

## TRIGONOMETRIC POLYNOMIALS WITH MANY REAL ZEROS AND A LITTLEWOOD-TYPE PROBLEM

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ABSTRACT. We examine the size of a real trigonometric polynomial of degree at most  $n$  having at least  $k$  zeros in  $K := \mathbb{R} \pmod{2\pi}$  (counting multiplicities). This result is then used to give a new proof of a theorem of Littlewood concerning flatness of unimodular trigonometric polynomials. Our proof is shorter and simpler than Littlewood's. Moreover our constant is explicit in contrast to Littlewood's approach, which is indirect.

### 1. INTRODUCTION

The set of all polynomials of degree  $n$  with coefficients  $\pm 1$  will be denoted by  $\mathcal{L}_n$ . Specifically

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{j=0}^n a_j z^j, a_j \in \{-1, 1\} \right\}.$$

Let  $D$  denote the closed unit disk of the complex plane. Let  $\partial D$  denote the unit circle of the complex plane. Littlewood made the following conjecture about  $\mathcal{L}_n$  in the fifties.

**Conjecture 1.1** (Littlewood). *There are at least infinitely many values of  $n \in \mathbb{N}$  for which there are polynomials  $p_n \in \mathcal{L}_n$  so that*

$$C_1(n+1)^{1/2} \leq |p_n(z)| \leq C_2(n+1)^{1/2}$$

for all  $z \in \partial D$ . Here the constants  $C_1$  and  $C_2$  are independent of  $n$ .

Since the  $L_2(\partial D)$  norm of a polynomial from  $\mathcal{L}_n$  is exactly  $(2\pi)^{1/2}(n+1)^{1/2}$ , the constants must satisfy  $C_1 \leq 1$  and  $C_2 \geq 1$ . See Problem 19 of [Li-68]. While there is much literature on this problem and its variants, this is still open. See [Saf-90] and [Bor-98]. In fact, finding polynomials that satisfy just the lower bound in Conjecture 1.1 is still open. The Rudin–Shapiro polynomials satisfy the upper bound.

There is a related conjecture of Erdős [Er-62].

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**Conjecture 1.2** (Erdős). *There is a constant  $\varepsilon > 0$  (independent of  $n$ ) so that*

$$\max_{z \in \partial D} |p_n(z)| \geq (1 + \varepsilon)(n + 1)^{1/2}$$

for every  $p_n \in \mathcal{L}_n$  and  $n \in \mathbb{N}$ . That is, the constant  $C_2$  in Conjecture 1.1 must be bounded away from 1 (independently of  $n$ ).

This conjecture is also open. Kahane [Kah-85], however, shows that if the polynomials are allowed to have complex coefficients of modulus 1, then Conjecture 1.1 holds and Conjecture 1.2 fails. That is, for every  $\varepsilon > 0$  there are infinitely many values of  $n \in \mathbb{N}$  for which there are polynomials  $p_n$  of degree  $n$  with complex coefficients of modulus 1 that satisfy

$$(1 - \varepsilon)(n + 1)^{1/2} \leq |p_n(z)| \leq (1 + \varepsilon)(n + 1)^{1/2}$$

for all  $z \in \partial D$ . Beck [Bec-95] proves Conjecture 1.1 for polynomials whose coefficients are 400th roots of unity.

Our main result is a reproving of Conjecture 1.2 for real trigonometric polynomials. This is Corollary 2.4 of the next section. Littlewood gives a proof of this in [Li-61] and explores related issues in [Li-62], [Li-66a], and [Li-66b]. The approach is via Theorem 2.1 which estimates the measure of the set where a real trigonometric polynomial of degree at most  $n$  with at least  $k$  zeros in  $K := \mathbb{R} \pmod{2\pi}$  is small. There are two reasons for doing this. First the approach is, we believe, easier, and secondly it leads to explicit constants.

## 2. NEW RESULTS

Let  $K := \mathbb{R} \pmod{2\pi}$ . For the sake of brevity the uniform norm of a continuous function  $p$  on  $K$  will be denoted by  $\|p\|_K := \|p\|_{L_\infty(K)}$ . Let  $\mathcal{T}_n$  denote the set of all real trigonometric polynomials of degree at most  $n$ , and let  $\mathcal{T}_{n,k}$  denote the subset of those elements of  $\mathcal{T}_n$  that have at least  $k$  zeros in  $K$  (counting multiplicities).

**Theorem 2.1.** *Suppose  $p \in \mathcal{T}_n$  has at least  $k$  zeros in  $K$  (counting multiplicities). Let  $\alpha \in (0, 1)$ . Then*

$$m\{t \in K : |p(t)| \leq \alpha \|p\|_K\} \geq \frac{\alpha k}{e n},$$

where  $m(A)$  denotes the one-dimensional Lebesgue measure of  $A \subset K$ .

**Theorem 2.2.** *We have*

$$2\pi \left(1 - \frac{c_2 k}{n}\right) \leq \sup_{p \in \mathcal{T}_{n,k}} \frac{\|p\|_{L_1(K)}}{\|p\|_{L_\infty(K)}} \leq 2\pi \left(1 - \frac{c_1 k}{n}\right)$$

for some absolute constants  $0 < c_1 < c_2$ .

**Theorem 2.3.** *Assume that  $p \in \mathcal{T}_n$  satisfies*

$$(2.1) \quad \|p\|_{L_2(K)} \leq A n^{1/2}$$

and

$$(2.2) \quad \|p'\|_{L_2(K)} \geq B n^{3/2}.$$

Then there is a constant  $\varepsilon > 0$  depending only on  $A$  and  $B$  such that

$$(2.3) \quad \|p\|_K^2 \geq (2\pi - \varepsilon)^{-1} \|p\|_{L_2(K)}^2.$$

Here

$$\varepsilon = \frac{\pi^3}{1024e} \frac{B^6}{A^6}$$

works.

**Corollary 2.4.** *Let  $p \in \mathcal{T}_n$  be of the form*

$$p(t) = \sum_{k=1}^n a_k \cos(kt - \gamma_k), \quad a_k = \pm 1, \quad \gamma_k \in \mathbb{R}, \quad k = 1, 2, \dots, n.$$

Then there is a constant  $\varepsilon > 0$  such that

$$\|p\|_K^2 \geq (2\pi - \varepsilon)^{-1} \|p\|_{L_2(K)}^2.$$

Here

$$\varepsilon := \frac{\pi^3}{1024e} \frac{1}{27}$$

works.

### 3. PROOFS

To prove Theorem 2.1 we need the lemma below that is proved in [Bor-95, E.11 of Section 5.1 on pages 236–237].

**Lemma 3.1.** *Let  $p \in \mathcal{T}_n$ ,  $t_0 \in K$ , and  $r > 0$ . Then  $p$  has at most  $enr|p(t_0)|^{-1}\|p\|_K$  zeros in the interval  $[t_0 - r, t_0 + r]$ .*

*Proof of Theorem 2.1.* Suppose  $p \in \mathcal{T}_n$  has at least  $k$  zeros in  $K$ , and let  $\alpha \in (0, 1)$ . Then

$$\{t \in K : |p(t)| \leq \alpha\|p\|_K\}$$

can be written as the union of pairwise disjoint intervals  $I_j$ ,  $j = 1, 2, \dots, m$ . Each of the intervals  $I_j$  contains a point  $y_j \in I_j$  such that

$$|p(y_j)| = \alpha\|p\|_K.$$

Also, each zero of  $p$  from  $K$  is contained in one of the intervals  $I_j$ . Let  $\mu_j$  denote the number of zeros of  $p$  in  $I_j$ . Since  $p \in \mathcal{T}_n$  has at least  $k$  zeros in  $K$ , we have  $\sum_{j=1}^m \mu_j \geq k$ . Note also that Lemma 3.1 implies that

$$\mu_j \leq en|I_j|(\alpha\|p\|_K)^{-1}\|p\|_K = \frac{en}{\alpha}|I_j|.$$

Therefore

$$k \leq \sum_{j=1}^m \mu_j \leq \frac{en}{\alpha} \sum_{j=1}^m |I_j| \leq \frac{en}{\alpha} m(\{t \in K : |p(t)| \leq \alpha\|p\|_K\}),$$

and the result follows. □

*Proof of Theorem 2.2.* The upper bound of the theorem follows from Theorem 2.1 applied with  $\alpha = 1/2$ . The lower bound follows by considering

$$p(t) := D_m(0)^2 - D_m(kt)^2 \in \mathcal{T}_{n,k} \quad \text{with} \quad m = \left\lfloor \frac{n}{2(k+1)} \right\rfloor,$$

where

$$D_m(t) = \frac{1}{2} + \sum_{j=1}^m \cos jt$$

is the Dirichlet kernel of degree  $m$ . □

*Proof of Theorem 2.3.* First note that by Bernstein's inequality for real trigonometric polynomials in  $L_2(K)$ , we have  $B \leq A$ . Assume that  $p \in \mathcal{T}_n$  satisfies (2.1) and (2.2) but (2.3) does not hold with  $\varepsilon = \pi$ . Then

$$(3.1) \quad M := \|p\|_K \leq (2\pi - \pi)^{-1/2} \|p\|_{L_2(K)} \leq \pi^{-1/2} A n^{1/2}.$$

Combining this with Bernstein's inequality we obtain

$$(3.2) \quad \|p'\|_K \leq n \|p\|_K \leq \pi^{-1/2} A n^{3/2}.$$

Using (2.2), we obtain

$$\begin{aligned} B^2 n^3 &\leq \|p'\|_{L_2(K)}^2 = \int_K |p'(t)|^2 dt \\ &\leq \|p'\|_K \int_K |p'(t)| dt \leq \pi^{-1/2} A n^{3/2} \|p'\|_{L_1(K)}, \end{aligned}$$

that is,

$$(3.3) \quad \|p'\|_{L_1(K)} \geq \pi^{1/2} \frac{B^2}{A} n^{3/2}.$$

Associated with  $p \in \mathcal{T}_n$ ,  $M = \|p\|_K$ , and  $\gamma \in [0, 1]$ , let

$$(3.4) \quad A_\gamma = A_\gamma(p) = \{t \in K : |p(t)| \leq (1 - \gamma)M\}$$

and

$$(3.5) \quad B_\gamma = B_\gamma(p) = \{t \in K : |p(t)| > (1 - \gamma)M\}.$$

Since every horizontal line  $y = c$  intersects the graph of  $p \in \mathcal{T}_n$  in at most  $2n$  points with  $x$  coordinates in  $K$ , we have

$$(3.6) \quad \int_{B_\gamma} |p'(t)| dt \leq 4n\gamma M \leq 4n\gamma\pi^{-1/2} A n^{1/2} \leq \frac{\pi^{1/2}}{2} \frac{B^2}{A} n^{3/2}$$

if

$$4\gamma\pi^{-1/2} A \leq \frac{\pi^{1/2}}{2} \frac{B^2}{A}, \quad \text{that is, if} \quad \gamma \leq \frac{\pi}{8} \frac{B^2}{A^2}.$$

Now (3.3)–(3.6) give

$$\int_{A_\gamma} |p'(t)| dt \geq \frac{\pi^{1/2}}{2} \frac{B^2}{A} n^{3/2} \quad \text{with} \quad \gamma = \frac{\pi}{8} \frac{B^2}{A^2}.$$

From this, with the help of (3.1) we can deduce that there is a

$$\delta \in (-(1 - \gamma)M, (1 - \gamma)M)$$

such that  $p - \delta$  has at least

$$\frac{\frac{\pi^{1/2}}{2} \frac{B^2}{A} n^{3/2}}{2(1 - \gamma)M} \geq \frac{\pi^{1/2}}{4} \frac{B^2}{A} \frac{n^{3/2}}{M} \geq \frac{\pi^{1/2}}{4} \frac{B^2}{A} \frac{n^{3/2}}{\pi^{-1/2} A n^{1/2}} = \frac{\pi}{4} \frac{B^2}{A^2} n$$

zeros in  $K$ . Therefore Theorem 2.1 yields that

$$\begin{aligned} m \left\{ t \in K : |p(t)| \leq \left(1 - \frac{\gamma}{2}\right) \|p\|_K \right\} &\geq m \left\{ t \in K : |p(t) - \delta| \leq \frac{\gamma}{2} \|p\|_K \right\} \\ &\geq m \left\{ t \in K : |p(t) - \delta| \leq \frac{\gamma}{4} \|p - \delta\|_K \right\} \\ &\geq \frac{1}{e} \frac{\gamma}{4} \frac{\pi}{4} \frac{B^2}{A^2} \frac{n}{n} \geq \frac{\pi^2}{128e} \frac{B^4}{A^4}. \end{aligned}$$

Therefore

$$\begin{aligned} 2\pi \|p\|_K^2 - \|p\|_{L_2(K)}^2 &= \int_K (\|p\|_K^2 - |p(t)|^2) dt \geq \frac{\pi^2}{128e} \frac{B^4}{A^4} \frac{\gamma}{2} \|p\|_K^2 \\ &= \frac{\pi^3}{1024e} \frac{B^6}{A^6} \|p\|_K^2. \end{aligned}$$

We now conclude that

$$\|p\|_{L_2(K)}^2 \leq \left(2\pi - \frac{\pi^3}{1024e} \frac{B^6}{A^6}\right) \|p\|_K^2,$$

and the result follows.  $\square$

*Proof of Corollary 2.4.* Let  $p \in \mathcal{T}_n$  be of the given form. We have

$$\|p\|_{L_2(K)}^2 = \pi \sum_{k=1}^n a_k^2 = \pi n,$$

that is,

$$\|p\|_{L_2(K)} = \pi^{1/2} n^{1/2}.$$

Also

$$\|p'\|_{L_2(K)}^2 = \pi \sum_{k=1}^n k^2 a_k^2 = \pi \frac{n(n+1)(2n+1)}{6} \geq \frac{\pi}{3} n^3,$$

that is,

$$\|p'\|_{L_2(K)} \geq \left(\frac{\pi}{3}\right)^{1/2} n^{3/2}.$$

Now the result follows from Theorem 2.3 with  $A := \pi^{1/2}$  and  $B := (\pi/3)^{1/2}$ .  $\square$

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