramsey theorem, a fairly simple proof of the following dichotomy:

Theorem 1.1. *Let $\mathcal{F}$, $\mathcal{G}$ be hereditary families of finite subsets of $\mathbb{N}$ and N an infinite subset of $\mathbb{N}$. Then there exists $M \in [N]$ so that either $\mathcal{G} \cap [M]^{<\infty} \subset \mathcal{F}$, or $\mathcal{F} \cap [M]^{<\infty} \subset \mathcal{G}$.*

Recall that $\mathcal{F}$ is *hereditary* if for every F, G with $F \in \mathcal{F}$ and $G \subset F$, we have that also $G \in \mathcal{F}$. $[M]^{<\infty}$ denotes the set of all finite subsets of M and $[N]$ stands for the set of all infinite subsets of N.

An immediate consequence of Theorem 1.1 is our next

Corollary 1.2. *Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$, ξ a countable ordinal, and N an infinite subset of $\mathbb{N}$. Then there exists $M \in [N]$ so that either $S_\xi \cap [M]^{<\infty} \subset \mathcal{F}$, or $\mathcal{F} \cap [M]^{<\infty} \subset S_\xi$.*

Under the assumptions of Corollary 1.2, a somewhat weaker dichotomy was proven by R. Judd [10] with the use of Schreier games: There exist $M \in [N]$,

Received by the editors February 19, 1999 and, in revised form, May 5, 1999.
1991 *Mathematics Subject Classification.* Primary 46B03; Secondary 06A07, 03E02.
Key words and phrases. Ramsey theory, Schreier sets, dichotomy.

$M = (m_i)$, and $L \in [\mathbb{N}]$ so that either $\{m_i : i \in F\} \in \mathcal{F}$, for every $F \in S_\xi$, or, $\{m_i : i \in F\} \in S_\xi$, for every $F \in \mathcal{F}$ such that $F \subset L$.

Finally, we would like to mention that Theorem 1.1 as well as Corollary 1.2 are also obtained by P. Kyriakouli [11]. A weaker version of Corollary 1.2 is proven by V. Farmaki [9]. Their methods however are different from ours and the proofs are rather complicated.

We wish to thank Dale Alspach and Rob Judd for useful discussions regarding this paper.

2. Proof of the main result

The proof of Theorem 1.1 is an immediate consequence of an important principle of infinitary combinatorics known as the infinite Ramsey theorem [12], [8], [18], [7], [19]. We recall the statement of the theorem. $[\mathbb{N}]$ is endowed with the topology of pointwise convergence.

Theorem 2.1. *Let A be an analytic subset of $[\mathbb{N}]$. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that either $[L] \subset A$, or $[L] \subset [M] \setminus A$.*

In the sequel, any set satisfying the conclusion of Theorem 2.1 will be called (completely) Ramsey. For applications of Theorem 2.1 in Banach space theory we refer to [14].

Proof of Theorem 1.1. Given $N \in [\mathbb{N}]$ suppose there exists $M \in [N]$ such that $[M]^{<\infty} \subset \mathcal{G}$; then obviously $\mathcal{F} \cap [M]^{<\infty} \subset \mathcal{G}$ and we are done. Alternatively, suppose there is no such M. In this case, since $\mathcal{G}$ is hereditary, it must be that for any $L \in [N]$, $L = (l_i)$, there exists a $r \in \mathbb{N}$ such that $\{l_1, \dots, l_r\} \notin \mathcal{G}$. Let $g(L)$ be the least $r \in \mathbb{N}$ so that $\{l_1, \dots, l_r\} \notin \mathcal{G}$. Similarly, we are done if there exists $M \in [N]$ such that $[M]^{<\infty} \subset \mathcal{F}$, so we can define $f(L)$ to be the least $r \in \mathbb{N}$ so that $\{l_1, \dots, l_r\} \notin \mathcal{F}$. Define

$$\mathcal{D} = \{L \in [N] : g(L) \leq f(L)\}.$$

By Theorem 2.1, there exists $M \in [N]$ such that either $[M] \subset \mathcal{D}$, or $[M] \subset [N] \setminus \mathcal{D}$. If $[M] \subset \mathcal{D}$, then $\mathcal{G} \cap [M]^{<\infty} \subset \mathcal{F}$. If $[M] \subset [N] \setminus \mathcal{D}$, then $\mathcal{F} \cap [M]^{<\infty} \subset \mathcal{G}$. These are true since any $A \in [M]^{<\infty}$ is an initial segment of some $L \in [M]$. The proof of Theorem 1.1 is complete. □

3. Applications to Schreier families

This section is devoted to some applications of Theorem 1.1 in the study of Schreier families. The main one, Corollary 1.2, is of course a special case of Theorem 1.1. We wish however to give an alternative argument for the proof which provides an "algorithm" for determining if a certain Schreier family embeds into a given hereditary family of finite subsets of $\mathbb{N}$.

We start by recalling the definition of the Schreier families $\{S_\xi\}_{\xi<\omega_1}$ [1]. If E, F are finite sets of integers, we shall adopt the notation $E < F$ to denote the relation $\max E < \min F$. The Schreier families are defined by transfinite induction as follows:

$$S_0 = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}.$$

Suppose S_ζ has been defined for every $\zeta < \xi$. If ξ is a successor ordinal, say $\xi = \zeta + 1$, we set

$$S_\xi = \Big\{\bigcup_{i=1}^n F_i : n \leq \min F_1,\, F_1 < \cdots < F_n,\, F_i \in S_\zeta\,(i \leq n),\, n \in \mathbb{N}\Big\} \cup \{\emptyset\}.$$

If ξ is a limit ordinal, let (ξ_n) be a preassigned increasing sequence of ordinals whose limit is ξ. We set

$$S_\xi = \bigcup_{n=1}^\infty \{F \in S_{\xi_n} : n \leq \min F\} \cup \{\emptyset\}.$$

S_1 was first considered by Schreier [17] in order to provide an example of a weakly null sequence without Cesaro summable subsequence.

The Schreier families have played a central role in the development of modern Banach space theory. They are used in the construction of mixed Tsirelson spaces which are asymptotic ℓ_1 and arbitrarily distortable [3]. The distortion of mixed Tsirelson spaces has been extensively studied in [2]. In that paper as well as in [16], the moduli $(\delta_\alpha)_{\alpha<\omega_1}$ were introduced measuring the complexity of the asymptotic ℓ_1 structure of a Banach space. The definition of those moduli also involves the Schreier families. Other applications can be found in [4] and [6] where the Schreier families form the main tool for determining the structure of those convex combinations of a weakly null sequence that tend to zero in norm, or are equivalent to the unit vector basis of c_0. For applications of the Schreier families in the construction of hereditarily indecomposable Banach spaces, we refer to [3] and [5].

By identifying elements of $[\mathbb{N}]^{<\infty}$ with their indicator functions, every subset of $[\mathbb{N}]^{<\infty}$ can be endowed with the topology of pointwise convergence. A subset $\mathcal{F}$ of $[\mathbb{N}]^{<\infty}$ is said to be *adequate* if it is hereditary and compact in the topology of pointwise convergence. Given $M \in [\mathbb{N}]$, $M = (m_i)$, and $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ we set $\mathcal{F}[M] = \{F \in \mathcal{F} : F \subset M\}$. We also set $\mathcal{F}(M) = \{M(F) : F \in \mathcal{F}\}$, where $M(F) = \{m_i : i \in F\}$. It is easily seen that $\mathcal{F}[M]$ and $\mathcal{F}(M)$ are hereditary (resp. adequate) if $\mathcal{F}$ is.

It is proven in [1] that S_ξ is an adequate family for all $\xi < \omega_1$. In fact, $S_\xi^{(\omega^\xi)} = \{\emptyset\}$ (given A, a subset of a topological space and α, a countable ordinal, we denote by $A^{(\alpha)}$ the α-th derived set of A) and thus S_ξ is homeomorphic to the ordinal interval $[1, \omega^{\omega^\xi}]$, by the Mazurkiewicz-Sierpinski theorem [13]. An important property shared by the Schreier families is that they are *spreading*: If $\{p_1, \ldots, p_k\} \in S_\xi$, $p_1 < \cdots < p_k$, and $q_1 < \cdots < q_k$ are so that $p_i \leq q_i$ for all $i \leq k$, then $\{q_1, \ldots, q_k\} \in S_\xi$. It follows that if $N \in [\mathbb{N}]$, then $S_\xi(M) \subset S_\xi(N)$, for all $M \in [N]$. Since $S_\xi(N)$ is naturally homeomorphic to S_ξ and $S_\xi(N) \subset S_\xi[N] \subset S_\xi$, the Mazurkiewicz-Sierpinski theorem [13] yields that $S_\xi[N]$ is also homeomorphic to S_ξ for all $N \in [\mathbb{N}]$.

Of particular interest are the maximal (under inclusion) members of S_ξ. A characterization of those sets is given in our next lemma.

Lemma 3.1. *Let $F \in S_\xi$, $\xi < \omega_1$. The following are equivalent:*

1. *F is a maximal S_ξ set. That is, if $F \subset G$ and $G \in S_\xi$ then $F = G$.*
2. *$F \cup \{n\} \notin S_\xi$, for every $n \in \mathbb{N}$ such that* $\max F < n$.
3. *$F \cup \{n\} \notin S_\xi$, for some $n \in \mathbb{N}$ with* $\max F < n$.

Proof. By transfinite induction on ξ. The case $\xi = 0$ is trivial. Assume all three assertions are equivalent for every $\zeta < \xi$. The implications 1. $\Rightarrow$ 2. and 2. $\Rightarrow$ 3. are clear so we need only show that 3. implies 1. Note that F is non-empty since the singletons are contained in every Schreier family. We let $k = \min F$.

Suppose first that ξ is a limit ordinal and let (ξ_i) be the sequence of ordinals associated to ξ in the definition of the family S_ξ. It follows that $F \cup \{n\} \notin S_{\xi_i}$ for each $i \leq k$ such that $F \in S_{\xi_i}$. By the induction hypothesis we obtain that F is a maximal S_{ξ_i} set for every $i \leq k$ such that $F \in S_{\xi_i}$. But this means exactly that F is a maximal S_ξ set and thus 3. implies 1. when ξ is a limit ordinal.

Next assume that ξ is a successor ordinal, say $\xi = \zeta + 1$. We claim that if $p \leq k$, $F_1 < \cdots < F_p$ are in S_ζ and $F = \bigcup_{i=1}^p F_i$, then $p = k$ and F_i is a maximal S_ζ set for every $i \leq p$. Indeed, our assumption that $F \cup \{n\} \notin S_\xi$ immediately yields that $p = k$ and $F_p \cup \{n\} \notin S_\zeta$. It follows by the induction hypothesis that F_p is a maximal S_ζ set and that $F_p^{'} = (F_p \setminus \{\min F_p\}) \cup \{n\} \in S_\zeta$. Thus, $F_{p-1} \cup \{\min F_p\} \notin S_\zeta$ for otherwise, $F \cup \{n\} = \bigcup_{i<p-1} F_i \cup (F_{p-1} \cup \{\min F_p\}) \cup F_p^{'}$ would belong to S_ξ contradicting our assumptions. Hence F_{p-1} is also a maximal S_ζ set. Successive applications of this argument yield our claim.

To complete the proof, let $G \in S_\xi$ such that $F \subset G$. We choose $p \leq \min G \leq k$ and $G_1 < \cdots < G_p$ in S_ζ so that $G = \bigcup_{i=1}^p G_i$. Put $I = \{i \leq p : G_i \bigcap F \neq \emptyset\}$. Since $F = \bigcup_{i \in I} G_i \cap F$ and $G_i \cap F \in S_\zeta$, for all $i \in I$, our previous claim yields that $|I| = k$ and that $G_i \cap F$ is a maximal S_ζ set, for all $i \in I$. Therefore, $p = k$ and $G_i \cap F = G_i$, for all $i \leq p$. It follows now that $F = G$. ☐

Corollary 3.2. *Let $M \in [\mathbb{N}]$ and $\xi < \omega_1$. There exists a (necessarily) unique decomposition of M into consecutive maximal S_ξ sets.*

Notation. Given $M \in [\mathbb{N}]$ and $\xi < \omega_1$, we denote by $\{F_n^\xi(M)\}_{n=1}^\infty$ the decomposition of M resulting from Corollary 3.2.

Proof of Corollary 3.2. Suppose $M = (m_i)$. Because S_ξ is adequate, there is a largest $k_1 \in \mathbb{N}$ such that $\{m_1, \ldots, m_{k_1}\} \in S_\xi$. Since $\{m_1, \ldots, m_{k_1}, m_{k_1+1}\} \notin S_\xi$, Lemma 3.1 yields that $\{m_1, \ldots, m_{k_1}\}$ is a maximal S_ξ set. Next choose $k_2 > k_1$ maximal with respect to $\{m_{k_1+1}, \ldots, m_{k_2}\} \in S_\xi$. Once again, Lemma 3.1 yields that $\{m_{k_1+1}, \ldots, m_{k_2}\}$ is a maximal S_ξ set. Continuing in this fashion we obtain the desired decomposition. ☐

Remark. The following stability properties of $\{F_n^\xi(M)\}_{n=1}^\infty$ are easily verified:

1. If $k_1 < k_2 < \ldots$ and $N = \bigcup_{n=1}^\infty F_{k_n}^\xi(M)$, then $F_n^\xi(N) = F_{k_n}^\xi(M)$, for all $n \in \mathbb{N}$.
2. Let $M = (m_i)$ and $N = (n_i)$ be infinite subsets of $\mathbb{N}$. Assume that for some $p \in \mathbb{N}$, $m_i = n_i$ for all $i \leq p$. If $F_k^\xi(M)$ is contained in $\{m_i : i \leq p\}$, then $F_i^\xi(M) = F_i^\xi(N)$ for all $i \leq k$.

Proof of Corollary 1.2. We let

$$\mathcal{D} = \{L \in [N] : F_1^\xi(L) \in \mathcal{F}\}.$$

It follows easily by our previous remark that $\mathcal{D}$ is closed in $[N]$ and therefore Ramsey. Theorem 2.1 yields $M \in [N]$ so that either $[M] \subset \mathcal{D}$, or $[M] \subset [N] \setminus \mathcal{D}$. If the former, then $S_\xi[M] \subset \mathcal{F}$. Indeed, if $A \in S_\xi[M]$ we can apply Lemma 3.1 to obtain $B \subset M$, a maximal S_ξ set containing A. Clearly $B = F_1^\xi(L)$, where $L = B \cup \{m \in M : m > \max B\}$, and thus $A \in \mathcal{F}$ as $\mathcal{F}$ is hereditary.

In case the latter alternative holds let $A = \{m_{i_1}, \dots, m_{i_k}\} \in \mathcal{F}[M]$. We claim that $A \in S_\xi$. Indeed, choose $r \le k$ maximal with respect to $\{m_{i_1}, \dots, m_{i_r}\} \in S_\xi$. Then $r = k$ for if not, letting $B = \{m_{i_1}, \dots, m_{i_r}\}$ we obtain through Lemma 3.1 that B is a maximal S_ξ subset of M. Once again it follows that $B = F_1^\xi(L)$ where $L = B \cup \{m \in M : m > \max B\}$. Hence $B \notin \mathcal{F}$ and thus also $A \notin \mathcal{F}$ since $B \subset A$. This contradiction shows that $r = k$ and so $A \in S_\xi$. The proof is now complete. □

Another application of Theorem 1.1 is the following

Corollary 3.3. *Let $\xi < \omega_1$ and $\mathcal{F}$ be an adequate family. Also let $N \in [\mathbb{N}]$ and assume that $\emptyset \in \mathcal{F}[L]^{(\omega^\xi)}$, for every $L \in [N]$. Then there exists $M \in [N]$ so that $F \setminus \{\min F\} \in \mathcal{F}$ for every $F \in S_\xi[M]$.*

Remark. 1. Corollary 3.3 was obtained in [2] for the case of $\mathcal{F} = S_\xi(N)$. In fact, as shown in [2], there exists $M \in [N]$ satisfying the conclusion of Corollary 3.3 for all $\xi < \omega_1$.

2. Adhering to the terminology of [4], the assumption that $\emptyset \in \mathcal{F}[L]^{(\omega^\xi)}$, for every $L \in [N]$, has an equivalent formulation in terms of the strong Cantor-Bendixson index. The precise statement is that the strong Cantor-Bendixson index of $\mathcal{F}[N]$ exceeds ω^ξ.

Proof of Corollary 3.3. We let

$$\mathfrak{D} = \big\{F \in [\mathbb{N}]^{<\infty} : F \setminus \{\min F\} \in \mathcal{F}\big\}.$$

It is clear that $\mathfrak{D}$ is hereditary and therefore Theorem 1.1 yields $M \in [N]$ so that either $S_\xi[M] \subset \mathfrak{D}$, or $\mathfrak{D}[M] \subset S_\xi$. We shall show that the second alternative cannot hold and from this deduce the assertion of the corollary. Indeed, for each $m \in M$, we set

$$K_m = \Big\{\{m\} \cup F : F \in \mathcal{F}[M], \text{ and } m < \min F\Big\} \cup \Big\{\{m\}\Big\}.$$

Evidently, $K_m \subset \mathfrak{D}[M]$, for all $m \in M$. If it were the case that $\mathfrak{D}[M] \subset S_\xi$, then we can easily verify that K_m is closed in S_ξ. Moreover, it is homeomorphic to a clopen neighborhood of $\emptyset$ in $\mathcal{F}[M]$. But then $K = \bigcup_{m \in M} K_m \cup \{\emptyset\}$ is a compact subset of S_ξ whose $(\omega^\xi + 1)$-derived set is non-empty. This is a contradiction since S_ξ is homeomorphic to $[1, \omega^{\omega^\xi}]$. □

Acknowledgments

We express our sincere appreciation to the referee for the elegant argument in the proof of Theorem 1.1. In an earlier version of this paper, Corollary 1.2 appeared as the main result and it was the referee who pointed out that an argument simpler than our original one actually yields the more general result stated in Theorem 1.1.

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