PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 4, Pages 1161–1168
S 0002-9939(00)05587-8
Article electronically published on September 20, 2000

CONSTRUCTING THE KÄHLER AND THE SYMPLECTIC STRUCTURES FROM CERTAIN SPINORS ON 4-MANIFOLDS

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(Communicated by Ronald A. Fintushel)

Abstract. We show that, on an oriented Riemannian 4-manifold, existence of a non-zero parallel spinor with respect to a spin^c structure implies that the underlying smooth manifold admits a Kähler structure. A similar but weaker condition is obtained for the 4-manifold to admit a symplectic structure. We also show that the *spin*^{c} structure in which the non-zero parallel spinor lives is equivalent to the canonical spin^c structure associated to the Kähler structure.

1. Introduction

Let X be a Kähler manifold and consider the canonical *spin*^{c} structure. Then the spin bundle can be identified with the exterior algebra of the complex antilinear cotangent bundle, $\bigwedge^{0,*}(X,\mathbb{C})$ (cf. §3.4, [2]). Furthermore, the Levi-Civita connection on $\bigwedge^{0,*}(X,\mathbb{C})$ corresponds to a *spin*^{c} *connection* and it follows that a constant function is a parallel spinor. Thus a Kähler manifold admits a non-zero parallel spinor. If X is only symplectic, which is somewhat weaker than being Kähler, still the spin bundle can be identified with $\bigwedge^{0,*}(X,\mathbb{C})$ assuming a choice of an almost complex structure on the tangent bundle TX. Then for an appropriate choice of *spin*^{c} connection, the constant function is a harmonic spinor ([6]). The main purpose of this paper is to show that the converses for these two observations hold for 4-manifolds.

Note that any closed oriented smooth Riemannian 4-manifold admits a *spin*^{c} structure, even if we will not demand the closedness condition for the manifolds.

Theorem 1. *Assume an oriented Riemannian 4-manifold X admits a spin^c structure with a non-zero spinor which is parallel with respect to a spin^c connection. Then the smooth manifold X admits a Kähler structure.*

In the above we allow the orientation of the Kähler structure to be opposite to the original one, which applies to Theorem 2 below as well.

Received by the editors April 30, 1999 and, in revised form, June 21, 1999.
1991 *Mathematics Subject Classification.* Primary 53C05, 53C07.
Key words and phrases. Parallel positive spinor, Kähler manifold, symplectic manifold, *spin*^{c} structure.
The first author was partially supported by the Hanyang University Research Fund. The second author was supported in part by 1998-015-D00044. The third author was partially supported by the Dongguk University Research Fund. The fourth author was supported in part by GARC.

Theorem 2. *Assume an oriented Riemannian 4-manifold X has a spin^c structure for which there is a non-zero positive harmonic spinor ψ such that*

$$\langle \tilde{\nabla}_v \psi, \psi \rangle = 0$$

for any $v \in TX$, where $\langle \cdot, \cdot \rangle$ denotes the Hermitian metric for the spin bundle and $\tilde{\nabla}$ is the spin^c connection. Then the smooth manifold X admits a symplectic structure.

Here that ψ is harmonic means that it is in the kernel of a Dirac operator. The main ideas for both the statement and the proof of Theorem 2 are due to C. H. Taubes [6].

It is interesting to compare Theorem 1 with a statement by A. Moroianu [3]: *A complete simply connected spin^c-manifold carrying a parallel spinor is isometric to the Riemannian product between a Kähler manifold and a spin manifold carrying a parallel spinor.* On the other hand it is shown by M. Wang ([7] and also see [1]) that *a complete, simply-connected irreducible spin 4-manifold having a non-zero spin field is Kähler.* Therefore, their results combined imply that a simply-connected complete 4-manifold carrying a parallel spinor is Kähler. However the simply-connectedness condition and the completeness one are essential for their arguments which exploit such theorems as the de Rham decomposition theorem. Our paper assumes neither completeness nor simply-connectedness for the manifolds.

Note that in each of Theorem 1 and 2, two *spin^c* structures, one of them being implicit, are involved. In fact, we will see that, given a nowhere vanishing positive spinor ψ in a spin^c structure on a 4-manifold X, it determines a 2-form ω which is nondegenerate at every point of X. In turn ω gives rise to a *spin^c* structure well-defined up to an equivalence, which we will refer to as the *spin^c* structure determined by ψ. The details of this construction are provided in §4, below Lemma 4.1.

For the special case when ψ is parallel, we have:

Theorem 3. *Let X be an oriented Riemannian 4-manifold which admits a spin^c structure. Assume there is a non-zero positive spinor ψ which is parallel with respect to a spin^c connection. Then the spin^c structure determined by ψ is equivalent to the original one.*

At the moment we do not know whether the above holds for general nowhere vanishing positive spinors.

2. Preliminaries

Let (X, g) be an oriented Riemannian 4-manifold, which is not necessarily compact and P denote the oriented tangent frame bundle of X. A spin^c structure on X is a smooth map $\tilde{P} \to P$, in which $\tilde{P}$ is a principal $Spin^c(4)$ bundle over X and the map respects the actions of $Spin^c(4)$, $SO(4)$ by means of the homomorphism $Spin^c(4) \to SO(4)$. Fix an irreducible complex representation of the complexified Clifford algebra, $\rho : Cl(4) \otimes \mathbb{C} \to End_{\mathbb{C}} S_{\mathbb{C}}(4)$. We also choose a Hermitian metric on $S_{\mathbb{C}}(4)$ so that ρ maps $Spin^c(4)$ into the group of unitary automorphisms of $S_{\mathbb{C}}(4)$. Then we obtain a Hermitian complex vector bundle $S_{\mathbb{C}}(\tilde{P}) = \tilde{P} \times_\rho S_{\mathbb{C}}(4)$, which is a $Cl(TX) \otimes \mathbb{C}$ module. $S_{\mathbb{C}}(\tilde{P})$ is commonly referred to as the spin bundle on X associated to the spin^c structure $\tilde{P} \to P$.

Let ∇ denote the Levi-Civita connection on TX and let A be a connection on the determinant line bundle $\mathcal{L} = \tilde{P} \times_{Spin^c(4)} \mathbb{C}$, in which $Spin^c(4)$ acts on $\mathbb{C}$ via the homomorphism $Spin^c(4) \to S^1$. Then there is a unique connection ∇^A on $S_{\mathbb{C}}(\tilde{P})$ determined by both ∇ and A, which is a spin^c connection (cf. [4], [6]) in the sense that the Hermitian metric on $S_{\mathbb{C}}(\tilde{P})$ is parallel and the following form of Leibniz rule holds:

$$\nabla^A_v(\omega \cdot \psi) = (\nabla_v \omega) \cdot \psi + \omega \cdot \nabla^A_v \psi,$$

where v is a tangent vector of X, ω is a complex valued k-form on X and ψ is a smooth section of $S_{\mathbb{C}}(\tilde{P})$, that is, a spinor. Here '$\cdot$' denotes the Clifford multiplication of a spin vector by a complex valued k-form, which is defined by a natural complex bundle isomorphism, $\bigwedge^*(T^*X) \otimes \mathbb{C} \to Cl(T^*X) \otimes \mathbb{C}$, followed by the isomorphism $Cl(T^*X) \otimes \mathbb{C} \to Cl(TX) \otimes \mathbb{C}$ determined by the Riemannian metric g. In fact every spin^c connection is ∇^A for some A (cf. [4]).

Note that the volume element of the Clifford algebra $Cl(TX) \otimes \mathbb{C}$ corresponds to $-\Omega$, where Ω denotes the volume form of (X, g). Then $S_{\mathbb{C}}(\tilde{P})$ can be decomposed into $S^+_{\mathbb{C}}(\tilde{P})$ and $S^-_{\mathbb{C}}(\tilde{P})$, which respectively consist of ± 1 eigenspaces of the volume element.

Now let $\Omega(S_{\mathbb{C}}(\tilde{P}))$ and $\Omega(S^+_{\mathbb{C}}(\tilde{P}))$, $\Omega(S^-_{\mathbb{C}}(\tilde{P}))$ denote the sets of smooth sections. Then the Dirac operator, $D^A : \Omega(S_{\mathbb{C}}(\tilde{P})) \to \Omega(S_{\mathbb{C}}(\tilde{P}))$, is defined by

$$(D^A \psi)_x = \sum_{i=1}^{4} e^i \cdot \nabla^A_{e_i} \psi$$

for any $\psi \in \Omega(S_{\mathbb{C}}(\tilde{P}))$ and any $x \in X$, where $e_i, i = 1, 2, 3, 4$, denotes an orthonormal basis of $T_x X$ and $e^i, i = 1, 2, 3, 4$, its dual. Note that the Dirac operator restricts to

$$D^A : \Omega(S^{\pm}_{\mathbb{C}}(\tilde{P})) \to \Omega(S^{\mp}_{\mathbb{C}}(\tilde{P})) \ .$$

Finally, a symplectic structure on X is given by a closed 2-form ω such that $\omega_x : T_x X \times T_x X \to \mathbb{R}$ is non-degenerate for each $x \in X$. A Kähler structure on X consists of a symplectic 2-form $\omega \in \Omega^2(X, \mathbb{R})$ together with an almost complex structure J so that the expression $g_{\omega,J}(v, w) = \omega(v, Jw)$, $v, w \in T_x X$, defines a Riemannian metric and J is parallel with respect to the Levi-Civita connection associated to $g_{\omega,J}$.

3. The Kähler and the symplectic structures

In this section we prove Theorems 1 and 2.

We start from the observation that a positive spinor determines a self-dual real-valued 2-form: Let $\psi \in \Omega(S^+_{\mathbb{C}}(\tilde{P}))$ where $\tilde{P}$ is a spin^c structure on an oriented Riemannian 4-manifold X. Then we consider the endomorphism of $S^+_{\mathbb{C}}(\tilde{P})$ given by

$$q(\psi) = \psi \otimes \psi^* - \frac{1}{2} \mid \psi \mid^2 Id \ .$$

Here we exploit the fact that the traceless self-adjoint endomorphisms of $S^+_{\mathbb{C}}(\tilde{P})$ are in 1-1 correspondence with the self-dual purely imaginary 2-forms on X where the 1-1 correspondence is given by the Clifford multiplication (cf. pp.55–57, [2]). It is straightforward to see that $q(\psi)$ is traceless and self-adjoint. Thus there exists a real-valued self-dual 2-form ω corresponding to $-iq(\psi)$.

Terminology. We will say that ω in the above is determined by ψ.

Lemma 3.1. *Let ψ be a positive spinor on an oriented Riemannian 4-manifold X, presuming a spin^c structure on X, and let ω be the self-dual real-valued 2-form determined by ψ. Then, for any $x \in X$ such that $\psi_x \neq 0$, we have that $\omega_x : T_xX \times T_xX \to \mathbb{R}$ is nondegenerate.*

Proof. It is enough to note that $\omega \wedge \omega = \omega \wedge (*\omega) = \mid \omega \mid^2 \Omega$, where Ω denotes the volume form of X, and that ω_x is non-zero if ψ_x is non-zero. □

Proof of Theorem 1. Let ψ be a non-zero section in a spin bundle $S_{\mathbb{C}}(\tilde{P})$ on X which is parallel with respect to a spin^c connection ∇^A.

Then ψ^+, the $S^+_{\mathbb{C}}(\tilde{P})$-component of ψ, is a non-zero parallel spinor, reversing the orientation if necessary. Thus without loss of generality we may assume ψ itself is a non-zero positive parallel spin field.

The proof is divided into two steps.

The Kähler form: Let ω denote the self-dual real-valued 2-form determined by ψ.

Since the Hermitian metric on $S_{\mathbb{C}}(\tilde{P})$ is parallel with respect to ∇^A, it follows that $\mid \psi \mid$ is constant: we have $d\langle\psi,\psi\rangle = \langle\nabla^A\psi,\psi\rangle + \langle\psi,\nabla^A\psi\rangle = 0$. In particular, it follows that ω is non-degenerate at every point of X by Lemma 3.1.

Note that $\omega\cdot\psi = -\frac{i}{2} \mid \psi \mid^2 \psi = -ik\psi$, writing k for the constant $\frac{1}{2} \mid \psi \mid^2$. Therefore, we have

$$0 = \nabla^A_v(\omega\cdot\psi) = (\nabla_v\omega)\cdot\psi_x,$$

where ∇ denotes the Levi-Civita connection and v is a tangent vector based at $x \in X$. On the other hand, ∇ preserves the self-dual 2-forms. Thus $\nabla_v\omega$ acts as a *traceless* skew adjoint endomorphism of $S^+_{\mathbb{C}}(\tilde{P})_x$ by the Clifford multiplication. Since it has turned out that one of its eigenvalues is zero, the other eigenvalue must be zero as well. Thus we have: $\nabla\omega = 0$.

The almost complex structure: Let $A : TX \to TX$ be the endomorphism defined by $g(v, Aw) = \omega(v,w)$. Then A is a skew adjoint automorphism of TX. Furthermore, A is parallel with respect to the Levi-Civita connection ∇ associated to g since g and ω are parallel.

Here we recall the following fact: Let $E \to X$ be a smooth real vector bundle of finite rank with a Riemannian metric h and let $S : E \to E$ be a smooth bundle isomorphism which is self-adjoint and positive definite. Then there is a unique self-adjoint positive definite smooth bundle isomorphism $\sqrt{S} : E \to E$ satisfying $\sqrt{S}^2 = S$. Furthermore, for any connection ∇' on E such that $\nabla' S = 0$, it holds that $\nabla'\sqrt{S} = 0$ since, by applying the covariant derivative to the both sides of the equality, $\sqrt{S}^2 = S$, we have

$$(\nabla'_v\sqrt{S})\sqrt{S}_x + \sqrt{S}_x(\nabla'_v\sqrt{S}) = 0,$$

for any tangent vector v of X based at $x \in X$. By applying both sides of this equation to each of the eigenvectors of $\sqrt{S}_x$ we conclude that $\nabla'_v\sqrt{S} = 0$.

Now we set $J = -A\sqrt{-A^2}^{-1}$ as the almost complex structure. First of all, we have $J^2 = -Id$. Also writing $\hat{A} = \sqrt{-A^2}$, we have

$$\omega(v, Jw) = g(v, \hat{A}w).$$

Since $\hat{A}$ is a self-adjoint positive definite endomorphism, J is compatible with ω. Furthermore, since $\nabla \hat{A} = 0$, the Levi-Civita connection associated to (ω, J) is exactly ∇ itself. Finally, since $\nabla J = 0$, we conclude that (ω, J) defines a Kähler structure on X. □

Proof of Theorem 2. Again we consider the self-dual real-valued 2-form ω determined by ψ. Then the argument of the proof of Theorem 1 is still valid to show that ω is nondegenerate since still we have $d\langle\psi, \psi\rangle = \langle\nabla^A\psi, \psi\rangle + \langle\psi, \nabla^A\psi\rangle = 0$.

It remains to show that

$$d\omega = 0 \ .$$

By definition we have $\omega \cdot \psi = -ik\psi$, where k is the constant $\frac{1}{2} \mid \psi \mid^2$. Apply the Dirac operator D^A to the both sides of this equality to obtain the equality:

$$\sum_{i=1}^{4} e^i \cdot (\nabla_{e_i}\omega \cdot \psi + \omega \cdot \nabla^A_{e_i}\psi) = 0 \ ,$$

where $e_i, i = 1, 2, 3, 4$, is an orthonormal basis of T_xX and $e^i, i = 1, 2, 3, 4$, is its dual.

Note that we have

$$\sum_{i=1}^{4} e^i \cdot \nabla_{e_i}\omega = \sum_{i=1}^{4} (e^i \wedge (\nabla_{e_i}\omega) - e_i \angle (\nabla_{e_i}\omega)) = d\omega - d^*\omega = (1 - *)d\omega \ .$$

Furthermore, we have $*\theta = (-\Omega) \cdot \theta$ and for any 3-form θ where Ω is the volume form of X. Note that $-\Omega$ corresponds to the volume element of the Clifford algebra bundle of T^*X. Since $d\omega \cdot \psi$ is a section of $S^-_{\mathbb{C}}(\tilde{P})$, we conclude that

$$2(d\omega) \cdot \psi = -\sum_{i=1}^{4} e^i \cdot (\omega \cdot \nabla^A_{e_i}\psi) \ .$$

Now by definition we have

$$\omega \cdot \nabla^A_{e_i}\psi = -iq(\psi)(\nabla^A_{e_i}\psi) = ik\nabla^A_{e_i}\psi$$

since by assumption $\langle\nabla^A_v\psi, \psi\rangle = 0$ for any $v \in TX$. Thus we have

$$\sum_{i=1}^{4} e^i \cdot (\omega \cdot \nabla^A_{e_i}\psi) = ikD^A\psi = 0 \ .$$

Therefore we conclude that

$$2(d\omega) \cdot \psi = 0.$$

Since a 1-form θ, $\theta_x \neq 0$, $x \in X$, acts on the fiber $S_{\mathbb{C}}(\tilde{P})_x$ as an isomorphism by the Clifford multiplication, it follows that the same is true for a 3-form since X is a four manifold. Thus we conclude that $d\omega = 0$. □

4. The spin^c structures

Let $\tilde{P}_g \to P_g$, $\tilde{P}_{g'} \to P_{g'}$ be two spin^c structures on an oriented n-manifold X, where g, g' are Riemannian metrics on X and P_g, $P_{g'}$ denote the respective principal $SO(n)$ bundles consisting of the oriented orthonormal frames.

Then it is well known that $\tilde{P}_g, \tilde{P}_{g'}$ are isomorphic to each other as principal $Spin^c(n)$ bundles over X if and only if their respective determinant S^1 bundles are isomorphic to each other, which are given by the homomorphism, $\det : Spin^c(n) \to$

S^1. Also it is well known that the complex line bundle associated to the determinant S^1 bundle of a $Spin^c(n)$ bundle is isomorphic to the determinant line bundle of the positive spin bundle. Therefore we have:

Lemma 4.1. *Let X be a smooth oriented n-manifold, g, g', two Riemannian metrics on X and P_g, $P_{g'}$, the respective oriented tangent frame bundles. Assume there are spin^c structures $\tilde{P}_g \to P_g$, $\tilde{P}_{g'} \to P_{g'}$. Then the two spin^c structures are equivalent to each other if and only if $S^+_{\mathbb{C}}(\tilde{P}_g)$, $S^+_{\mathbb{C}}(\tilde{P}_{g'})$ are isomorphic to each other as complex vector bundles.*

In the above by writing that two spin^c structures are equivalent to each other we mean the two principal $Spin^c(n)$ bundles, $\tilde{P}_g, \tilde{P}_{g'}$, are isomorphic to each other.

Now let (X, g) be an oriented Riemannian 4-manifold which admits a spin^c structure, $\tilde{P}_g \to P_g$, and assume there is a nowhere vanishing positive spinor ψ with respect to the spin^c structure. Then the 2-form ω on X determined by ψ is nondegenerate at each point of X by Lemma 3.1. Let J denote an almost complex structure on X which is compatible with ω in the sense the expression $h(v, w) = \omega(v, Jw)$ defines a Riemannian metric h on X. Such almost complex structures always exist and they form a contractible space (cf. Proposition 4.1, [5]). Then J determines a spin^c structure, say, $\tilde{P}_{\omega,J} \to P_{\omega,J}$ on (X, h) whose spin bundle can be identified with $\bigwedge^{0,*}(X, \mathbb{C})$. Note that the spin^c structure, $\tilde{P}_{\omega,J} \to P_{\omega,J}$, does not depend on the choice of J up to equivalence.

Now we may restate Theorem 3.

Theorem 4. *Assume ψ is parallel. Then the two spin^c structures, $\tilde{P}_g \to P_g$, $\tilde{P}_{\omega,J} \to P_{\omega,J}$ are equivalent to each other.*

Proof. The proof is divided into two cases.

Case 1: The vector space of parallel positive spinors is of rank 1. Note that $\bigwedge^{0,even}(X, \mathbb{C})$ is isomorphic to $S^+_{\mathbb{C}}(\tilde{P}_{\omega,J})$ as a $(Cl_{h,0}(T^*X) \otimes \mathbb{C})^+$ module. On the other hand, $\bigwedge^{0,2}(X, \mathbb{C})$ can be also viewed as a subbundle of $(Cl_{h,0}(T^*X) \otimes \mathbb{C})^+$ by the canonical identification of $\bigwedge^*(X, \mathbb{C})$ with $Cl_h(T^*X) \otimes \mathbb{C}$.

Let h denote the Riemannian metric on X determined by ω, J. Then first consider the endomorphism $A : TX \to TX$ defined by $h(v, w) = g(v, Aw)$ and subsequently set $S : TX \to TX$ as the 'square-root' of A (see the proof Theorem 1). Note that S satisfies $h(v, w) = g(Sv, Sw)$ for any $v, w \in T_xX$, $x \in X$ and also that S is self-adjoint with respect to both g, h, positive definite and parallel with respect to the Levi-Civita connection associated g, h.

Let $S : Cl_h(T^*X) \otimes C \to Cl_g(T^*X) \otimes C$ denote the isomorphism induced by $S : TX \to TX$, allowing ourselves a slight abuse of notation. Then S maps $(Cl_{h,0}(X) \otimes \mathbb{C})^+$ isomorphically into $(Cl_{g,0}(T^*X) \otimes \mathbb{C})^+$. Therefore, we have a well defined bundle homomorphism:

$$S^+_{\mathbb{C}}(\tilde{P}_{\omega,J}) \equiv \bigwedge{}^{0,even}(X, \mathbb{C}) \to S^+_{\mathbb{C}}(\tilde{P}_g),\ a \to S(a) \cdot \psi\ .$$

Considering Lemma 4.1, the following is enough to show that this is an isomorphism between complex vector bundles.

Claim. If $a \in \bigwedge^{0,2}(X, \mathbb{C}), a \neq 0$, then $S(a) \cdot \psi$ and $\psi = S(1) \cdot \psi$ are linearly independent over $\mathbb{C}$ at every point of X.

Proof. Let e_1, Je_1, e_2, Je_2 be an orthonormal frame of T_xX with respect to the metric h. Write $\bar{\varepsilon}_k = \frac{1}{\sqrt{2}}(e_k^* - i(Je_k)^*)$ for $k = 1, 2$. We may assume without loss of generality

$$a = \bar{\varepsilon}_1 \wedge \bar{\varepsilon}_2 \ \in \bigwedge\nolimits^{0,2}(X, \mathbb{C}).$$

Note that we have $\omega_x = e_1^* \wedge (Je_1)^* + e_2^* \wedge (Je_2)^*$. Then a direct calculation shows that $\omega_x \cdot a = 2ia$, where '$\cdot$' denotes the Clifford multiplication within the Clifford algebra with respect to the metric h.

On the other hand, we have $S(\omega) = \frac{4}{|\psi|^2}\omega$: First of all, since $S(\omega)$ is a parallel 2-form, we have that $S(\omega)\cdot\psi$ is a parallel field of $S^+_{\mathbb{C}}(\tilde{P}_g)$. Therefore, $S(\omega)\cdot\psi = z\psi$ for some complex number z by the condition that the vector space of parallel positive spin fields is of dimension 1. Since $S(\omega)$ is a real-valued self-dual 2-form, it should act as a skew adjoint traceless endomorphism. Thus z must be a purely imaginary number, say, $-ri$ for some real number r. Note that both $\omega, S(\omega)$ act as traceless skew adjoint endomorphisms and that $\omega \cdot \psi = -i\frac{|\psi|^2}{2}\psi$. Thus we conclude that $S(\omega) = \frac{2r}{|\psi|^2}\omega$. In fact, since S is positive definite, r must be a positive real number. Furthermore we note that $S(\omega) \cdot S(\omega) = S(\omega \cdot \omega) = S(2\Omega_h - 2) = 2\Omega_g - 2$, which acts as just the multiplication by -4 on $S^+_{\mathbb{C}}(\tilde{P}_g)$. Thus we have $r = 2$.

Note that $S(\omega_x) \cdot (S(a) \cdot \psi_x) = S(\omega \cdot a)\psi_x = 2iS(a) \cdot \psi_x$ and $S(\omega_x) \cdot \psi_x = \frac{4}{|\psi|^2}\omega \cdot \psi_x = -2i\psi_x$. Thus $S(a)\cdot\psi_x$, ψ_x are respectively in eigenspaces of $S(\omega)$ with eigenvalues $\pm 2i$. Thus it is enough to show that $S(a) \cdot \psi_x \neq 0$.

Let $\psi_x^c \in S^+_{\mathbb{C}}(\tilde{P}_g)_x$ be such that $\langle \psi_x, \psi_x^c \rangle = 0$, $\psi_x^c \neq 0$. Note that ψ_x^c is an eigenvector with eigenvalue $2i$ of $S(\omega_x) = \frac{4}{|\psi|^2}\omega$. Then since $S(\omega_x) \cdot (S(a) \cdot \psi_x^c) = S(\omega \cdot a)\psi_x^c = 2iS(a) \cdot \psi_x^c$ as well, we must have $S(a) \cdot \psi_x = z_1\psi_x^c$, $S(a) \cdot \psi_x^c = z_2\psi_x^c$ for some complex numbers z_1, z_2. On the other hand a direct calculation shows that $a \cdot a = 0$. Since $S(a) \in \mathrm{End}_{\mathbb{C}}(S^+_{\mathbb{C}}(\tilde{P}_g))_x$ is not the zero homomorphism, we must have $z_1 \neq 0$, $z_2 = 0$. Thus $S(a) \cdot \psi_x \neq 0$. This completes the proof of Case 1. □

Case 2: The vector space of parallel positive spin fields is of rank 2. Considering Lemma 4.1, it is enough to show that $S^+_{\mathbb{C}}(\tilde{P}_{\omega,J}) \equiv \bigwedge^{0,even}(X, \mathbb{C})$ is a trivial complex bundle.

Let ψ_1, ψ_2 be parallel positive spinors in $S^+_{\mathbb{C}}(\tilde{P}_g)$ which are orthonormal with respect to the Hermitian metric. Then the endomorphisms of $S^+_{\mathbb{C}}(\tilde{P}_g)$, $q(\sqrt{2}\psi_1)$, $q(\psi_1 + \psi_2)$ and $iq(\sqrt{2}\psi_1) \circ q(\psi_1 + \psi_2)$, where '$\circ$' means the composition, are represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

with respect to the frame field ψ_1, ψ_2. These are self-adjoint traceless endomorphisms of $S^+_{\mathbb{C}}(\tilde{P}_g)$ and therefore corresponds to self-dual purely imaginary two forms, say, ω_1, ω_2 and ω_3, which form a parallel frame field for $\bigwedge^2_{+,g}(X; i\mathbb{R})$.

On the other hand, there is the isomorphism $S^{-1} : (T^*X, g) \to (T^*X, h)$ between Riemannian vector bundles (see the 2nd paragraph of *Case 1* above), which induces an isomorphism $S^{-1} : \bigwedge^2_{+,g}(X; i\mathbb{R}) \to \bigwedge^2_{+,h}(X; i\mathbb{R})$. Parallel sections are preserved by this map. Thus, $S^{-1}(\omega_1)$, $S^{-1}(\omega_2)$ and $S^{-1}(\omega_3)$ form a parallel frame for $\bigwedge^2_{+,h}(X; i\mathbb{R})$.

Now note that the canonical spin^c structure of the Kähler structure (ω, J) admits a non-zero positive spinor ψ_0, that is, the non-zero constant function which is understood as a section of $\bigwedge^{0,even}(X, \mathbb{C}) \equiv S^+_{\mathbb{C}}(\tilde{P}_{\omega,J})$.

By letting $S^{-1}(\omega_1)$, $S^{-1}(\omega_2)$ and $S^{-1}(\omega_3)$ act on ψ_0, we have three parallel positive spin fields which are linearly independent over $\mathbb{R}$. It follows that the complex positive spin bundle $S^+_{\mathbb{C}}(\tilde{P}_{\omega,J})$, which is of rank 2, has a parallel frame field and hence it is a trivial complex vector bundle. $\square$

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