

CHARACTERIZATION OF DISCRIMINATOR VARIETIES

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ABSTRACT. We prove that a variety \mathcal{V} is a discriminator variety if and only if \mathcal{V} has the Fraser-Horn property and every member of \mathcal{V} is representable as a Boolean product whose factors are directly indecomposable or trivial.

A variety has the *Fraser-Horn property* [3] if every congruence on a product $A_1 \times A_2$ is of the form $\theta_1 \times \theta_2$. Examples of varieties with this property are congruence distributive varieties and congruence modular varieties in which no non-trivial member has a trivial subalgebra [8]. In this paper we prove the following:

Theorem. *For a variety \mathcal{V} the following are equivalent:*

- (a) \mathcal{V} is a discriminator variety.
- (b) \mathcal{V} has the Fraser-Horn property and every member of \mathcal{V} is representable as a Boolean product whose factors are directly indecomposable or trivial.

We remit to [2] for notation and basic facts on Boolean products and discriminator varieties. The concept of decomposition operation plays a deep role in the proof. A *decomposition operation* on an algebra A is a homomorphism $d : A \times A \rightarrow A$ satisfying:

$$\begin{aligned}d(x, x) &\approx x, \\d(d(x, y), z) &\approx d(x, z) \approx d(x, d(y, z)).\end{aligned}$$

Given a pair θ, δ of complementary factor congruences we have associated a decomposition operation defined by $d_{\theta, \delta}(a, b) =$ the unique $c \in A$ such that $(c, a) \in \theta$ and $(c, b) \in \delta$. Reciprocally, given a decomposition operation d , the relations $\theta_d = \{(x, y) : d(x, y) = y\}$ and $\delta_d = \{(x, y) : d(x, y) = x\}$ are a pair of complementary factor congruences. These maps are mutually inverse [4].

For an algebra A let Δ denote the diagonal congruence on A .

Proof of the Theorem. (a) \Rightarrow (b) Since \mathcal{V} is congruence distributive it has the Fraser-Horn property [1]. The Boolean representation is the well known result of Bulman-Fleming, Keimel and Werner [2].

(b) \Rightarrow (a) Since \mathcal{V} has the Fraser-Horn property, we have that:

(1) For every $A \in \mathcal{V}$, the set $FC(A)$ of all factor congruences forms a Boolean sublattice of the congruence lattice of A .

(2) If $\varphi : A \rightarrow B$ is an onto homomorphism and d is a decomposition operation on A , then the equation $\varphi(d)(\varphi(a), \varphi(b)) = \varphi(d(a, b))$ defines a decomposition operation $\varphi(d)$ on B .

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(1) is proved in [1, 1.4]. To prove (2) suppose (w.l.o.g.) that $A = A_1 \times A_2$ and $d((a_1, a_2), (b_1, b_2)) = (a_1, b_2)$, and use the fact that $\ker \varphi$ factorizes.¹

Next let $A \subseteq \Pi\{A_i : i \in I\}$ be a Boolean product. For $i \in I$, let $\theta_i = \{(x, y) \in A^2 : x(i) = y(i)\}$. Suppose that there is a set $J \subseteq I$ such that every A_i , with $i \in J$, is directly indecomposable and $\bigcap\{\theta_i : i \in J\} = \Delta$. We will prove that

(3) $\{\theta_i : i \in I\} - \{\Delta\} = \{\cup \mathbf{m} : \mathbf{m} \text{ is a maximal ideal of } FC(A)\} - \{\Delta\}$.

For a set B define $\eta^B(a, b, c, d) = c$, if $a = b$ and $\eta^B(a, b, c, d) = d$, if $a \neq b$. By [6, 2.3] there is a 4-ary operation h on A such that the algebra $\langle A, h \rangle$ generates a discriminator variety and

(4) $\langle A, h \rangle \subseteq \Pi\{\langle A_i, \eta^{A_i} \rangle : i \in I\}$ is a Boolean product with simple or trivial factors.

Let θ, θ^* be a pair of complementary factor congruences of A . Since $\theta_i = (\theta_i \vee \theta) \cap (\theta_i \vee \theta^*)$ [3] and each θ_i , with $i \in J$, is indecomposable, we have that $\theta \subseteq \theta_i$ or $\theta^* \subseteq \theta_i$, for every $i \in J$. Since $\bigcap\{\theta_i : i \in J\} = \Delta$, we obtain that $\bigcap\{\theta_i : \theta \subseteq \theta_i\}$ and $\bigcap\{\theta_i : \theta^* \subseteq \theta_i\}$ are a pair of complementary factor congruences of A , which by (1) implies that $\theta = \bigcap\{\theta_i : \theta \subseteq \theta_i\}$. Thus θ is a congruence on $\langle A, h \rangle$ (note that the θ_i are congruences of $\langle A, h \rangle$) and we have proved that

(5) $FC(A) = FC(\langle A, h \rangle)$.

Since principal congruences in discriminator varieties are factor congruences [10, (6) of 2.2], (5) implies that

(6) For $(x, y) \in A^2$ there is a least factor congruence containing the pair (x, y) .

Since $\langle A, h \rangle$ generates a discriminator variety, we have that

(7) The natural map

$$\langle A, h \rangle \rightarrow \Pi\{\langle A, h \rangle / \cup \mathbf{m} : \mathbf{m} \text{ is a maximal ideal of } FC(A)\}$$

is a Boolean representation with simple or trivial factors.

Thus (3) follows from (4), (7) and the uniqueness of Boolean representations in discriminator varieties [5, 8.3]. We note that (3) and (b) imply that

(8) For every $A \in \mathcal{V}$, $A / \cup \mathbf{m}$ is directly indecomposable or trivial, for every maximal ideal of $FC(A)$.

Next we will prove² that

(9) If A is directly indecomposable and $B \subseteq A$, then B is directly indecomposable or trivial.

Suppose to the contrary that B is decomposable. Let x_0 be a point in the Cantor Discontinuum C . Give A the discrete topology. Let $D = \{f \in A^C : f \text{ is continuous and } f(x_0) \in B\}$. Note that $D \subseteq \Pi\{D_x : x \in C\}$ is a Boolean product where $D_x = A$ if $x \neq x_0$ and $D_{x_0} = B$. Further note that $\bigcap_{x \in C - \{x_0\}} \theta_x = \Delta$ ($C - \{x_0\}$ is dense). Thus (3) and (8) produce a contradiction since $D / \theta_{x_0} \cong B$.

Let c_1, c_2 be new constant symbols and let d_1, d_2, \dots be new binary function symbols. Let \mathcal{V}_e be the variety axiomatized by a set of identities defining \mathcal{V} plus the following identities:

$$d_i(x, x) \approx x,$$

$$d_i(d_i(x, y), z) \approx d_i(x, z) \approx d_i(x, d_i(y, z)),$$

$$d_i(f(\vec{x}), f(\vec{y})) \approx f(d_i(x_1, y_1), \dots, d_i(x_n, y_n)),$$

$$d_i(c_1, c_2) \approx c_2, f \text{ an } n\text{-ary function symbol of } \mathcal{V}, n \geq 1, i = 1, 2, \dots$$

Note that $\langle A, a, b, d_1, \dots \rangle \in \mathcal{V}_e$ iff $A \in \mathcal{V}$ and each d_i is a decomposition operation on A such that $(a, b) \in \theta_{d_i}$. Let F be the free \mathcal{V}_e -algebra freely generated by z and

¹Indeed condition (2) is equivalent to the Fraser-Horn property.

²The argument we will use was extracted from [1].

let F_r be the reduct of F to the language of \mathcal{V} . Since $F_r \in \mathcal{V}$, by (6) there exists a factor congruence θ , which is the least factor congruence on F_r containing the pair (c_1, c_2) . Let δ be the complement of θ in $FC(F_r)$. Note that

$$(10) \quad d_{\theta, \delta}(c_1, c_2) = c_2.$$

Since each d_i^F is a decomposition operation on F_r satisfying $(c_1, c_2) \in \theta_{d_i^F}$, we have

$$(11) \quad \forall a, b \in F_r \quad d_{\theta, \delta}(a, b) = b \rightarrow d_i^F(a, b) = b.$$

For each term $t(z)$ in the language of \mathcal{V}_e , define a term $t(z)^*$ with variables in $\{x, y, z\}$ as follows:

$$\begin{aligned} c_1^* &= x, \\ c_2^* &= y, \\ s^* &= s, \text{ for } s \in \{z\} \cup \{\text{constant symbols of } \mathcal{V}\}, \\ f(t_1, \dots, t_n)^* &= f(t_1^*, \dots, t_n^*), \text{ } f \text{ an } n\text{-ary function symbol of } \mathcal{V}, \\ d_i(t_1, t_2)^* &= t_2^*, \text{ } i = 1, 2, \dots \end{aligned}$$

We will follow the usual custom of thinking of $d_{\theta, \delta}(z, c_1) \in F$ as a term $t(z)$, which we prefer to write as $t(c_1, c_2, z)$ to refer to all the occurrences of c_1 and c_2 in $t(z)$. The goal will be to show that $d_{\theta, \delta}(z, c_1)^* = t(c_1, c_2, z)^*$ provides a discriminator term for \mathcal{V} , i.e. we will prove that for every directly indecomposable $A \in \mathcal{V}$ and every $a, b, c \in A$, we have that $t(c_1, c_2, z)^*(a, b, c) = c$, if $a = b$ and $t(c_1, c_2, z)^*(a, b, c) = a$, otherwise. Note that by (9) we can suppose that A is generated by $\{a, b, c\}$. Let $\pi_1, \pi_2 : A \times A \rightarrow A$ be the canonical projections. Let k be such that each d_i , with $i > k$, does not occur in $t(c_1, c_2, z)$. First suppose that

$a = b$. Since $\langle A, a, b, \overbrace{\pi_2, \dots, \pi_2}^k, \pi_1, \pi_1, \dots \rangle \in \mathcal{V}_e$, there is a morphism

$$\begin{aligned} \varphi : F &\rightarrow \langle A, a, b, \pi_2, \dots, \pi_2, \pi_1, \pi_1, \dots \rangle \\ z &\rightarrow c. \end{aligned}$$

Since $d_{\theta, \delta}(z, d_{\theta, \delta}(z, c_1)) = d_{\theta, \delta}(z, c_1)$, by (11) we have that $d_i^F(z, d_{\theta, \delta}(z, c_1)) = d_{\theta, \delta}(z, c_1)$, $i = 1, 2, \dots$. Further note that

$$\varphi(d_{\theta, \delta}(z, c_1)) = \varphi(t(c_1, c_2, z)) = t(c_1, c_2, z)^*(a, b, c).$$

Thus we have

$$t(c_1, c_2, z)^*(a, b, c) = \varphi(d_{k+1}(z, d_{\theta, \delta}(z, c_1))) = \pi_1(c, \varphi(d_{\theta, \delta}(z, c_1))) = c.$$

Finally suppose that $a \neq b$. Let

$$\begin{aligned} \gamma : F &\rightarrow \langle A, a, b, \pi_2, \dots, \pi_2, \pi_2, \pi_2, \dots \rangle \\ z &\rightarrow c. \end{aligned}$$

Note that by (2) we have a decomposition operation $\gamma(d_{\theta, \delta})$. Since $(c_1, c_2) \in \theta = \theta_{d_{\theta, \delta}}$, we have that $(a, b) \in \theta_{\gamma(d_{\theta, \delta})}$, which implies that $\theta_{\gamma(d_{\theta, \delta})} = A \times A$, because A is directly indecomposable. Thus $\gamma(d_{\theta, \delta}) = \pi_2$ and hence

$$t(c_1, c_2, z)^*(a, b, c) = \gamma(d_{\theta, \delta}(z, c_1)) = \gamma(d_{\theta, \delta})(\gamma(z), \gamma(c_1)) = \pi_2(c, a) = a. \quad \square$$

(b) \Rightarrow (a) was proved in [7] with the restriction that the Boolean representations have no trivial factors. That proof is based on the concept of central element [9] which cannot be used in general since central elements exist only in varieties in which no non-trivial algebra has a trivial subalgebra. We conclude the paper with the following question:

Is it possible to replace in the above theorem the Fraser-Horn property by the property that for every $A \in \mathcal{V}$, the set $FC(A)$ forms a Boolean sublattice of the congruence lattice of A ?

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