PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 3, Pages 663–666
S 0002-9939(00)05627-6
Article electronically published on August 30, 2000

CHARACTERIZATION OF DISCRIMINATOR VARIETIES

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(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. We prove that a variety $\mathcal{V}$ is a discriminator variety if and only if $\mathcal{V}$ has the Fraser-Horn property and every member of $\mathcal{V}$ is representable as a Boolean product whose factors are directly indecomposable or trivial.

A variety has the *Fraser-Horn property* [3] if every congruence on a product $A_1 \times A_2$ is of the form $\theta_1 \times \theta_2$. Examples of varieties with this property are congruence distributive varieties and congruence modular varieties in which no non-trivial member has a trivial subalgebra [8]. In this paper we prove the following:

Theorem. *For a variety $\mathcal{V}$ the following are equivalent:*

(a) *$\mathcal{V}$ is a discriminator variety.*

(b) *$\mathcal{V}$ has the Fraser-Horn property and every member of $\mathcal{V}$ is representable as a Boolean product whose factors are directly indecomposable or trivial.*

We remit to [2] for notation and basic facts on Boolean products and discriminator varieties. The concept of decomposition operation plays a deep role in the proof. A *decomposition operation* on an algebra A is a homomorphism $d : A \times A \to A$ satisfying:

$d(x, x) \approx x$,

$d(d(x, y), z) \approx d(x, z) \approx d(x, d(y, z))$.

Given a pair θ, δ of complementary factor congruences we have associated a decomposition operation defined by $d_{\theta,\delta}(a, b) =$ the unique $c \in A$ such that $(c, a) \in \theta$ and $(c, b) \in \delta$. Reciprocally, given a decomposition operation d, the relations $\theta_d = \{(x, y) : d(x, y) = y\}$ and $\delta_d = \{(x, y) : d(x, y) = x\}$ are a pair of complementary factor congruences. These maps are mutually inverse [4].

For an algebra A let Δ denote the diagonal congruence on A.

Proof of the Theorem. (a)$\Rightarrow$(b) Since $\mathcal{V}$ is congruence distributive it has the Fraser-Horn property [1]. The Boolean representation is the well known result of Bulman-Fleming, Keimel and Werner [2].

(b)$\Rightarrow$(a) Since $\mathcal{V}$ has the Fraser-Horn property, we have that:

(1) *For every $A \in \mathcal{V}$, the set $FC(A)$ of all factor congruences forms a Boolean sublattice of the congruence lattice of A.*

(2) *If $\varphi : A \to B$ is an onto homomorphism and d is a decomposition operation on A, then the equation $\varphi(d)(\varphi(a), \varphi(b)) = \varphi(d(a, b))$ defines a decomposition operation $\varphi(d)$ on B.*

Received by the editors March 2, 1999 and, in revised form, May 14, 1999.
2000 *Mathematics Subject Classification.* Primary 08A05, 08A30, 08A40, 08B10, 06E15.
The author's research was supported by CONICOR and SECYT (UNC).

(1) is proved in [1, 1.4]. To prove (2) suppose (w.l.o.g.) that $A = A_1 \times A_2$ and $d((a_1, a_2), (b_1, b_2)) = (a_1, b_2)$, and use the fact that $\ker\varphi$ factorizes.[1]

Next let $A \subseteq \Pi\{A_i : i \in I\}$ be a Boolean product. For $i \in I$, let $\theta_i = \{(x, y) \in A^2 : x(i) = y(i)\}$. Suppose that there is a set $J \subseteq I$ such that every A_i, with $i \in J$, is directly indecomposable and $\bigcap\{\theta_i : i \in J\} = \Delta$. We will prove that

(3) $\{\theta_i : i \in I\} - \{\Delta\} = \{\cup\mathbf{m} : \mathbf{m}$ *is a maximal ideal of* $FC(A)\} - \{\Delta\}$.

For a set B define $\eta^B(a, b, c, d) = c$, if $a = b$ and $\eta^B(a, b, c, d) = d$, if $a \neq b$. By [6, 2.3] there is a 4-ary operation h on A such that the algebra $\langle A, h\rangle$ generates a discriminator variety and

(4) $\langle A, h\rangle \subseteq \Pi\{\langle A_i, \eta^{A_i}\rangle : i \in I\}$ *is a Boolean product with simple or trivial factors.*

Let θ, θ^* be a pair of complementary factor congruences of A. Since $\theta_i = (\theta_i \vee \theta) \cap (\theta_i \vee \theta^*)$ [3] and each θ_i, with $i \in J$, is indecomposable, we have that $\theta \subseteq \theta_i$ or $\theta^* \subseteq \theta_i$, for every $i \in J$. Since $\bigcap\{\theta_i : i \in J\} = \Delta$, we obtain that $\bigcap\{\theta_i : \theta \subseteq \theta_i\}$ and $\bigcap\{\theta_i : \theta^* \subseteq \theta_i\}$ are a pair of complementary factor congruences of A, which by (1) implies that $\theta = \bigcap\{\theta_i : \theta \subseteq \theta_i\}$. Thus θ is a congruence on $\langle A, h\rangle$ (note that the θ_i are congruences of $\langle A, h\rangle$) and we have proved that

(5) $FC(A) = FC(\langle A, h\rangle)$.

Since principal congruences in discriminator varieties are factor congruences [10, (6) of 2.2], (5) implies that

(6) *For* $(x, y) \in A^2$ *there is a least factor congruence containing the pair* (x, y).

Since $\langle A, h\rangle$ generates a discriminator variety, we have that

(7) *The natural map*

$$\langle A, h\rangle \to \Pi\{\langle A, h\rangle / \cup\mathbf{m} : \mathbf{m} \text{ is a maximal ideal of } FC(A)\}$$

is a Boolean representation with simple or trivial factors.

Thus (3) follows from (4), (7) and the uniqueness of Boolean representations in discriminator varieties [5, 8.3]. We note that (3) and (b) imply that

(8) *For every* $A \in \mathcal{V}$, $A/\cup\mathbf{m}$ *is directly indecomposable or trivial, for every maximal ideal of* $FC(A)$.

Next we will prove[2] that

(9) *If* A *is directly indecomposable and* $B \subseteq A$, *then* B *is directly indecomposable or trivial.*

Suppose to the contrary that B is decomposable. Let x_0 be a point in the Cantor Discontinuum C. Give A the discrete topology. Let $D = \{f \in A^C : f$ is continuous and $f(x_0) \in B\}$. Note that $D \subseteq \Pi\{D_x : x \in C\}$ is a Boolean product where $D_x = A$ if $x \neq x_0$ and $D_{x_0} = B$. Further note that $\bigcap_{x \in C - \{x_0\}} \theta_x = \Delta$ ($C - \{x_0\}$ is dense). Thus (3) and (8) produce a contradiction since $D/\theta_{x_0} \cong B$.

Let c_1, c_2 be new constant symbols and let $d_1, d_2, \ldots$ be new binary function symbols. Let $\mathcal{V}_e$ be the variety axiomatized by a set of identities defining $\mathcal{V}$ plus the following identities:

$d_i(x, x) \approx x$,

$d_i(d_i(x, y), z) \approx d_i(x, z) \approx d_i(x, d_i(y, z))$,

$d_i(f(\vec{x}), f(\vec{y})) \approx f(d_i(x_1, y_1), \ldots, d_i(x_n, y_n))$,

$d_i(c_1, c_2) \approx c_2$, f an n-ary function symbol of $\mathcal{V}$, $n \geq 1$, $i = 1, 2, \ldots$.

Note that $\langle A, a, b, d_1, \ldots\rangle \in \mathcal{V}_e$ iff $A \in \mathcal{V}$ and each d_i is a decomposition operation on A such that $(a, b) \in \theta_{d_i}$. Let F be the free $\mathcal{V}_e$-algebra freely generated by z and

[1] Indeed condition (2) is equivalent to the Fraser-Horn property.

[2] The argument we will use was extracted from [1].

let F_r be the reduct of F to the language of $\mathcal{V}$. Since $F_r \in \mathcal{V}$, by (6) there exists a factor congruence θ, which is the least factor congruence on F_r containing the pair (c_1, c_2). Let δ be the complement of θ in $FC(F_r)$. Note that

(10) $d_{\theta,\delta}(c_1, c_2) = c_2$.

Since each d_i^F is a decomposition operation on F_r satisfying $(c_1, c_2) \in \theta_{d_i^F}$, we have

(11) $\forall a, b \in F_r \;\; d_{\theta,\delta}(a, b) = b \rightarrow d_i^F(a, b) = b$.

For each term $t(z)$ in the language of $\mathcal{V}_e$, define a term $t(z)^*$ with variables in $\{x, y, z\}$ as follows:

$c_1^* = x$,
$c_2^* = y$,
$s^* = s$, for $s \in \{z\} \cup \{\text{constant symbols of } \mathcal{V}\}$,
$f(t_1, ..., t_n)^* = f(t_1^*, ..., t_n^*)$, f an n-ary function symbol of $\mathcal{V}$,
$d_i(t_1, t_2)^* = t_2^*$, $i = 1, 2, ...$

We will follow the usual custom of thinking of $d_{\theta,\delta}(z, c_1) \in F$ as a term $t(z)$, which we prefer to write as $t(c_1, c_2, z)$ to refer to all the occurrences of c_1 and c_2 in $t(z)$. The goal will be to show that $d_{\theta,\delta}(z, c_1)^* = t(c_1, c_2, z)^*$ provides a discriminator term for $\mathcal{V}$, i.e. we will prove that for every directly indecomposable $A \in \mathcal{V}$ and every $a, b, c \in A$, we have that $t(c_1, c_2, z)^*(a, b, c) = c$, if $a = b$ and $t(c_1, c_2, z)^*(a, b, c) = a$, otherwise. Note that by (9) we can suppose that A is generated by $\{a, b, c\}$. Let $\pi_1, \pi_2 : A \times A \rightarrow A$ be the canonical projections. Let k be such that each d_i, with $i > k$, does not occur in $t(c_1, c_2, z)$. First suppose that $a = b$. Since $\langle A, a, b, \overbrace{\pi_2, ..., \pi_2}^{k}, \pi_1, \pi_1, ...\rangle \in \mathcal{V}_e$, there is a morphism

$$\begin{array}{rcl} \varphi : F & \rightarrow & \langle A, a, b, \pi_2, ..., \pi_2, \pi_1, \pi_1, ...\rangle \\ z & \rightarrow & c. \end{array}$$

Since $d_{\theta,\delta}(z, d_{\theta,\delta}(z, c_1)) = d_{\theta,\delta}(z, c_1)$, by (11) we have that $d_i^F(z, d_{\theta,\delta}(z, c_1)) = d_{\theta,\delta}(z, c_1)$, $i = 1, 2, ...$ Further note that

$$\varphi(d_{\theta,\delta}(z, c_1)) = \varphi(t(c_1, c_2, z)) = t(c_1, c_2, z)^*(a, b, c).$$

Thus we have

$$t(c_1, c_2, z)^*(a, b, c) = \varphi(d_{k+1}(z, d_{\theta,\delta}(z, c_1))) = \pi_1(c, \varphi(d_{\theta,\delta}(z, c_1))) = c.$$

Finally suppose that $a \neq b$. Let

$$\begin{array}{rcl} \gamma : F & \rightarrow & \langle A, a, b, \pi_2, ..., \pi_2, \pi_2, \pi_2, ...\rangle \\ z & \rightarrow & c. \end{array}$$

Note that by (2) we have a decomposition operation $\gamma(d_{\theta,\delta})$. Since $(c_1, c_2) \in \theta = \theta_{d_{\theta,\delta}}$, we have that $(a, b) \in \theta_{\gamma(d_{\theta,\delta})}$, which implies that $\theta_{\gamma(d_{\theta,\delta})} = A \times A$, because A is directly indecomposable. Thus $\gamma(d_{\theta,\delta}) = \pi_2$ and hence

$$t(c_1, c_2, z)^*(a, b, c) = \gamma(d_{\theta,\delta}(z, c_1)) = \gamma(d_{\theta,\delta})(\gamma(z), \gamma(c_1)) = \pi_2(c, a) = a. \quad \Box$$

(b)⇒(a) was proved in [7] with the restriction that the Boolean representations have no trivial factors. That proof is based on the concept of central element [9] which cannot be used in general since central elements exist only in varieties in which no non-trivial algebra has a trivial subalgebra. We conclude the paper with the following question:

Is it possible to replace in the above theorem the Fraser-Horn property by the property that for every $A \in \mathcal{V}$, the set $FC(A)$ forms a Boolean sublattice of the congruence lattice of A?

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