

THE DIRICHLET-JORDAN TEST AND MULTIDIMENSIONAL EXTENSIONS

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ABSTRACT. If \mathcal{F} is a foliation of an open set $\Omega \subset \mathbb{R}^n$ by smooth $(n-1)$ -dimensional surfaces, we define a class of functions $\mathcal{B}(\Omega, \mathcal{F})$, supported in Ω , that are, roughly speaking, smooth along \mathcal{F} and of bounded variation transverse to \mathcal{F} . We investigate geometrical conditions on \mathcal{F} that imply results on pointwise Fourier inversion for these functions. We also note similar results for functions on spheres, on compact 2-dimensional manifolds, and on the 3-dimensional torus. These results are multidimensional analogues of the classical Dirichlet-Jordan test of pointwise convergence of Fourier series in one variable.

Suppose $f \in L^1(\mathbb{R}^n)$, with Fourier transform

$$(1) \quad \hat{f}(\xi) = (2\pi)^{-n/2} \int f(x)e^{-ix \cdot \xi} dx.$$

We set

$$(2) \quad S_R f(x) = (2\pi)^{-n/2} \int_{|\xi| \leq R} \hat{f}(\xi)e^{ix \cdot \xi} d\xi.$$

When $n = 1$, the Dirichlet-Jordan test for pointwise convergence of $S_R f(x)$ as $R \rightarrow \infty$ states that, if f has bounded variation, then for each $x \in \mathbb{R}$,

$$(3) \quad \lim_{R \rightarrow \infty} S_R f(x) = \frac{1}{2} \lim_{\varepsilon \searrow 0} [f(x + \varepsilon) + f(x - \varepsilon)].$$

This can be established as follows. Pick a function $h(t)$, equal to 0 for $t < 0$, 1 for $0 < t \leq 1$, smooth on $(0, \infty)$, and equal to 0 for $t \geq 2$. Set $h(0) = 1/2$. By Riemann's localization principle there is no loss of generality in assuming f has compact support. If f has bounded variation, its distributional derivative $f' = \mu$ is a (signed) measure, and we have

$$(4) \quad f(x) = \int h(x - y) d\mu(y) + g(x),$$

with $g \in C_0^\infty(\mathbb{R})$. If $f(x)$ is adjusted to equal the right side of (3) at each point of discontinuity, then (4) holds for all $x \in \mathbb{R}$. Then we have

$$(5) \quad S_R f(x) = \int S_R h(x - y) d\mu(y) + S_R g(x).$$

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Obviously $S_Rg(x) \rightarrow g(x)$ for all x . The Dirichlet-Jordan result can then be proven using the following two properties of S_Rh :

$$(6) \quad S_Rh(x) \rightarrow h(x), \quad \text{for every } x \in \mathbb{R}$$

(including $x = 0$), and, for some $C < \infty$, independent of x, R ,

$$(7) \quad |S_Rh(x)| \leq C.$$

To establish (6), one can appeal to the Dini test, or use localization and smoothness for $x \neq 0$, plus a symmetrization argument to cover the case $x = 0$. The bound (7) is a consequence of the analysis of the Gibbs phenomenon for S_Rh . From this, the Dirichlet-Jordan result can be deduced via Lebesgue’s dominated convergence theorem. Let us state an abstract version of this last segment of the argument.

Let (Y, \mathfrak{B}) be a set with sigma algebra, let μ be a finite signed measure on \mathfrak{B} , and let X be a set. Let $h_R : X \times Y \rightarrow \mathbb{C}$ be given, for each $R \in (0, \infty)$. Assume that $h_R(x, \cdot)$ is \mathfrak{B} -measurable, for each $x \in X, R \in (0, \infty)$, that

$$(8) \quad |h_R(x, y)| \leq C, \quad \forall x \in X, y \in Y, R \in (0, \infty),$$

and that

$$(9) \quad \lim_{R \rightarrow \infty} h_R(x, y) = h(x, y), \quad \forall x \in X, y \in Y.$$

Then

$$(10) \quad \lim_{R \rightarrow \infty} \int_Y h_R(x, y) d\mu(y) = \int_Y h(x, y) d\mu(y), \quad \forall x \in X.$$

As mentioned, this is simply a consequence of the dominated convergence theorem. The role played by X here is, in essence, trivial, except for the fact that it arises in nontrivial contexts.

Multidimensional analogues of functions for which (6)–(7) hold arise as follows. Let Σ be a smooth $(n - 1)$ -dimensional surface in \mathbb{R}^n . Let $\mathcal{C}_1(\Sigma)$ denote the set of caustic points of order ≥ 1 , in the terminology used in §10 of [PT]. (This follows Definition 5.2.3 of [Dui], in the case where Λ is the Lagrangian flow-out of the unit normal bundle of Σ .) Let \mathcal{O}_Σ be an open neighborhood of $\mathcal{C}_1(\Sigma)$. Let $h(x)$ be a piecewise smooth function, with compact support, with simple jump across Σ . For $x \in \Sigma$, set $h(x)$ equal to the mean value of its limits from each side. The fact that

$$(11) \quad S_Rh(x) \rightarrow h(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}_\Sigma,$$

follows from Proposition 26 in §10 of [PT] (the result for $x \in \Sigma$ holding by the analysis in §11). The fact that, for any compact $K \subset \mathbb{R}^n \setminus \mathcal{O}_\Sigma$,

$$(12) \quad |S_Rh(x)| \leq C_K, \quad \forall R \in (0, \infty), x \in K,$$

follows from the analysis of the Gibbs phenomenon in §11 of [PT] (cf. also [CV]). We note that $\mathcal{C}_1(\Sigma)$ is empty when $n = 2$. Also, when $n = 3$, $\mathcal{C}_1(\Sigma)$ is empty if Σ is real analytic and not part of a sphere (as noted by [K]).

Now suppose we have a foliation of an open set $\Omega \subset \mathbb{R}^n$ by such surfaces. More precisely, suppose we have smooth functions u_1, \dots, u_{n-1}, v on Ω , producing a diffeomorphism

$$(13) \quad (u_1, \dots, u_{n-1}, v) : \Omega \rightarrow Q \subset \mathbb{R}^n,$$

where Q is the open cube $(-\pi, \pi) \times \cdots \times (-\pi, \pi)$. We consider the family of surfaces $\Sigma_c = \{x \in \Omega : v(x) = c\}$. Assume that \mathcal{O} is an open neighborhood of the union of the sets $\mathcal{C}_1(\Sigma_c)$. Fix $\varphi \in C_0^\infty(\Omega)$. Let $h_t : \Omega \rightarrow \mathbb{R}$ be given by

$$h_t(x) = \begin{cases} 1 & \text{if } v(x) > t, \\ \frac{1}{2} & \text{if } v(x) = t, \\ 0 & \text{if } v(x) < t. \end{cases}$$

Let K be any compact set in $\mathbb{R}^n \setminus \mathcal{O}$. Then, for each $g \in C^\infty(\Omega)$, we have

$$(14) \quad |S_R(g\varphi h_t)(x)| \leq C_K(g), \quad \forall R \in (0, \infty), x \in K, t \in I = (-\pi, \pi).$$

Hence, if we set $\Phi(g)(R, x, t) = S_R(g\varphi h_t)(x)$, we have

$$(15) \quad \Phi : C^\infty(\Omega) \rightarrow L^\infty((0, \infty) \times K \times (-\pi, \pi)).$$

Now, if we compose this with the inclusion $\iota : L^\infty((0, \infty) \times K \times (-\pi, \pi)) \rightarrow L_{\text{loc}}^\infty((0, \infty) \times K \times (-\pi, \pi))$, it is easy to see that the map

$$\iota \circ \Phi : C^\infty(\Omega) \rightarrow L_{\text{loc}}^\infty((0, \infty) \times K \times (-\pi, \pi))$$

is continuous. It follows that the map Φ in (15) has closed graph. Hence, we can apply the closed graph theorem and deduce that

$$(16) \quad \sup_{x \in K, t \in I, R \in (0, \infty)} |S_R(g\varphi h_t)(x)| \leq C_K \|g\|_{H^\ell(\Omega)},$$

for some finite ℓ . This estimate can also be demonstrated by a recollection of what makes geometrical optics constructions work, up to any given finite order, and its implementation for the analysis of the Gibbs phenomenon in [PT]. (It would be of interest to study the optimal value of ℓ , but we will not pursue this here. We will stipulate that $\ell > n/2$.)

Now, if μ is a finite (signed) measure on I we can say that, for each $g \in H^\ell(\Omega)$,

$$(17) \quad f(x) = \int_I g(x)\varphi(x)h_t(x) d\mu(t) \Rightarrow S_R f(x) \rightarrow f(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}.$$

This class of synthesized functions is somewhat constrained, but it will serve as a starting point for an analysis of a much more natural class of functions, which we will now introduce.

Let $\Omega \subset \mathbb{R}^n$ be open and let $\mathcal{F} = \{\Sigma_c : c \in I\}$ be a foliation of Ω by smooth $(n - 1)$ -dimensional surfaces. Let $\mathcal{M}(\Omega)$ denote the space of finite (signed) Borel measures on Ω . We say

$$(18) \quad f \in \mathcal{B}(\Omega, \mathcal{F})$$

if f is a compactly supported element of $L^\infty(\Omega)$ with the property that

$$(19) \quad X_1 \cdots X_k f \in \mathcal{M}(\Omega),$$

for any k , and any smooth vector fields X_1, \dots, X_k on Ω , provided that *at most one* of them is not tangent to \mathcal{F} . One would have the same class of functions if one insisted the one exceptional vector field be X_1 (or that it be X_k). The following is our main result.

Theorem 1. *Given $f \in \mathcal{B}(\Omega, \mathcal{F})$, there exists a Borel measurable \tilde{f} , equal to f a.e., such that, as $R \rightarrow \infty$,*

$$(20) \quad S_R f(x) \rightarrow \tilde{f}(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O},$$

where \mathcal{O} is a neighborhood of the union of $\mathcal{C}_1(\Sigma_c)$, $c \in I$.

To begin the proof, we note that $\mathcal{B}(\Omega, \mathcal{F})$ is clearly a module over $C_0^\infty(\Omega)$. Hence, using a partition of unity, we can assume that Ω is as in (13), and $\Sigma_c = \{v = c\}$. Use the inverse of the diffeomorphism in (13) to pull f back to a compactly supported element $g \in L^\infty(Q)$, with the property on $\omega = \partial g / \partial x_n$ that

$$(21) \quad \Delta_T^M \omega \in \mathcal{M}(Q), \quad M = 0, 1, 2, \dots,$$

where

$$(22) \quad \Delta_T = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_{n-1}^2}.$$

For the first $n - 1$ factors of $(-\pi, \pi)$ in Q , throw in the endpoints and identify them, to regard ω as a compactly supported measure on $\mathbb{T}^{n-1} \times (-\pi, \pi)$. We have, for φ continuous on $[-\pi, \pi]$,

$$(23) \quad |\langle \varphi(t) e^{-ik \cdot x'}, \Delta_T^M \omega \rangle| \leq C_M \|\varphi\|_{L^\infty},$$

with $x' = (x_1, \dots, x_{n-1}) \in \mathbb{T}^{n-1}$, $k \in \mathbb{Z}^{n-1}$, so

$$(24) \quad |\langle \varphi(t) e^{-ik \cdot x'}, \omega \rangle| \leq C_M \langle k \rangle^{-M} \|\varphi\|_{L^\infty}.$$

Hence, we have measures μ_k on $(-\pi, \pi)$, supported on $[-a, a]$ for some $a < \pi$, such that

$$(25) \quad \|\mu_k\|_{\mathcal{M}(I)} \leq C_M \langle k \rangle^{-2M}, \quad \omega = \sum_k e^{ik \cdot x'} \mu_k,$$

where the norm denotes the total variation of μ_k . Hence

$$(26) \quad g(x', y) = \int_{-\pi}^y \sum_k e^{ik \cdot x'} d\mu_k(t),$$

so

$$(27) \quad \begin{aligned} f(x) &= \sum_k e^{ik \cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t) \\ &= \varphi(x) \sum_k e^{ik \cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t) \\ &= \sum_k \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t), \end{aligned}$$

where we choose $\varphi \in C_0^\infty(\Omega)$ equal to 1 on the support of f , and set $g_k(x) = e^{ik \cdot u(x)}$, with $u(x) = (u_1(x), \dots, u_{n-1}(x))$. The estimates done above imply convergence in sup-norm of the infinite series, to a function $\tilde{f}(x)$ equal a.e. to $f(x)$. The analysis done above also shows that, for

$$(28) \quad f_k(x) = \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t),$$

we have

$$(29) \quad S_R f_k(x) \rightarrow f_k(x), \quad x \in \mathbb{R}^n \setminus \mathcal{O},$$

and, for each compact $K \subset \mathbb{R}^n \setminus \mathcal{O}$,

$$(30) \quad \sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq C_K \|g_k\|_{H^\epsilon(\Omega)} \|\mu_k\|_{\mathcal{M}(I)}.$$

Now

$$(31) \quad \|g_k\|_{H^\ell(\Omega)} \leq C\langle k \rangle^\ell,$$

so, given N , we can produce $M = M(\ell, N)$ and apply (25) to obtain

$$(32) \quad \sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq C_{KN} \langle k \rangle^{-N}.$$

Thus, from (27), we have, for $x \in K$,

$$(33) \quad \lim_{R \rightarrow \infty} S_R f(x) = \sum_k f_k(x) = \tilde{f}(x),$$

and the theorem is proven.

It is clear what sort of representative of the class of $f \in \mathcal{B}(\Omega, F)$ the function $\tilde{f}(x)$ is. If $x_0 \in \Sigma_c \subset \Omega$, then $\tilde{f}(x_0)$ is the mean of the limit of $\tilde{f}(x)$ as $x \rightarrow x_0$ from within $\{v(x) > c\}$ and as $x \rightarrow x_0$ from within $\{v(x) < c\}$. In particular, for each $x_0 \in \Omega$,

$$(34) \quad \tilde{f}(x_0) = \lim_{r \searrow 0} \frac{1}{V_n r^n} \int_{|y| < r} f(x_0 + y) dy,$$

where V_n is the volume of the unit ball in \mathbb{R}^n .

There are other Riemannian manifolds M besides \mathbb{R}^n for which there are analogues of Theorem 1, with

$$(35) \quad S_R f(x) = \chi_R(\sqrt{-\Delta})f(x),$$

where Δ is the Laplace-Beltrami operator on M and $\chi_R(\lambda)$ is 1 for $|\lambda| < R$, 0 for $|\lambda| > R$, and $1/2$ for $|\lambda| = R$. One class of examples is the class of “strongly scattering manifolds,” in the terminology of [PT], §10. Using the “compactification” trick from §6 of [PT], we can extend Theorem 1 to the case where M is a sphere S^n , or other compact rank-one symmetric space. Using results of [BC], we can extend Theorem 1 to compact 2-dimensional manifolds (and then \mathcal{O} is empty). Using Theorem 5.4 of [T], we can extend Theorem 1 to the case $M = \mathbb{T}^3$, as long as all the leaves Σ_c of \mathcal{F} have nonzero Gauss curvature in Ω .

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