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DIVERGENT LAGUERRE SERIES

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ABSTRACT. We prove failure of a.e. convergence of partial sums of Laguerre expansions of L^p functions for $p > 4$. The idea which is used goes back to Stanton and Tomas. We follow Meaney's paper (1983), where divergence results were proved in the Jacobi polynomial case.

1. INTRODUCTION

Let $L_n^a(x)$ denote the nth Laguerre polynomial of order $a > -1$ [Sz, p. 101]. For any p, $1 \le p \le \infty$, and h in $L^p(x^a e^{-x}dx)$ (all Lebesgue spaces we consider live on $(0,\infty)$ and dx denotes the Lebesgue measure there) we associate to h its Laguerre polynomial series

$$\sum_0^\infty b_n L_n^a(x)$$

provided the coefficients

$$b_n = \frac{\Gamma(n+1)}{\Gamma(n+a+1)} \int_0^\infty h(x) L_n^a(x) x^a e^{-x} dx$$

exist (this is the case when $1 < p \le \infty$). It is known, [Sz, p. 261(6)], that for $a > -1/2$ the Laguerre polynomial series of the function x^μ, $-(a+1) < \mu \le -(2a+3)/4$, is divergent for every $x > 0$. In particular, $h(x) = x^{-(2a+3)/4}$, is an example of a function in $L^p(x^a e^{-x}dx)$, $1 \le p < \frac{4(a+1)}{2a+3}$, whose Laguerre polynomial series is divergent everywhere. In fact, more can be said. For any a and p, $a > -1$, $1 \le p < 2$, there is a function in $L^p(x^a e^{-x}dx)$ whose Laguerre polynomial series diverges everywhere. Such is the function e^{cx}, $\frac{1}{2} < c < \frac{1}{p}$ whose Laguerre polynomial series is even not summable in the Abel sense for every $x > 0$. This example was furnished by Pollard [Po].

On the other hand, it is also known that for any $a > -1$ the Laguerre polynomial series of any function f in $L^2(x^a e^{-x}dx)$, hence in $L^p(x^a e^{-x}dx)$, $2 < p \le \infty$, converges to f a.e. This follows from Muckenhoupt's equiconvergence theorem [M] and the celebrated Carleson's result on a.e. convergence of partial sums of Fourier series.

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2. The result

We will be concerned with the a.e. convergence problem of expansions with respect to the system of Laguerre functions

$$\mathcal{L}_n^a(x) = (\Gamma(n+1)/\Gamma(n+a+1))^{1/2} e^{-x/2} x^{a/2} L_n^a(x).$$

For any p, $1 \le p \le \infty$, and f in $L^p(dx)$ we associate to f its Laguerre series

$$\sum_0^\infty c_n \mathcal{L}_n^a(x) \tag{2.1}$$

provided the coefficients

$$c_n = \langle f, \mathcal{L}_n^a \rangle_{L^2(dx)} = \int_0^\infty f(x) \mathcal{L}_n^a(x) dx \tag{2.2}$$

exist (this is the case when $a \ge 0$ and $1 \le p \le \infty$ or $-1 < a < 0$ and $(1+a/2)^{-1} < p \le \infty$). Muckenhoupt's equiconvergence theorem also implies a.e. convergence of partial sums of the series (2.1) for every f in $L^p(dx)$ provided $a > -1/2$ and $p \in (4/3, 4)$ or $-1 < a \le -1/2$ and $(1+a/2)^{-1} < p < 4$; cf. [St, Proposition 2.1]. The negative result contained in [St, Corollary 2.5] suggested the absence of a.e. convergence of partial sums for functions in $L^p(dx)$ for $p \in (1, \infty)$ lying outside the interval $(4/3, 4)$ in the case $a > -1/2$. In fact, the function $f(x) = h(x) e^{-x/2} x^{a/2}$, $h(x) = x^{-(2a+3)/4}$, $a > -1/2$, is in $L^p(dx)$, $1 \le p < 4/3$, and its Laguerre series (2.1) diverges for every $x > 0$.

Our result almost clarifies the situation on the right of the interval $(4/3, 4)$.

Theorem 2.1. *Assume $a > -1$ and $4 < p \le \infty$. There is a function f in $L^p(dx)$ such that its Laguerre series (2.1) diverges for almost every $x > 0$.*

The proof of Theorem 2.1 is based on precise asymptotics of the Laguerre functions $\mathcal{L}_n^a(x)$.

Lemma 2.2 ([Ma, Lemma 1])**.** *Let $a \ge 0$ and $1 \le p \le \infty$ or $-1 < a < 0$ and $1 \le p < -2/a$. Then*

$$\|\mathcal{L}_n^a\|_p \sim \begin{cases} n^{1/p-1/2}, & 1 \le p < 4, \\ n^{-1/4}(\log n)^{1/4}, & p = 4, \\ n^{-1/p}, & 4 < p \le \infty. \end{cases} \tag{2.3}$$

Here $\tau_n \sim \sigma_n$ means that $C^{-1} \le \tau_n/\sigma_n \le C$ for a constant $C > 0$. An immediate consequence of Hölder's inequality and (2.3) is, that for a and p such that $a \ge 0$ and $1 \le p \le \infty$ or $-1 < a < 0$ and $(1+a/2)^{-1} < p \le \infty$, f in $L^p(dx)$ and c_n's given by (2.2) we have

$$c_n = o(n^{3/4}) \tag{2.4}$$

(we are slightly generous here since, in fact, there are better estimates: $c_n = o(n^{1/2})$ if $1 \le p < \infty$ and $c_n = O(n^{1/2})$ if $p = \infty$). Note, that if, in addition to these assumptions on a and p, we further restrict p to $p < 4$, then Hölder's inequality and (2.3) give $c_n = o(n^{1/4})$.

We include the proof of the next lemma for the sake of completeness.

Lemma 2.3 ([M, Lemma 5])**.** *Assume $a > -1$ and a sequence $\{c_n\}$ is such that $c_n = o(n^{3/4})$ and $\lim_{n\to\infty} c_n \mathcal{L}_n^a(x) = 0$ for all x in a subset $E \subset (0, \infty)$ of positive Lebesgue measure. Then $c_n = o(n^{1/4})$.*

Proof. By assumption, $\lim_{n\to\infty} c_n \mathcal{L}_n^a(x^2) = 0$ for all $x \in E^{1/2}$. Given x in $E^{1/2}$ we write Hilb's asymptotic formula, [Sz, 8.22.4], in the form

$$\mathcal{L}_n^a(x^2) = J_a(2n^{1/2}x) + O(n^{-3/4}).$$

The assumption made on c_n gives $\lim_{n\to\infty} c_n J_a(2n^{1/2}x) = 0$. For the Bessel function $J_a(t)$ we have

$$J_a(t) = \left(\frac{2}{\pi t}\right)^{1/2} \cos(t + A_a) + O(t^{-3/2}), \qquad t \geq 1,$$

$A_a = -\frac{a\pi}{2} - \frac{\pi}{4}$. Hence, given x in $E^{1/2}$ we write

$$J_a(2n^{1/2}x) = \pi^{-1/2}x^{-1/2}n^{-1/4}\cos(2n^{1/2}x + A_a) + O(n^{-3/4}),$$

and again, by using the hypothesis made on c_n, we obtain

$$\lim_{n\to\infty} c_n n^{-1/4}\cos(2n^{1/2}x + A_a) = 0.$$

Assuming, a contrario, that $|c_{n_k} n_k^{-1/4}| \geq \varepsilon$ for some $\varepsilon > 0$ and an increasing subsequence n_k we conclude, that $\lim_{n\to\infty} \cos(2n_k^{1/2}x + A_a) = 0$ for all x in $E^{1/2}$, a set of positive measure. This contradicts a variant of the Cantor-Lebesgue theorem; cf. [M, Lemma 4]. □

Proof of Theorem 2.1. Suppose, a contrario, that for any f in $L^p(dx)$, $4 < p \leq \infty$, the series (2.1) converges on a set E_f of positive measure. Then, by (2.4) and Lemma 2.3, $\langle f, \mathcal{L}_n^a\rangle_{L^2(dx)} = o(n^{1/4})$. We will show, however, that there is a function f in $L^p(dx)$ that does not possess this property. Consider the functionals

$$T_n f = n^{-1/4-\delta}\langle f, \mathcal{L}_n^a\rangle_{L^2(dx)}$$

on $L^p(dx)$ ($\delta > 0$ is to be small enough; for instance, $\delta = \frac{1}{2}(\frac{1}{4} - \frac{1}{p})$ will be sufficient). Assume, that for every f in $L^p(dx)$ we have $\sup_n |T_n f| < \infty$. This implies, by the Banach-Steinhaus theorem, that $\sup_n ||T_n|| < \infty$. By (2.3), this is equivalent to

$$\sup_n n^{\frac{1}{4} - \frac{1}{p} - \delta} < \infty$$

which is not possible for δ sufficiently small. Hence, there is an f in $L^p(dx)$ such that

$$\sup_n n^{-\delta} \cdot n^{-\frac{1}{4}} |\langle f, \mathcal{L}_n^a\rangle_{L^2(dx)}| = \infty$$

and this contradicts the conclusion of Lemma 2.3. □

Remark 2.4. We do not know what happens with a.e. convergence if $a > -1/2$ and $p = 4/3$ or $p = 4$ or, if $-1 < a \leq -1/2$ and $p = 4$.

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