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A SHORT PROOF OF ELLENTUCK'S THEOREM

PIERRE MATET

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ABSTRACT. We combine standard arguments to give a shorter proof of Ellentuck's Theorem.

ω denotes the set of nonnegative integers. Given $A \subseteq \omega$, $[A]^\omega$ (respectively, $[A]^{<\omega}$) denotes the set of all infinite (respectively, finite) subsets of A.

Given $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, we let $\langle a, A\rangle$ denote the set of all $B \in [\omega]^\omega$ such that $a \subseteq B \subseteq a \cup \{n \in A \colon \forall m \in a\ n > m\}$.

$W \subseteq [\omega]^\omega$ is *completely Ramsey* if for all $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, there is a $B \in [A]^\omega$ such that either $\langle a, B\rangle \subseteq W$, or else $\langle a, B\rangle \cap W = \varnothing$.

Given a topological space X, an open set O is *dense* if $O \cap U \neq \varnothing$ for every open set $U \neq \varnothing$. Given $W \subseteq X$, W is *nowhere dense* if it is disjoint from some dense open set. W is *meager* if it is a countable union of nowhere dense sets. W has the *Baire property* if $(O - W) \cup (W - O)$ is meager for some open set O.

The Ellentuck topology on $[\omega]^\omega$ is defined by taking as basic open sets $\varnothing$ and all $\langle a, A\rangle$ for $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$.

Ellentuck's Theorem ([2]). *Given $W \subseteq [\omega]^\omega$, W is completely Ramsey if and only if W has the Baire property with respect to the Ellentuck topology.*

Ellentuck's proof of his theorem reuses part of the proof of an earlier, weaker result of Galvin and Prikry ([1]). The (shorter) proof given below is obtained by combining well-known arguments. For instance the proof of the following is reminiscent of that of Proposition 1.5 in [3].

Lemma 1. *Let $a \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$ and $W_i \subseteq [\omega]^\omega$ for $i \in \omega$ be such that for every $B \in [A]^\omega$, $\langle a, B\rangle \not\subseteq \bigcap_{i\in\omega} W_i$. Then there are $i \in \omega$, $c \in [\omega]^{<\omega}$ and $C \in [\omega]^\omega$ such that $\langle c, C\rangle \subseteq \langle a, A\rangle$, $|c| \leq |a| + i$ and $\langle d, D\rangle \not\subseteq W_i$ for all $d \in [\omega]^{<\omega}$ and $D \in [\omega]^\omega$ such that $\langle d, D\rangle \subseteq \langle c, C\rangle$.*

Proof. For $d \in [\omega]^{<\omega}$ and $i \in \omega$, let $S(d, i) = \{D \in [\omega]^\omega \colon \langle d, D\rangle \subseteq W_i\}$. Assume w.l.o.g. that $\min(A) > k$ for all $k \in a$. Let $A_{-1} = A$, $n_{-1} = 0$. Define A_j and n_j for $j \in \omega$ so that:

(i) $A_j \in [A_{j-1}]^\omega$;

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(ii) for each pair (d,i) where $a \subseteq d \subseteq a \cup \{n_k\colon k < j\}$ and $i \leq j$, either A_j is in $S(d,i)$, or else no subset of A_j is in $S(d,i)$;
(iii) $n_j \in A_j$ and $n_j > n_{j-1}$.

Now let $B = \{n_j\colon j \in \omega\}$. There is an $i \in \omega$ such that $\langle a, B\rangle \nsubseteq W_i$. Select $E \in \langle a, B\rangle - W_i$, and put $c = a \cup (E \cap \{n_j\colon j < i\})$ and $H = E - c$. For each $b \in [H]^{<\omega}$, set $T_b = \{m \in H\colon \langle c \cup b \cup \{m\}, H\rangle \subseteq W_i\}$ and $K_b = \{D \in [H]^\omega\colon \langle c \cup b, D\rangle \subseteq W_i\}$.

Notice that if $b \in [H]^{<\omega}$ is such that T_b is infinite, then $T_b \in K_b$, as $\langle c \cup b, T_b\rangle \subseteq \bigcup_{m \in T_b} \langle c \cup b \cup \{m\}, H\rangle$.

Moreover, we have the following. Suppose $b \in [H]^{<\omega}$ is such that $K_b \neq \varnothing$. Select $D \in K_b$. Let $x \in [\omega]^{<\omega}$ be such that $(c \cup b) - a = \{n_j\colon j \in x\}$, and let p be the least nonnegative integer $\geq i$ such that $j < p$ for every $j \in x$. As $\{m \in D\colon m \geq n_p\} \in P(A_p) \cap S(c \cup b, i)$, we have that $A_p \in S(c \cup b, i)$. Hence $\langle c \cup b, H\rangle \subseteq W_i$, and therefore $b \neq \varnothing$ and $\max(b) \in T_{b - \{\max(b)\}}$.

Now define m_j by induction for $j \in \omega$ so that:

(0) $m_0 \in H - T_\varnothing$;
(1) $m_j < m_{j+1}$;
(2) T_b is finite for all $b \subseteq \{m_r\colon r \leq j\}$;
(3) $m_{j+1} \in H - \bigcup_{b \subseteq \{m_r\colon r \leq j\}} T_b$.

Then set $C = \{m_j\colon j \in \omega\}$. Given $b \in [C]^{<\omega}$, we clearly have that $K_b = \varnothing$ and therefore for every $D \in [C]^\omega$, $\langle c \cup b, D\rangle \nsubseteq W_i$. □

We define $\mathcal{N} \subseteq P([\omega]^\omega)$ by letting $W \in \mathcal{N}$ if and only if for all $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, there are $b \in [\omega]^{<\omega}$ and $B \in [\omega]^\omega$ such that $\langle b, B\rangle \subseteq \langle a, A\rangle - W$.

The following is immediate from Lemma 1.

Lemma 2 ([3])**.** *$\mathcal{N}$ is closed under countable unions.*

We define $\mathcal{C} \subseteq P([\omega]^\omega)$ by letting $W \in \mathcal{C}$ if and only if for all $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, there are $b \in [\omega]^{<\omega}$ and $B \in [\omega]^\omega$ such that $\langle b, B\rangle \subseteq \langle a, A\rangle$ and either $\langle b, B\rangle \subseteq W$, or else $\langle b, B\rangle \cap W = \varnothing$.

Lemma 3 ([3], Proposition 4.12)**.** *Every member of $\mathcal{C}$ is completely Ramsey.*

Proof. Let $W \in \mathcal{C}$, $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ be such that for every $B \in [A]^\omega$, $\langle a, B\rangle \nsubseteq W$. Define W_i for $i \in \omega$ by letting $W_i = W$ in case $i = 0$, and $W_i = [\omega]^\omega$ otherwise. By Lemma 1, there is a $C \in [A]^\omega$ such that $\langle d, D\rangle \nsubseteq W_0$ for all $\langle d, D\rangle \subseteq \langle a, C\rangle$. Now set $\overline{W}_i = [\omega]^\omega - W$ for all $i \in \omega$. Since $W \in \mathcal{C}$, there is by Lemma 1 a $B \in [C]^\omega$ such that $\langle a, B\rangle \subseteq \bigcap_{i \in \omega} \overline{W}_i$. □

Proof of Ellentuck's Theorem. For each $Y \subseteq [\omega]^\omega$, let O_Y be the open set defined by letting

$$O_Y = \bigcup \{\langle a, A\rangle\colon a \in [\omega]^{<\omega}, A \in [\omega]^\omega \text{ and } \langle a, A\rangle \subseteq Y\}.$$

If $W \subseteq [\omega]^\omega$ is completely Ramsey, then $O_W \cup O_{[\omega]^\omega - W}$ is dense and therefore W has the Baire property, as $(W - O_W) \cap (O_W \cup O_{[\omega]^\omega - W}) = \varnothing$.

Conversely, if $W \subseteq [\omega]^\omega$ has the Baire property, then there is an open set O such that $(O - W) \cup (W - O)$ is meager. As every nowhere dense set clearly lies in $\mathcal{N}$, we have by Lemma 2 that $(O - W) \cup (W - O) \in \mathcal{N}$. Therefore $W \in \mathcal{C}$, as clearly $O \in \mathcal{C}$. Hence W is completely Ramsey by Lemma 3. □

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Université de Caen-CNRS, ESA 6081, Laboratoire SDAD, Campus II, 14032 Caen Cedex, France

E-mail address: `matet@math.unicaen.fr`