PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 3, Pages 815–821
S 0002-9939(00)05621-5
Article electronically published on September 19, 2000

A NUMERICAL RANGE CHARACTERIZATION OF UNIFORMLY SMOOTH BANACH SPACES

ANGEL RODRIGUEZ PALACIOS

(Communicated by Dale Alspach)

ABSTRACT. We prove that a Banach space X is uniformly smooth if and only if, for every X-valued bounded function f on the unit sphere of X, the intrinsic numerical range of f is equal to the closed convex hull of the spatial numerical range of f.

1. INTRODUCTION

Let X be a Banach space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). We denote by S_X, B_X, and X^* the unit sphere of X, the closed unit ball of X, and the dual space of X, respectively. For u in S_X, we denote by $D(X,u)$ the set of all states of X relative to u, namely

$$D(X,u) := \{\phi \in S_{X^*} : \phi(u) = 1\},$$

and then, for x in X, we define the *numerical range* $V(X,u,x)$ of x relative to u as the nonempty, convex, and compact subset of $\mathbb{K}$ given by the equality

$$V(X,u,x) := \{\phi(x) : \phi \in D(X,u)\}.$$

Given a mapping f from S_X into X, we can consider the so-called *spatial numerical range* $W(f)$ of f, namely

$$W(f) := \bigcup \{V(X,u,f(u)) : u \in S_X\}.$$

If the mapping f above is bounded, then it also has an *intrinsic numerical range* $V(f)$, given by the equality

$$V(f) := V(B(S_X, X), \mathbf{1}, f),$$

where $B(S_X, X)$ denotes the Banach space of all bounded functions from S_X to X, and $\mathbf{1}$ stands for the natural embedding $S_X \hookrightarrow X$. In this case we have the inclusion

$$\overline{\text{co}}\, W(f) \subseteq V(f),$$

where $\overline{\text{co}}$ means closed convex hull. (Indeed, for u in S_X and ϕ in $D(X,u)$, the mapping $g \to \phi(g(u))$ from $B(S_X, X)$ to $\mathbb{K}$ is an element of $D(B(S_X, X), \mathbf{1})$.) The inclusion above is known to be an equality whenever either f is (the restriction to S_X of) a continuous linear operator on X [6], or $\mathbb{K} = \mathbb{C}$ and f is (the restriction to S_X of) a uniformly continuous function on B_X which is holomorphic on the interior of B_X [5].

Received by the editors October 19, 1998 and, in revised form, May 24, 1999.
2000 *Mathematics Subject Classification.* Primary 46B04, 46B20.

The aim of this note is to show that the equality $\overline{\text{co}}\, W(f) = V(f)$ can hold for every f in $B(S_X, X)$, but only if X lies in a very restrictive class of Banach spaces. Actually, we prove that the equality $\overline{\text{co}}\, W(f) = V(f)$ holds for every f in $B(S_X, X)$ if and only if X is uniformly smooth. Note that every uniformly smooth Banach space is super-reflexive, and that, in fact, super-reflexivity is the isomorphic side of the uniform smoothness, i.e., every super-reflexive Banach space has a uniformly smooth equivalent renorming (see for instance [2]).

2. The results

Let X be a Banach space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), and u an element in S_X. For x in X, the function $\alpha \to \|u + \alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex, hence there exists the number

$$M^u(x) := \lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} = \inf \left\{ \frac{\|u + \alpha x\| - 1}{\alpha} : \alpha \in \mathbb{R}^+ \right\},$$

and it is known that the equality

$$M^u(x) = \max\{\text{Re}(\lambda) \colon \lambda \in V(X, u, x)\}$$

holds (see for instance [3, Theorem V.9.5]). For α in $\mathbb{R}^+$, we put

$$\varphi(X, u, \alpha) := \sup \left\{ \frac{\|u + \alpha x\| - 1}{\alpha} - M^u(x) \colon x \in B_X \right\}.$$

Theorem 1. *Let I be an infinite set, $\{X_i\}_{i \in I}$ a family of Banach spaces over $\mathbb{K}$, and, for each i in I, let u_i be a fixed norm-one element in X_i. Denote by Y the l_∞-sum of the family $\{X_i\}_{i \in I}$, and by u the element of S_Y given by $u(i) := u_i$ for every i in I. Then the following assertions are equivalent:*

i) *the equality*

$$V(Y, u, y) = \overline{\text{co}} \left(\bigcup \{V(X_i, u(i), y(i)) \colon i \in I\} \right)$$

holds for every y in Y.

ii) $\lim_{(i,\alpha) \to (\infty, 0)} \varphi(X_i, u(i), \alpha) = 0$.

In assertion ii) above, the symbol $\lim_{(i,\alpha) \to (\infty, 0)}$ means the limit along the filter basis on $I \times \mathbb{R}^+$ consisting of all subsets of $I \times \mathbb{R}^+$ of the form $J \times (0, \delta)$, where J is a co-finite subset of I and δ is a positive number. We also note that, if the set I in the theorem fails to be infinite, then, for every y in Y, the equality

$$V(Y, u, y) = \text{co} \left(\bigcup \{V(X_i, u(i), y(i)) \colon i \in I\} \right)$$

is automatically true.

Proof of Theorem 1. ii) $\Rightarrow$ i) Fix y in S_Y, and let ε be a positive number. By assumption ii), there exists $\delta > 0$, and a co-finite subset J of I, such that the inequality

$$\frac{\|u(j) + \alpha x_j\| - 1}{\alpha} - M^{u(j)}(x_j) < \varepsilon$$

holds whenever j is in J, x_j is in B_{X_j} for such a j, and $0 < \alpha < \delta$. For k in $I \backslash J$, we can choose $\delta_k > 0$ such that

$$\frac{\|u(k) + \alpha y(k)\| - 1}{\alpha} - M^{u(k)}(y(k)) < \varepsilon$$

whenever $0 < \alpha < \delta_k$. By putting $\rho := \min\{\delta, \min\{\delta_k \colon k \in I \backslash J\}\}$, it follows that the inequality

$$\frac{\|u(i) + \alpha y(i)\| - 1}{\alpha} < M^{u(i)}(y(i)) + \varepsilon$$

is true for every i in I and $0 < \alpha < \rho$. Therefore, for $0 < \alpha < \rho$ we have

$$\frac{\|u + \alpha y\| - 1}{\alpha} \le \sup\{M^{u(i)}(y(i)) \colon i \in I\} + \varepsilon,$$

and, by letting $\alpha \to 0$, we obtain

$$M^u(y) \le \sup\{M^{u(i)}(y(i)) \colon i \in I\} + \varepsilon.$$

In view of the arbitrariness of ε, we actually have

$$M^u(y) \le \sup\{M^{u(i)}(y(i)) \colon i \in I\},$$

so that, replacing y by zy with z in $S_{\mathbb{K}}$, the inclusion

$$V(Y, u, y) \subseteq \overline{\mathrm{co}} \left(\bigcup \{V(X_i, u(i), y(i)) \colon i \in I\} \right)$$

follows. The reverse inclusion is trivial.

i) $\Rightarrow$ ii) Assume that ii) is not true. Then there is $\varepsilon > 0$ such that, for every co-finite subset J of I, and for every $\delta > 0$, there exists $0 < \alpha < \delta$, j in J, and x_j in B_{X_j} such that

$$\frac{\|u(j) + \alpha x_j\| - 1}{\alpha} - M^{u(j)}(x_j) \ge \varepsilon.$$

Take $0 < \alpha_1 < 1$, $i(1)$ in I, and $x_{i(1)}$ in $B_{X_{i(1)}}$ such that

$$\frac{\|u(i(1)) + \alpha_1 x_{i(1)}\| - 1}{\alpha_1} - M^{u(i(1))}(x_{i(1)}) \ge \varepsilon.$$

Assume that, for some n in $\mathbb{N}$, $\alpha_1, \ldots, \alpha_n$, $i(1), \ldots, i(n)$, $x_{i(1)}, \ldots, x_{i(n)}$ have been found such that $0 < \alpha_m < \frac{1}{m}$, $i(m)$ is in I, $x_{i(m)}$ belongs to $X_{i(m)}$,

$$\frac{\|u(i(m)) + \alpha_m x_{i(m)}\| - 1}{\alpha_m} - M^{u(i(m))}(x_{i(m)}) \ge \varepsilon$$

(for all m in $\{1, \ldots, n\}$), and $i(m) \ne i(m')$ (for all m, m' in $\{1, \ldots, n\}$ with $m \ne m'$). Then we can choose $0 < \alpha_{n+1} < \frac{1}{n+1}$, $i(n+1)$ in $I \backslash \{i(1), \ldots, i(n)\}$, and $x_{i(n+1)}$ in $B_{X_{i(n+1)}}$ satisfying

$$\frac{\|u(i(n+1)) + \alpha_{n+1} x_{i(n+1)}\| - 1}{\alpha_{n+1}} - M^{u(i(n+1))}(x_{i(n+1)}) \ge \varepsilon.$$

In this way, we have constructed a sequence $\{(\alpha_n, i(n), x_{i(n)})\}_{n \in \mathbb{N}}$ with $\alpha_n \in \mathbb{R}^+$, $\{\alpha_n\} \to 0$, $i(n) \in I$, $i(n) \ne i(m)$ for $n \ne m$, $x_{i(n)} \in B_{X_{i(n)}}$, and

$$\frac{\|u(i(n)) + \alpha_n x_{i(n)}\| - 1}{\alpha_n} - M^{u(i(n))}(x_{i(n)}) \ge \varepsilon.$$

Put $r := \limsup_{n \to \infty} \{M^{u(i(n))}(x_{i(n)})\}$, so that, by discarding a finite number of terms in the sequence $\{(\alpha_n, i(n), x_{i(n)})\}$, we can assume that, in addition to the above properties, we have

$$M^{u(i(n))}(x_{i(n)}) \le r + \frac{\varepsilon}{2}$$

for every n. Now, consider the element y in Y defined by $y(i) = x_{i(n)}$ if $i = i(n)$ for some n, and $y(i) = ru(i)$ otherwise. Then, for every n in $\mathbb{N}$ we have

$$\frac{\|u + \alpha_n y\| - 1}{\alpha_n} \geq \frac{\|u(i(n)) + \alpha_n y(i(n))\| - 1}{\alpha_n}$$
$$= \frac{\|u(i(n)) + \alpha_n x_{i(n)}\| - 1}{\alpha_n} \geq M^{u(i(n))}(x_{i(n)}) + \varepsilon.$$

By taking upper limits as $n \to \infty$, we obtain

$$M^u(y) \geq r + \varepsilon.$$

On the other hand, the definition of y yields the inequality

$$\sup\{M^{u(i)}(y(i)) \colon i \in I\} \leq r + \frac{\varepsilon}{2}.$$

It follows that

$$\sup\{M^{u(i)}(y(i)) \colon i \in I\} + \frac{\varepsilon}{2} \leq M^u(y),$$

so that the equality in assertion i) cannot be true for y. □

Let X be a Banach space. Given u in S_X, we say that *the norm of X is strongly subdifferentiable at u* if

$$M^u(x) = \lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} \text{ uniformly for } x \text{ in } B_X$$

(equivalently, if $\lim_{\alpha \to 0} \varphi(X, u, \alpha) = 0$). Given a nonempty set I, we denote by $B(I, X)$ the Banach space of all bounded functions from I to X. A straightforward consequence of Theorem 1 is the following.

Corollary 2 ([1]). *Let X be a Banach space, and let u be in S_X. For every nonempty set I, denote by $\hat{u}$ the element of $B(I, X)$ defined by $\hat{u}(i) = u$ for all i in I. Then the following assertions are equivalent:*

i) *For every nonempty set I and every f in $B(I, X)$, the equality*

$$V(B(I, X), \hat{u}, f) = \overline{\text{co}}\left(\bigcup\{V(X, u, f(i)) \colon i \in I\}\right)$$

holds.

ii) *There exists an infinite set I such that*

$$V(B(I, X), \hat{u}, f) = \overline{\text{co}}\left(\bigcup\{V(X, u, f(i)) \colon i \in I\}\right)$$

for every f in $B(I, X)$.

iii) $V(B(\mathbb{N}, X), \hat{u}, f) = \overline{\text{co}}(\bigcup\{V(X, u, f(n)) \colon n \in \mathbb{N}\})$ *for all f in $B(\mathbb{N}, X)$.*

iv) *The norm of X is strongly subdifferentiable at u.*

Given a Banach space X, and a subset D of S_X, we say that *the norm of X is uniformly strongly subdifferentiable on D* if

$$M^u(x) = \lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} \text{ uniformly for } (u, x) \text{ in } D \times B_X.$$

Proposition 3. *Let E be a topological space without nonempty open finite subsets (for instance, every topological T_1-space without isolated points), X a Banach space, and $g \colon E \to S_X$ a continuous function. Then the following assertions are equivalent:*

i) *For every f in $B(E,X)$, the equality*

$$V(B(E,X),g,f) = \overline{\text{co}}\left(\bigcup\{V(X,g(t),f(t)) \colon t \in E\}\right)$$

holds.

ii) *The norm of X is uniformly strongly subdifferentiable on the range of g.*

Proof. ii) $\Rightarrow$ i) The uniform strong subdifferentiability of the norm of X on the range of g implies $\lim_{(t,\alpha)\to(\infty,0)} \varphi(X,g(t),\alpha) = 0$. Now, apply Theorem 1.

i) $\Rightarrow$ ii) Let $\varepsilon > 0$. By assumption i) and Theorem 1, there exist $\delta > 0$ and a co-finite subset J of E such that the inequality

$$\frac{\|g(t)+\alpha x\| - 1}{\alpha} - M^{g(t)}(x) \leq \frac{\varepsilon}{2}$$

is true whenever t is in J, x is in B_X, and $0 < \alpha < \delta$. Then, for $0 < \alpha < \delta$, $0 < \beta < \delta$, t in J, and x in B_X, we have

$$\left| \frac{\|g(t)+\alpha x\| - 1}{\alpha} - \frac{\|g(t)+\beta x\| - 1}{\beta} \right| \leq \varepsilon.$$

Since J is dense in E and g is continuous, the last inequality remains true for $0 < \alpha < \delta$, $0 < \beta < \delta$, t in E, and x in B_X. By letting $\beta \to 0$, we obtain

$$\frac{\|g(t)+\alpha x\| - 1}{\alpha} - M^{g(t)}(x) \leq \varepsilon$$

whenever t is in E, x is in B_X, and $0 < \alpha < \delta$. □

We note that Proposition 3 generalizes Corollary 2. Indeed, if I is an infinite set, and if we endow I with the trivial topology, then I becomes a topological space with no nonempty open finite subsets, and the unique continuous functions on I are the constant ones.

Let X be a Banach space. Recall that, given u in S_X, *the norm of* X is said to be *Fréchet differentiable at* u if there exists a continuous $\mathbb{R}$-linear mapping $\tau(u,\cdot) \colon X \to \mathbb{R}$ such that

$$\lim_{\|h\|\to 0} \frac{|\|u+h\| - 1 - \tau(u,h)|}{\|h\|} = 0.$$

It is easy to see that the norm of X is Fréchet differentiable at u if and only if X is smooth at u (i.e., $D(X,u)$ reduces to a singleton) and the norm of X is strongly subdifferentiable at u. If this is the case, then the unique element ϕ in $D(X,u)$ and the real-valued functional $\tau(u,\cdot)$ are mutually determined by means of the equality $\text{Re} \circ \phi = \tau(u,\cdot)$. If the norm of X is Fréchet differentiable at every point of S_X, and if we have

$$\lim_{\|h\|\to 0} \frac{|\|u+h\| - 1 - \tau(u,h)|}{\|h\|} = 0 \text{ uniformly for } u \text{ in } S_X,$$

then we say that X is *uniformly smooth*. The notion of uniform smoothness of X is apparently stronger than that of uniform strong subdifferentiability of the norm of X on S_X, but, as proved in [4, Proposition 4.1], both notions actually are equivalent. Noticing also that the uniform strong subdifferentiability of the norm of X on some subset D of S_X implies the uniform strong subdifferentiability of the norm of X on the closure of D (argue like in the proof of Proposition 3), we derive the next corollary.

Corollary 4. *Let E be a topological space without nonempty open finite subsets, X a Banach space, and g a continuous function from E onto a dense subset of S_X. Then the following assertions are equivalent:*

i) *For every f in $B(E, X)$, the equality*

$$V(B(E,X), g, f) = \overline{\text{co}}\left(\bigcup\{V(X, g(t), f(t)) \colon t \in E\}\right)$$

holds.

ii) *X is uniformly smooth.*

Taking in Corollary 4 $E = S_X$ and g equal to the inclusion mapping $S_X \hookrightarrow X$, we obtain the characterization of uniformly smooth Banach spaces announced in the introduction.

Theorem 5. *For a Banach space X, the following assertions are equivalent*:

i) *For every bounded function $f \colon S_X \to X$, the equality $\overline{\text{co}}\, W(f) = V(f)$ is true.*

ii) *X is uniformly smooth.*

To conclude the paper, let us show an elemental example of a Banach space X, together with a bounded function $f \colon S_X \to X$, such that the inclusion $\overline{\text{co}}\, W(f) \subseteq V(f)$ is strict.

Example 6. Take $X = \mathbb{R}^2$ with norm $\|(\lambda, \mu)\| := \max\{|\lambda|, |\mu|\}$. For x, y in X, put $]x, y[:= \{\alpha x + (1-\alpha)y \colon 0 < \alpha < 1\}$, and define a function $f \colon S_X \to X$ by

$$f(u) := \begin{cases} (0,0) & \text{if } u \in \{(1,1), (1,-1), (-1,-1), (-1,1)\}, \\ (0,1) & \text{if } u \in](1,1), (1,-1)[\cup](-1,-1), (-1,1)[, \\ (1,0) & \text{if } u \in](1,-1), (-1,-1)[\cup](-1,1), (1,1)[. \end{cases}$$

Then f is bounded with $\|f\| = 1$. Moreover, it is easily checked that the equality $\|\mathbf{1} + \alpha f\| = 1 + |\alpha|$ holds for every α in $\mathbb{R}$, and therefore $V(f)$ is equal to the closed real interval $[-1, 1]$. However, we have $V(X, u, f(u)) = \{0\}$ for every u in S_X, and hence $W(f) = \{0\}$.

Acknowledgements

The author is most grateful to J. Becerra, R. Payá, and W. Werner for fruitful discussions about the matter in this note. He also thanks the referee for his interesting comments.

References

[1] C. Aparicio, F. Ocaña, R. Paya, and A. Rodriguez, *A nonsmooth extension of Fréchet differentiability of the norm with applications to numerical ranges*, Glasgow Math. J. **28** (1986), 121–137. MR **87j:**46026

[2] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Math. **64**, 1993. MR **94d:**46012

[3] N. Dunford and J. T. Schwartz, *Linear operators, Part* I, Interscience Publishers, New York, 1958. MR **90g:**47001a

[4] C. Franchetti and R. Paya, *Banach spaces with strongly subdifferentiable norm*, Bolletino U. M. I. 7-B (1993), 45–70. MR **94d:**46015

[5] L. A. Harris, *The numerical range of holomorphic functions in Banach spaces*, Amer. J. Math. **93** (1971), 1005–1019. MR **46:**663

[6] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. **100** (1961), 29–43. MR **24A:**2860

Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain

E-mail address: apalacio@goliat.ugr.es