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CHARACTERISTIC NUMBERS AND COBORDISM CLASSES OF FIBERINGS WITH FIBER $RP(2k)$

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ABSTRACT. Some groups A_n^k of n-dimensional unoriented cobordism classes represented by the total space of a fibering with the real projective space $RP(2k)$ as fiber are determined.

1. INTRODUCTION

Being given a cobordism class $\alpha_n \in MO_n$, the unoriented cobordism group, we say that α_n fibers over N^{n-m} with fiber F^m if there is a differentiable fibering of closed manifolds

$$F^m \hookrightarrow M^n \longrightarrow N^{n-m}$$

where M^n belongs to the class α_n.

In the paper [1] the following question was pointed out by R.E. Stong. Given a closed manifold F, which classes α_n in MO_n are represented by a fibering with the prescribed fiber F? The set of classes in MO_n which can be so represented forms a subgroup in MO_n and all such classes in $MO_* = \sum MO_n$ form an ideal in MO_*.

In this paper we consider the real projective space $RP(2k)$ as fiber F and obtain some results extending Stong's work [1, Proposition 8.1, Proposition 8.5].

Let A_n^k $(n \geq 2k)$ denote the set of classes in MO_n which are represented by a fibering with the fiber $RP(2k)$. Then $A_*^k = \sum_{n \geq 2k} A_n^k$ is an ideal in MO_* generated by the manifolds $RP(n_1, n_2, \cdots, n_{2k+1})$ ([1]).

Let M^n be a closed $n-$dimensional manifold and $f(t) = f(t_1, t_2, \cdots, t_{n+1})$ a homogeneous and symmetric Z_2-polynomial in $n+1$ variables of degree n. If, in this polynomial, the $j-$th elementary symmetric function in $\{t_1, t_2, \cdots, t_{n+1}\}$, $\sigma_j(t)$, is replaced by the $j-$th Stiefel–Whitney class of M^n, the resulting cohomology class may be evaluated on the fundamental homology class of M^n, obtaining a characteristic number

$$f(t)[M].$$

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Assuming $\xi^{2k+1} \to N^{n-2k}$ to be a vector bundle and β to be a formal variable, we consider the expression

$$\frac{\beta f(\beta + y_1, \cdots, \beta + y_{2k+1}, z_1, \cdots, z_{n-2k})}{\prod_{i=1}^{2k+1}(\beta + y_i)}.$$

If, in the expression, we replace the $j-$th elementary symmetric function in

$\{y_1, \cdots, y_{2k+1}\}$ by the $j-$th Stiefel–Whitney class $w_j(\xi^{2k+1})$,

$\{z_1, \cdots, z_{n-2k}\}$ by the $j-$th Stiefel–Whitney class $w_j(N^{n-2k})$,

the result is a class in the cohomology of N^{n-2k} which may be evaluated on the fundamental homology class of N^{n-2k}. This gives a characteristic number

$$\frac{\beta f(\beta + y, z)}{\prod(\beta + y)}[N^{n-2k}],$$

which can be considered as an element (a polynomial in β) in the quotient field K of $Z_2[\beta]$.

Here the denominator terms $1/(\beta + y_i)$ are meant to be formally inverted, with $\beta \in K$ being invertible, as by

$$\frac{1}{\beta + y_i} = \frac{1}{\beta} + \frac{y_i}{\beta^2} + \frac{y_i^2}{\beta^3} + \cdots .$$

Then

$$\prod_{i=1}^{2k+1} \frac{1}{\beta + y_i} = \sum_{j=0}^{\infty} \frac{\tilde{w}_j(\xi^{2k+1})}{\beta^{2k+1+j}}$$

where $\tilde{w}$ is the dual Stiefel–Whitney class defined by $\tilde{w}(\xi)w(\xi) = 1$.

If, for any vector bundle $\xi^{2k+1} \to N^{n-2k}$,

$$\frac{\beta f(\beta + y, z)}{\prod(\beta + y)}[N^{n-2k}]$$

as a polynomial in β in K has always constant term equal to zero, then $f(t_1, t_2, \cdots, t_{n+1})$ is called a **good polynomial** in $n+1$ variables. Let P_n^k be the set of **good polynomials** in $n+1$ variables, and

$$B_n^k = \{[M^n] \in MO_n | \ \forall f(t) \in P_n^k, f(t)[M^n] = 0\}, \quad B_*^k = \sum B_n^k.$$

Now we arrive at the main results:

Theorem 1. *Let $f(t_1, t_2, \cdots, t_{n+1})$ be a homogeneous and symmetric Z_2-polynomial in $n+1$ variables of degree n and $RP(\xi^{2k+1}) \longrightarrow N^{n-2k}$ the projective space bundle associated to the vector bundle $\xi^{2k+1} \to N^{n-2k}$. Then*

$$\frac{\beta f(\beta + y_1, \cdots, \beta + y_{2k+1}, z_1, \cdots, z_{n-2k})}{\prod_{i=1}^{2k+1}(\beta + y_i)}[N^{n-2k}] \in K$$

is a polynomial in β and has constant term equal to

$$f(t)[RP(\xi^{2k+1})].$$

Theorem 2. *Let k be a positive integer and $n \geq 2k$. Then $A_n^k \subset B_n^k$.*

Theorem 3. *If $2k$ has the dyadic expansion $2^{r_u} + 2^{r_{u-1}} + \cdots + 2^{r_1}, r_u > r_{u-1} > \cdots > r_1 \geq 0$, then $A_n^k = B_n^k = MO_n$ for $n > 2^{r_u+1}(2^{r_u+2} - 1)(2k - 1)$.*

Theorem 4. *Let* $n_k = 2, 6, 12, 18$ *respectively for* $k = 1, 2, 3, 4$. *Then*
(1) $A_n^k = B_n^k = MO_n$ *for* $k = 1, n \geq 2k$;
(2) $A_n^k = B_n^k \neq MO_n$ *for* $k \neq 1, 2k \leq n \leq n_k$;
(3) $A_n^k = B_n^k = MO_n$ *for* $k \neq 1, n > n_k$.

2. Proofs of Theorem 1 and Theorem 2

Let us recall the facts about the projective space bundle $RP(\xi^{2k+1}) \longrightarrow N^{n-2k}$ associated to the vector bundle $\xi^{2k+1} \to N^{n-2k}$. The manifold $RP(\xi^{2k+1})$ has dimension n. The cohomology $H^*(RP(\xi^{2k+1}); Z_2)$ is the free $H^*(N; Z_2)$ module on $1, c, c^2, \cdots, c^{2k}$ where c is the first Stiefel–Whitney class of the standard line bundle. The ring structure is determined by the relation

$$c^{2k+1} + c^{2k}w_1(\xi) + c^{2k-1}w_2(\xi) + \cdots + w_{2k+1}(\xi) = 0.$$

The total Stiefel–Whitney class of $RP(\xi^{2k+1})$ is

$$w(RP(\xi^{2k+1})) = \left(\sum_{j=0}^{2k+1} (1+c)^{2k+1-j} w_j(\xi) \right) w(N),$$

so the $i-$th Stiefel–Whitney class of $RP(\xi^{2k+1})$ is

$$w_i(RP(\xi^{2k+1})) = \sum_{r+j+q=i} \binom{2k+1-j}{q} c^q w_r(N) w_j(\xi),$$

and if $a \in H^*(N; Z_2)$, we have from [2] that

$$c^q a[RP(\xi)] = \begin{cases} 0 & \text{if } q < 2k, \\ \tilde{w}_{q-2k}(\xi) a[N] & \text{if } q \geq 2k. \end{cases}$$

Proof of Theorem 1. It suffices to prove the theorem for

$$f(t) = \sigma_{i_1}(t)\sigma_{i_2}(t) \cdots \sigma_{i_m}(t), \;\; i_1 + i_2 + \cdots + i_m = n,$$

where $\sigma_{i_j}(t)$ is the i_j-th elementary symmetric function. We sometimes write $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}(t)$ for $\sigma_{i_1}(t)\sigma_{i_2}(t)\cdots\sigma_{i_m}(t)$. Let $\gamma = 1/\beta$;

$$\frac{\beta f(\beta + y_1, \cdots, \beta + y_{2k+1}, z_1, \cdots, z_{n-2k})}{\prod_{i=1}^{2k+1}(\beta + y_i)}[N^{n-2k}]$$

$$= \frac{\beta f(\beta + \beta\gamma y_1, \cdots, \beta + \beta\gamma y_{2k+1}, \beta\gamma z_1, \cdots, \beta\gamma z_{n-2k})}{\prod_{i=1}^{2k+1}(\beta + \beta\gamma y_i)}[N^{n-2k}]$$

$$= \frac{\beta^{n+1} f(1 + \gamma y_1, \cdots, 1 + \gamma y_{2k+1}, \gamma z_1, \cdots, \gamma z_{n-2k})}{\beta^{2k+1}\prod_{i=1}^{2k+1}(1 + \gamma y_i)}[N^{n-2k}]$$

in which the value of the resulting characteristic class is obtained as

$$\frac{\beta^{n+1} f(1 + y_1, \cdots, 1 + y_{2k+1}, z_1, \cdots, z_{n-2k})}{\beta^{2k+1}\prod_{i=1}^{2k+1}(1 + y_i)}\gamma^{n-2k}[N^{n-2k}],$$

for the only homogeneous term of the characteristic class having a non-zero value is that of degree equal to the dimension of N^{n-2k}.

Now $\frac{\beta^{n+1}}{\beta^{2k+1}}\gamma^{n-2k} = \frac{\beta^{n+1}}{\beta^{n+1}} = \beta^0$, so that

$$\frac{\beta f(\beta+y_1,\cdots,\beta+y_{2k+1},z_1,\cdots,z_{n-2k})}{\prod_{i=1}^{2k+1}(\beta+y_i)}[N^{n-2k}]$$
$$= \beta^0 \frac{f(1+y_1,\cdots,1+y_{2k+1},z_1,\cdots,z_{n-2k})}{\prod_{i=1}^{2k+1}(1+y_i)}[N^{n-2k}].$$

Thus, the theorem is equivalent to

$$f(t)[RP(\xi^{2k+1})] = \frac{f(1+y_1,\cdots,1+y_{2k+1},z_1,\cdots,z_{n-2k})}{\prod_{i=1}^{2k+1}(1+y_i)}[N^{n-2k}].$$

Now we have that

$$f(t)[RP(\xi^{2k+1})] = \sigma_{i_1}(t)\sigma_{i_2}(t)\cdots\sigma_{i_m}(t)[RP(\xi^{2k+1})] = w_{i_1}w_{i_2}\cdots w_{i_m}[RP(\xi^{2k+1})]$$
$$= \prod_{l=1}^{m}\left[\sum_{r+j+q=i_l}\binom{2k+1-j}{q} w_r(N)w_j(\xi)c^q\right][RP(\xi^{2k+1})]$$

in which the value of the resulting characteristic class is obtained as

$$\left[\prod_{l=1}^{m}\sum_{r+j+q=i_l}\binom{2k+1-j}{q} w_r(N)w_j(\xi)\right]\tilde{w}(\xi)[N^{n-2k}],$$

for

$$c^q a[RP(\xi)] = \begin{cases} 0 & \text{if } q < 2k, \\ \tilde{w}_{q-2k}(\xi)a[N] & \text{if } q \geq 2k \end{cases}$$

and $\sum_{j=1}^{m} i_j = n = dimRP(\xi)$.

From [2, Lemma, p. 317] we have

$$\sigma_{i_l}(1+y_1,\cdots,1+y_{2k+1},z_1,\cdots,z_{n-2k}) = \sum_{j+r\leq i_l}\binom{2k+1-j}{i_l-j-r}\sigma_j(y)\sigma_r(z)$$
$$= \sum_{r+j+q=i_l}\binom{2k+1-j}{q}\sigma_j(y)\sigma_r(z),$$

so that

$$\frac{f(1+y_1,\cdots,1+y_{2k+1},z_1,\cdots,z_{n-2k})}{\prod_{i=1}^{2k+1}(1+y_i)}[N^{n-2k}]$$

$$= \frac{\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}(1+y_1,\cdots,1+y_{2k+1},z_1,\cdots,z_{n-2k})}{1+\sigma_1(y)+\cdots+\sigma_{2k+1}(y)}[N^{n-2k}]$$

$$= \frac{\prod_{l=1}^{m}\left[\sum_{r+j+q=i_l}\binom{2k+1-j}{q}\sigma_j(y)\sigma_r(z)\right]}{1+\sigma_1(y)+\cdots+\sigma_{2k+1}(y)}[N^{n-2k}]$$

$$= \left[\prod_{l=1}^{m}\sum_{r+j+q=i_l}\binom{2k+1-j}{q} w_r(N)w_j(\xi)\right]\tilde{w}(\xi)[N^{n-2k}].$$

Therefore

$$f(t)[RP(\xi^{2k+1})] = \frac{f(1+y_1,\cdots,1+y_{2k+1},z_1,\cdots,z_{n-2k})}{\prod_{i=1}^{2k+1}(1+y_i)}[N^{n-2k}].$$

Proof of Theorem 2. Let $\alpha_n \in A_n^k$. Then from [1, Proposition 8.5] we have $\alpha_n = [RP(\xi^{2k+1})]$ for some vector bundle $\xi^{2k+1} \to N^{n-2k}$. If $f(t) \in P_n^k$, from Theorem 1 we obtain

$$f(t)[\alpha_n] = f(t)[RP(\xi^{2k+1})] = 0,$$

so $\alpha_n \in B_n^k$. The proof is completed.

3. Proof of Theorem 3

It is well known that the unoriented cobordism ring MO_* is a Z_2-polynomial algebra with a single generator in each dimension which is not of the form 2^r-1 ([3]). If a cobordism class $[M^n]$ can be expressed as a sum of products of lower dimensional cobordism classes, then $[M^n]$ is decomposable, otherwise it is indecomposable. The indecomposable classes can be used as generators of the polynomial algebra MO_*.

To identify indecomposability it is useful to recall Stong's result:

Let $RP(n_1,n_2,\cdots,n_l)$ be the projective space bundle of $\lambda_1\oplus\lambda_2\oplus\cdots\oplus\lambda_l$ over $RP(n_1)\times RP(n_2)\times\cdots\times RP(n_l)$, where λ_i is the pullback of the canonical line bundle over the $i-$th factor. Then for $l>1, RP(n_1,n_2,\cdots,n_l)$ is indecomposable in MO_* if and only if

$$\binom{n+l-2}{n_1}+\binom{n+l-2}{n_2}+\cdots+\binom{n+l-2}{n_l}\equiv 1 \mod 2,$$

where $n=n_1+n_2+\cdots+n_l$.

The manifold $RP(n_1,n_2,\cdots,n_l)$ has dimension $n+l-1$. If $n_{i+1}=n_{i+2}=\cdots=n_l=0$, then $RP(n_1,n_2,\cdots,n_l)$ will sometimes be written as $RP(n_1,\cdots,n_i;l)$.

Following [4], let J_n^r be the set of cobordism classes in MO_n which are represented by a manifold admitting an involution with fixed point set of dimension $n-r$ and $J_*^r=\sum J_n^r$. Then $J_*^{r_1}J_*^{r_2}\subset J_*^{r_1+r_2}$, $J_n^{2r}\subset A_n^r$, $J_*^{2r}\subset A_*^r$, $J_*^6\subset J_*^4\subset J_*^2$ ([5], [6]).

Lemma 1. *There exist indecomposable classes $x_n \in A_n^k$ if one of the conditions remains true:*

(1) *n even, $n\geq 2k$;*

(2) *$n=2^p(2q+1)-1(p,q>0)$, $n-2^p>2k$.*

Proof. In the first case, $x_n=[RP((n-2k)/2,(n-2k)/2;2k+1)]$ is as required. In the second case, $x_n=[RP(2^p,n-2^p-2k;2k+1)]$ is as required.

Lemma 2. *If $2k$ has the dyadic expansion $2^{r_u}+2^{r_{u-1}}+\cdots+2^{r_1}, r_u>r_{u-1}>\cdots>r_1\geq 0$, then there exist indecomposable classes x_n $(n\neq 2^r-1)$ such that*

(1) *$x_n\in A_n^k$ for $n\geq 2^{r_u+2}$;*

(2) *$x_n^{2k}\in A_{2nk}^k$ for $2\leq n<2^{r_u+2}$.*

Proof. (1) If n is even, then $n\geq 2^{r_u+2}\geq 2k$. If n is odd, $n=2^p(2q+1)-1(p,q>0)$. From $n\geq 2^{r_u+2}$ we have $2^{p+1}q\geq 2^{r_u+2}$, so $n-2^p=2^{p+1}q-1\geq 2^{r_u+1}>2k$. By Lemma 1 we know the result is true.

(2) Take $x_5=[RP(2,0,0,0)]$, $x_6=[RP(3,0,0,0)]$, $x_{2^r}=[RP(2^r)]$. Then $\{x_5,x_6,x_{2^r}^2\}\subset J_*^2$. If $9\leq i<2^{r_u+2}$, i not of the form 2^r or 2^r-1, we take x_i as in [4, Lemma 3.1]. Then $x_i\in J_*^4\subset J_*^2$. For each x_n, $x_n^{2k}\in \underbrace{J_*^2\cdots J_*^2}_{k}\subset J_*^{2k}\subset A_*^k$.

Proof of Theorem 3. Choose indecomposable classes $x_n (n \neq 2^r - 1)$ as in Lemma 2 as generators of MO_*. Since A^k_* is an ideal, we know $x^{l_1}_{i_1} x^{l_2}_{i_2} \cdots x^{l_m}_{i_m} \in A^k_n$ for $n = i_1 l_1 + i_2 l_2 + \cdots + i_m l_m > (2k-1)(1+2+\cdots+(2^{r_u+2}-1)) = 2^{r_u+1}(2^{r_u+2}-1)(2k-1)$.

4. Proof of Theorem 4

$\omega = (i_1, i_2, \cdots, i_m)$ is called a partition of n if $\sum_{j=1}^m i_j = n$. Let $\omega' = (i'_1, i'_2, \cdots, i'_r)$ be a partition of n'. We use $\omega\omega'$ to denote the partition $(i_1, \cdots, i_m, i'_1, \cdots, i'_r)$ of $n + n'$. If ω_j is a partition of i_j for every j $(1 \leq j \leq m)$, then $\omega_1 \omega_2 \cdots \omega_m$ is called a refinement of ω. We sometimes write $|\omega|$ for $\sum_{j=1}^m i_j$.

Given a closed manifold M^n, let $\omega = (i_1, i_2, \cdots, i_m)$ be a partition of n and $s_\omega(t) = \sum t^{i_1}_{j_1} t^{i_2}_{j_2} \cdots t^{i_m}_{j_m}$ the usual smallest symmetric polynomial in variables $\{t_1, t_2, \cdots, t_r \ (r \geq n)\}$. $s_\omega(M^n)$ denotes the cohomology class associated to $s_\omega(t)$ and $s_\omega[M^n]$ denotes the characteristic number associated to $s_\omega(M^n)$.

From the formula in [7, p. 192] we have the following:

Lemma 3. *Let x_n $(n \neq 2^r - 1)$ be generators of MO_*. Then*

$$s_\omega[x_{i_1} x_{i_2} \cdots x_{i_m}] = \begin{cases} 0 & \text{if } \omega \text{ is not a refinement of } (i_1, i_2, \cdots, i_m), \\ 1 & \text{if } \omega = (i_1, i_2, \cdots, i_m). \end{cases}$$

Let $\theta : MO_* \to Z_2[t_1, t_2, t_3, \cdots]$ be the ring homomorphism defined by

$$\theta([M^n]) = \sum_{|\omega|=n} s_\omega[M^n] \cdot t_\omega$$

where $t_{(i_1, \cdots, i_m)} = t_{i_1} \cdots t_{i_m}$, and reduce modulo the ideal generated by the t_i for $i > 17$, to give a ring homomorphism $\theta' : MO_* \to Z_2[t_1, t_2, \cdots, t_{17}]$.

Lemma 4. *Let the total Stiefel–Whitney class $w(M)$ of a closed manifold M^n $(n \leq 17)$ be expressed as the form $\prod_{i=1}^r (1 + a_i)$ with $a_i \in H^1(M^n; Z_2)$. Then $\theta'([M^n])$ is the value of the $n-$dimensional component of $\prod_{i=1}^r (1 + a_i t_1 + a_i^2 t_2 + \cdots + a_i^{17} t_{17})$ on the fundamental class of M^n.*

Proof. From $w(M) = \prod_{i=1}^r (1 + a_i)$, we claim

$$s_\omega(M^n) = \begin{cases} s_\omega(a_1, a_2, \cdots, a_r) & \text{if } r \geq n, \\ s_\omega(a_1, a_2, \cdots, a_r, \underbrace{0, \cdots, 0}_{n-r}) & \text{if } r < n, \end{cases}$$

so that the result is true.

Proof of Theorem 4. *Case* $k = 1$. [1, Proposition 8.1] and Theorem 2 show the theorem.

Case $k = 2$. Take $x_2 = [RP(2)], x_4 = [RP(4)], x_5 = [RP(2, 0, 0, 0)]$. From Lemma 1 there exist indecomposable classes $x_n \in A^2_n$ for $n \geq 6 (n \neq 2^r - 1)$. The above x_n form the generators of MO_*. $\{x_2^2, x_5\} \subset J^2_*$, $\{x_4, x_n (n \geq 6)\} \subset A^2_*$, $x_2 x_5 = [RP(2, 1; 5)] \in A^2_*$. Since A^2_* is an ideal and $J^2_* J^2_* \subset J^4_* \subset A^2_*$, we obtain that A^2_* contains products, which have dimension n greater than $2k - 1$, of generators of MO_* except the following classes

$$x_2^2, x_5, x_2^3.$$

By making use of Theorem 1 we have $\sigma_2^2(t) \in P^2_4, \sigma_2\sigma_3(t) \in P^2_5$ and $\sigma_2^3(t) \in P^2_6$. A simple computation shows that $\sigma_2^2(t)[x_2^2] \neq 0, \sigma_2\sigma_3(t)[x_5] \neq 0, \sigma_2^3(t)[x_2^3] \neq 0$; hence $x_2^2, x_5, x_2^3 \notin B^2_*$. The theorem is established.

Case $k = 3$. Take $x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}$ respectively as $[RP(2)]$, $[RP(4)]$, $[RP(2,0,0,0)]$, $[RP(6)]$, $[RP(2;7)]$, $[RP(2,1;7)]$, $[RP(1,1,1,1;7)]$, $[RP(4,1;7)]$,$[RP(1,1,1,1,1,1,0)]$. For $n \geq 13$ take x_n as in Lemma 1. Then x_n form the generators of MO_*. $\{x_2^2, x_5\} \subset J_*^2$, $\{x_4^2, x_5^2\} \subset J_*^4 \subset J_*^2, x_n (n \geq 6) \in A_*^3$.

To complete the proof we consider the following cases.

Dimension $n = 6$. Since $x_6 \in A_6^3 \subset B_6^3$, we need only show that the classes

$$x_2^3, x_2 x_4$$

are distinguished by characteristic numbers associated to $f(t) \in P_6^3$. By Theorem 1 a direct computation shows that $f_1(t) = \sigma_3^2(t) + \sigma_6(t)$ and $f_2(t) = \sigma_3^2(t) + \sigma_1^6(t)$ are in P_6^3. Characteristic numbers are given by

	x_2^3	x_2x_4
$f_1(t)$	0	1
$f_2(t)$	1	1

Since this matrix is nonsingular, these classes are distinguished.

Dimension $n = 7$. $x_2 x_5$ is the only non-zero class. Similarly from Theorem 1 we have $f_3(t) = \sigma_2\sigma_5(t) \in P_7^3$. Since $f_3(t)[x_2x_5] \neq 0$, the result is established.

Dimension $n = 8$. Since $x_8, x_6x_2 \in A_8^3 \subset B_8^3$, we need only show that the classes

$$x_2^4, x_2^2 x_4, x_4^2$$

are distinguished by characteristic numbers associated to $f(t) \in P_8^3$. In P_8^3 take $f_4(t) = \sigma_8(t) + \sigma_5\sigma_3(t) + \sigma_7\sigma_1(t)$, $f_5(t) = \sigma_8(t) + \sigma_5\sigma_1^3(t) + \sigma_2^4(t)$, $f_6(t) = \sigma_1^8(t) + \sigma_2^4(t)$. We have the nonsingular matrix

	x_2^4	$x_2^2x_4$	x_4^2
$f_4(t)$	1	1	1
$f_5(t)$	1	0	0
$f_6(t)$	0	0	1

so that the classes are distinguished.

Dimension $n = 9$. Since $x_9 \in A_9^3 \subset B_9^3$, we need only show that the classes

$$x_2^2 x_5, x_4 x_5$$

are distinguished by characteristic numbers associated to $f(t) \in P_9^3$. In P_9^3 take $f_7(t) = \sigma_8\sigma_1(t) + \sigma_6\sigma_1^3(t) + \sigma_5\sigma_4(t)$, $f_8(t) = \sigma_9(t) + \sigma_6\sigma_3(t) + \sigma_3\sigma_2^3(t)$. We have the nonsingular matrix

	$x_2^2x_5$	x_4x_5
$f_7(t)$	1	0
$f_8(t)$	0	1

so that the classes are distinguished.

Dimension $n = 10$. Since A^3_* is an ideal, A^3_{10} contains products of generators except the classes

$$x_2^5, x_5^2, x_2^3x_4, x_2x_4^2.$$

By Lemma 3 and Lemma 4 a direct computation shows that

$$x_2^5 + x_5^2 = [RP(2,2;7)] + [RP(1,1,1,1;7)] \in A^3_{10}.$$

In P^3_{10} we take $f_9(t) = \sigma_9\sigma_1(t) + \sigma_8\sigma_2(t) + \sigma_6\sigma_4(t) + \sigma_2\sigma_1^8(t)$, $f_{10}(t) = \sigma_6\sigma_2^2(t) + \sigma_8\sigma_2(t) + \sigma_7\sigma_1^3(t) + \sigma_7\sigma_2\sigma_1(t) + \sigma_5^2(t) + \sigma_2^5(t)$, $f_{11}(t) = \sigma_{10}(t) + \sigma_9\sigma_1(t) + \sigma_7\sigma_1^3(t) + \sigma_6\sigma_4(t) + \sigma_6\sigma_2^2(t) + \sigma_3\sigma_2^3\sigma_1(t)$. Characteristic numbers are given by

	x_2^5	x_5^2	$x_2^3x_4$	$x_2x_4^2$
$f_9(t)$	1	1	0	0
$f_{10}(t)$	0	0	1	0
$f_{11}(t)$	1	1	0	1

Since the rank of the matrix consisting of i_1-th, i_2-th, $\cdots$, i_j-th columns in the above number matrix is equal to j $(1 \leq j \leq 3)$ except the case $\{1,2\} \subset \{i_1, i_2, \cdots, i_j\}$, the classes are distinguished.

Dimension $n = 11$. Similarly A^3_{11} contains products of generators except the classes

$$x_2^3x_5, x_2x_4x_5.$$

By Lemma 3 and Lemma 4 a direct computation shows that

$$x_2^3x_5 + x_2x_4x_5 = x_5x_6 + x_2x_9 + [RP(2,1,1,1;7)] \in A^3_{11}.$$

In P^3_{11} we take $f_{12}(t) = \sigma_5\sigma_2^3(t) + \sigma_7\sigma_3\sigma_1(t) + \sigma_4^2\sigma_3(t) + \sigma_9\sigma_2(t) + \sigma_8\sigma_3(t) + \sigma_{10}\sigma_1(t) + \sigma_2\sigma_1^9(t) + \sigma_7\sigma_4(t)$. From $f_{12}(t)[x_2^3x_5] = f_{12}(t)[x_2x_4x_5] = 1$ we obtain the result.

Dimension $n = 12$. A^3_{12} contains products of generators except the class x_4^3. Take $f_{13}(t) = \sigma_3^4(t) + \sigma_1^{12}(t) \in P^3_{12}$. Since $f_{13}(t)[x_4^3] = 1$, the theorem is proved.

Dimension $n > 12$. By Lemma 3 and Lemma 4 we have

$$x_2x_4^3 = x_{10}x_4 + y_{14} + [RP(4,2,1,1;7)] \in A^3_{14},$$

where y_{14} is a sum of some monomials from the following

$$x_9x_5, x_8x_6, x_{10}x_2^2, x_6^2x_2, x_6x_4^2, x_5^2x_4,$$
$$x_8x_2^3, x_6x_4x_2^2, x_2^2x_5^2, x_2^4x_6, x_2^3x_4^2, x_2^5x_4, x_2^7.$$

Since A^3_* forms an ideal and $J_*^{2r_1}J_*^{2r_2} \subset J_*^{2r_1+2r_2} \subset A_*^{r_1+r_2}$, A^3_* contains all the products, which have dimension $n > 12$, of generators of MO_*. So $A^3_n = B^3_n = MO_n$ for $n > 12$.

The proof is completed.

Case $k = 4$. Take $x_2, x_4, x_5, x_6, x_9, x_{11}$ respectively as $[RP(2)], [RP(4)]$, $[RP(2,0,0,0)]$, $[RP(3,0,0,0)], [RP(2;8)], [RP(4;8)]$. For other $n (n \neq 2^r - 1)$ we choose x_n as in Lemma 1. Then x_n form the generators of MO_*. $\{x_2^2, x_5, x_6\} \subset J^2_*$, $\{x_4^2, x_9, x_{11}\} \subset J^4_* \subset J^2_*, x_6^2 \in J^6_* \subset J^4_*, x_n \in A^4_*$ if $n \geq 8, n \neq 9, 11$ and $2^r - 1$.

Similarly we consider the following cases.

Dimension $n = 8$. Since $x_8 \in A_8^4 \subset B_8^4$, we need only show that the classes

$$x_2^4, x_2^2 x_4, x_4^2, x_6 x_2$$

are distinguished by characteristic numbers associated to $f(t) \in P_8^4$. In P_8^4 take $f_1(t) = \sigma_4^2(t)$, $f_2(t) = \sigma_2^4(t)$, $f_3(t) = \sigma_5\sigma_3(t)$, $f_4(t) = \sigma_6\sigma_2(t)$. We have the nonsingular matrix

	x_4^2	x_2^4	$x_2^2 x_4$	$x_2 x_6$
$f_1(t)$	0	1	0	1
$f_2(t)$	1	0	0	0
$f_3(t)$	0	0	0	1
$f_4(t)$	0	0	1	0

so that the classes are distinguished.

Dimension $n = 9$. In P_9^4 we take $f_5(t) = \sigma_3\sigma_2^3(t)$, $f_6(t) = \sigma_4\sigma_2\sigma_1^3(t)$, $f_7(t) = \sigma_6\sigma_3(t) + \sigma_4\sigma_2\sigma_1^3(t) + \sigma_3\sigma_2^3(t)$. The nonsingular matrix is given by

	$x_2^2 x_5$	$x_4 x_5$	x_9
$f_5(t)$	1	0	0
$f_6(t)$	0	1	0
$f_7(t)$	0	0	1

so that the result is true.

Dimension $n = 10$. A_{10}^4 contains products of generators except the classes

$$x_2^5, x_2 x_4^2, x_2^3 x_4, x_2^2 x_6, x_4 x_6, x_5^2.$$

In P_{10}^4 we take $f_8(t) = \sigma_7\sigma_3(t)$, $f_9(t) = \sigma_6\sigma_4(t)$, $f_{10}(t) = \sigma_5^2(t)$, $f_{11}(t) = \sigma_4\sigma_2^3(t)$, $f_{12}(t) = \sigma_3^2\sigma_2\sigma_1^2(t)$, $f_{13}(t) = \sigma_3\sigma_2^3\sigma_1(t)$. Characteristic numbers are given by

	x_2^5	$x_2 x_4^2$	$x_2^3 x_4$	$x_2^2 x_6$	$x_4 x_6$	x_5^2
$f_8(t)$	0	0	1	1	1	0
$f_9(t)$	1	0	1	0	0	1
$f_{10}(t)$	1	0	0	1	0	0
$f_{11}(t)$	0	1	1	0	1	0
$f_{12}(t)$	0	0	1	0	1	0
$f_{13}(t)$	0	1	1	1	0	0

This matrix is nonsingular, so that the classes are distinguished.

Dimension $n = 11$. A_{11}^4 contains products of generators except the classes

$$x_5 x_2^3, x_2 x_4 x_5, x_5 x_6, x_2 x_9, x_{11}.$$

By Lemma 3 and Lemma 4 a computation shows that

(1) $$x_2^3x_5 + x_2x_4x_5 + x_5x_6 + x_2x_9 = [RP(2,1;9)] \in A_{11}^4.$$

In P_{11}^4 we take $f_{14}(t) = \sigma_7\sigma_4(t),\ f_{15}(t) = \sigma_6\sigma_4\sigma_1(t),\ f_{16}(t) = \sigma_5\sigma_2^3(t),\ f_{17}(t) = \sigma_4\sigma_2^3\sigma_1(t)$. Characteristic numbers are given by

	$x_5x_2^3$	$x_2x_4x_5$	x_5x_6	x_2x_9	x_{11}
$f_{14}(t)$	1	0	0	1	0
$f_{15}(t)$	0	1	1	0	0
$f_{16}(t)$	1	0	1	0	1
$f_{17}(t)$	1	1	0	0	0

Since the rank of the matrix consisting of i_1-th, i_2-th, $\cdots$, i_j-th columns in the above number matrix is equal to j $(1 \le j \le 4)$ except the case $\{1,2,3,4\} \subset \{i_1, i_2, \cdots, i_j\}$, the classes are distinguished.

Dimension $n = 12$. A_{12}^4 contains products of generators except the classes

$$x_6^2, x_2x_4x_6, x_6x_2^3, x_2x_5^2, x_4^3, x_2^2x_4^2, x_4x_2^4, x_2^6.$$

From Lemma 3 and Lemma 4, we have

(2) $$x_6^2 + x_2x_5^2 + x_4x_2^4 = x_{12} + [RP(1,1,1,1;9)] \in A_{12}^4.$$

In P_{12}^4 we take $f_{18}(t) = \sigma_5\sigma_3^2\sigma_1(t),\ f_{19}(t) = \sigma_3^2\sigma_2^3(t),\ f_{20}(t) = \sigma_3^4(t),\ f_{21}(t) = \sigma_4^2\sigma_2^2(t),\ f_{22}(t) = \sigma_4\sigma_2^4(t),\ f_{23}(t) = \sigma_4^3(t),\ f_{24}(t) = \sigma_6\sigma_4\sigma_2(t)$. Characteristic numbers are given by

	x_6^2	$x_2x_4x_6$	$x_6x_2^3$	$x_2x_5^2$	x_4^3	$x_2^2x_4^2$	$x_4x_2^4$	x_2^6
$f_{18}(t)$	0	0	1	0	0	0	0	0
$f_{19}(t)$	0	1	1	0	0	0	0	0
$f_{20}(t)$	0	0	0	0	1	0	0	0
$f_{21}(t)$	0	1	1	0	0	1	0	1
$f_{22}(t)$	0	0	0	0	1	1	0	0
$f_{23}(t)$	1	1	0	0	0	0	1	1
$f_{24}(t)$	1	1	1	1	0	0	0	1

Similarly combining this matrix with the equation (2), we know that the classes are distinguished.

Dimension $n = 13$. A_{13}^4 contains products of generators except the classes

$$x_2x_{11}, x_4x_9, x_2^2x_9, x_2x_5x_6, x_4^2x_5, x_2^2x_4x_5, x_2^4x_5.$$

A computation shows that

(3) $$x_2x_{11} + x_4x_9 + x_2^2x_9 + x_2^2x_4x_5 + x_2^4x_5 = [RP(4,1;9)] \in A_{13}^4.$$

From $x_2 \cdot (1)$, we have

$$x_2^2x_9 + x_2x_5x_6 + x_2^2x_4x_5 + x_2^4x_5 \in A_{13}^4.$$

In P^4_{13} we take $f_{25}(t)=\sigma_6\sigma_4\sigma_3(t),\ f_{26}(t)=\sigma_4\sigma_3\sigma_2^3(t),\ f_{27}(t)=\sigma_4^2\sigma_3\sigma_2(t),\ f_{28}(t)=\sigma_5\sigma_4\sigma_2^2(t),\ f_{29}(t)=\sigma_3\sigma_2^5(t)$. Characteristic numbers are given by

	x_2x_{11}	x_4x_9	$x_2^2x_9$	$x_2x_5x_6$	$x_4^2x_5$	$x_2^2x_4x_5$	$x_2^4x_5$
$f_{25}(t)$	0	1	1	1	0	1	1
$f_{26}(t)$	1	0	0	1	1	1	0
$f_{27}(t)$	1	0	0	1	0	0	1
$f_{28}(t)$	1	0	1	1	1	0	0
$f_{29}(t)$	0	0	0	0	1	0	0

Combining this matrix with the equations (3) and $x_2\cdot(1)$, we know that the classes are distinguished.

Dimension $n=14$. A computation shows that $x_4x_5^2=x_2x_{12}+[RP(2,2,2;9)]\in A^4_{14}$. Hence A^4_{14} contains products of generators except the classes

$$x_5x_9, x_2x_6^2, x_4^2x_6, x_2^2x_4x_6, x_2^4x_6, x_2^2x_5^2, x_2^3x_4^2, x_2^5x_4, x_2^7, x_2x_4^3.$$

Furthermore we have

$$x_5x_9+x_4^2x_6+x_2^2x_4x_6+x_2^3x_4^2+x_2^5x_4=x_4x_5^2$$

(4) $$+x_4x_{10}+x_2^2x_{10}+x_2^3x_8+[RP(4,2;9)]\in A^4_{14},$$

(5) $$x_2^2x_5^2+x_2^4x_6+x_2^7=x_{14}+x_2x_{12}+x_4x_{10}+x_2^2x_{10}$$
$$+x_4x_5^2+[RP(4,1,1,;9)]\in A^4_{14}.$$

From $x_2\cdot(2)$, we have

$$x_2x_6^2+x_2^2x_5^2+x_2^5x_4\in A^4_{14}.$$

In P^4_{14} we take $f_{30}(t)=\sigma_5\sigma_4\sigma_3\sigma_2(t),\ f_{31}(t)=\sigma_4\sigma_3^3\sigma_1(t),\ f_{32}(t)=\sigma_2\sigma_4^3(t),\ f_{33}(t)=\sigma_2^2\sigma_5^2(t)+\sigma_2^2\sigma_4\sigma_6(t),\ f_{34}(t)=\sigma_2^3\sigma_4^2(t),\ f_{35}(t)=\sigma_2^4\sigma_3^2(t),\ f_{36}(t)=\sigma_4^2\sigma_3\sigma_2\sigma_1(t)$. Characteristic numbers are given by

	x_5x_9	$x_2x_6^2$	$x_4^2x_6$	$x_2^2x_4x_6$	$x_2^4x_6$	$x_2^2x_5^2$	$x_2^3x_4^2$	$x_2^5x_4$	x_2^7	$x_2x_4^3$
$f_{30}(t)$	1	0	1	1	1	0	1	0	1	0
$f_{31}(t)$	0	0	0	0	0	0	0	0	0	1
$f_{32}(t)$	0	1	1	1	0	0	1	1	0	0
$f_{33}(t)$	0	0	1	1	1	1	1	1	0	1
$f_{34}(t)$	0	0	1	0	1	0	1	0	1	0
$f_{35}(t)$	0	0	1	0	0	0	1	0	0	0
$f_{36}(t)$	0	0	0	1	1	0	1	0	1	1

Combining this matrix with equations $(4),(5)$ and $x_2\cdot(2)$, we know that the classes are distinguished.

Dimension $n=15$. A computation shows that

(6) $$x_6x_9+x_2^3x_9=x_2x_{13}+x_5x_{10}+x_2x_5x_8+[RP(2,1,1,1,1,1;9)]\in A^4_{15},$$

(7) $x_5^3 + x_2^5x_5 = x_6x_9 + x_2^3x_9 + x_2x_{13} + x_5x_{10} + x_2x_5x_8 + [RP(2,2,2,1;9)] \in A_{15}^4$,

(8) $x_4x_{11} + x_2x_4x_9 + x_4x_5x_6 + x_5^3 = x_5x_{10} + x_2x_5x_8 + [RP(4,1,1,1;9)] \in A_{15}^4$,

(9) $x_6x_9 + x_2x_4x_9 + x_2^2x_5x_6 + x_2^3x_4x_5 = x_5^3 + x_2^5x_5 + [RP(5,2;9)] \in A_{15}^4$.

From $x_2 \cdot (3), x_2^2 \cdot (1)$ and $x_4 \cdot (1)$, we know

$$x_2^2x_{11} + x_2x_4x_9 + x_2^3x_9 + x_2^3x_4x_5 + x_2^5x_5 \in A_{15}^4,$$

$$x_2^3x_9 + x_2^2x_5x_6 + x_2^3x_4x_5 + x_2^5x_5 \in A_{15}^4,$$

$$x_2x_4x_9 + x_4x_5x_6 + x_2x_4^2x_5 + x_2^3x_4x_5 \in A_{15}^4.$$

A_{15}^4 contains products of generators except the classes

$$x_4x_{11}, x_2^2x_{11}, x_6x_9, x_2x_4x_9, x_2^3x_9, x_4x_5x_6, x_2^2x_5x_6, x_5^3, x_2x_4^2x_5, x_2^3x_4x_5, x_2^5x_5.$$

In P_{15}^4 we take $f_{37}(t) = \sigma_6\sigma_3^3(t),\ f_{38}(t) = \sigma_5\sigma_4\sigma_3^2(t),\ f_{39}(t) = \sigma_4^3\sigma_3(t),\ f_{40}(t) = \sigma_6\sigma_4\sigma_3\sigma_2(t) + \sigma_4^2\sigma_3\sigma_2\sigma_1^2(t)$. Characteristic numbers are given by

	x_4x_{11}	$x_2^2x_{11}$	x_6x_9	$x_2x_4x_9$	$x_2^3x_9$	$x_4x_5x_6$
$f_{37}(t)$	0	1	1	0	1	0
$f_{38}(t)$	1	1	1	0	1	1
$f_{39}(t)$	1	1	0	0	0	1
$f_{40}(t)$	0	0	0	1	0	0

	$x_2^2x_5x_6$	x_5^3	$x_2x_4^2x_5$	$x_2^3x_4x_5$	$x_2^5x_5$
$f_{37}(t)$	1	0	0	0	0
$f_{38}(t)$	1	0	1	0	0
$f_{39}(t)$	1	0	0	1	0
$f_{40}(t)$	1	1	1	0	1

Combining the matrixes with equations (6), (7), (8), (9) and $x_2\cdot(3), x_2^2\cdot(1), x_4\cdot(1)$, we get the result.

Dimension $n = 16$. From $x_2^2 \cdot (2), x_4 \cdot (2), x_2 \cdot (5), x_5 \cdot (1)$ and $x_2 \cdot (4)$, we know

$$x_2^3x_5^2 + x_2^6x_4 \in A_{16}^4, \quad x_4x_6^2 \in A_{16}^4, \quad x_2^3x_5^2 + x_2^5x_6 \in A_{16}^4,$$

$$x_2x_5x_9 \in A_{16}^4, \quad x_2^6x_4 + x_2x_4^2x_6 + x_2^3x_4x_6 \in A_{16}^4.$$

A computation shows that

(10) $x_2^3x_5^2 + x_5^2x_6 = x_{16} + x_4x_{12} + x_4^4 + x_2^4x_8 + x_2^8 + [RP(4,2,2;9)] \in A_{16}^4$,

(11) $$x_5x_{11} + x_2^5x_6 = x_{16} + x_2x_{14} + x_4x_{12} + x_6x_{10} + x_2x_6x_8 + x_2^2x_{12} + x_2^3x_{10} + x_2^2x_4x_8 + + x_5^2x_6 + x_2x_4^2x_6 + x_2^3x_4x_6 + x_2x_4x_5^2 + x_4^4 + x_2^8 + x_2x_5x_9 + [RP(4,2,1,1;9)] \in A_{16}^4.$$

A_{16}^4 contains products of generators except the classes

$$x_5x_{11}, x_2x_4^2x_6, x_2^3x_4x_6, x_2^3x_5^2, x_2^5x_6, x_6x_5^2, x_2^2x_4^3, x_2^6x_4.$$

In P_{16}^4 we take $f_{41}(t) = \sigma_6\sigma_4\sigma_3^2(t) + \sigma_5^2\sigma_3^2(t)$, $f_{42}(t) = \sigma_3^2\sigma_4^2\sigma_2(t)$, $f_{43}(t) = \sigma_3^4\sigma_2^2(t)$. Characteristic numbers are given by

	x_5x_{11}	$x_2x_4^2x_6$	$x_2^3x_4x_6$	$x_2^3x_5^2$	$x_2^5x_6$	$x_6x_5^2$	$x_2^2x_4^3$	$x_2^6x_4$
$f_{41}(t)$	1	1	0	1	1	1	1	1
$f_{42}(t)$	0	1	1	0	0	0	0	0
$f_{43}(t)$	0	0	0	0	0	0	1	0

Combining this matrix with equations $(10), (11)$ and $x_2^2 \cdot (2), x_2 \cdot (5), x_2 \cdot (4)$, we obtain the result.

Dimension $n = 17$. A computation shows that

$$x_6x_{11} = x_{17} + 0 \cdot x_5x_4^3 + y_{17} + [RP(4,1,1,1,1,1;9)] \in A_{17}^4,$$

where y_{17} can be expressed as a sum of products of generators without x_6x_{11}, x_{17} and $x_5x_4^3$.

A_{17}^4 contains products of generators except the class $x_5x_4^3$. We take $f_{44}(t) = \sigma_3^5\sigma_2(t) \in P_{17}^4$, obtaining $f_{44}(t)[x_5x_4^3] \neq 0$. Therefore the result is true.

Dimension $n = 18$. From $x_4 \cdot (4)$, we get $x_2^3x_4^3 + x_4^3x_6 \in A_{18}^4$. A_{18}^4 contains products of generators except the classes $x_4^3x_2^3, x_4^3x_6$. We take $f_{45}(t) = \sigma_3^4\sigma_2^3(t) \in P_{18}^4$, obtaining $f_{45}(t)[x_4^3x_2^3] = f_{45}(t)[x_4^3x_6] = 1$. Therefore the result is true.

Dimension $n > 18$. Since A_*^4 forms an ideal and $J_*^{2r_1}J_*^{2r_2} \subset J_*^{2r_1+2r_2} \subset A_*^{r_1+r_2}$, A_*^4 contains all the products, which have dimension $n > 18$, of generators of MO_*. So $A_n^4 = B_n^4 = MO_n$ for $n > 18$.

The proof is completed.

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