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THE DIRICHLET-JORDAN TEST AND MULTIDIMENSIONAL EXTENSIONS

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ABSTRACT. If $\mathcal{F}$ is a foliation of an open set $\Omega \subset \mathbb{R}^n$ by smooth $(n-1)$-dimensional surfaces, we define a class of functions $\mathcal{B}(\Omega, \mathcal{F})$, supported in Ω, that are, roughly speaking, smooth along $\mathcal{F}$ and of bounded variation transverse to $\mathcal{F}$. We investigate geometrical conditions on $\mathcal{F}$ that imply results on pointwise Fourier inversion for these functions. We also note similar results for functions on spheres, on compact 2-dimensional manifolds, and on the 3-dimensional torus. These results are multidimensional analogues of the classical Dirichlet-Jordan test of pointwise convergence of Fourier series in one variable.

Suppose $f \in L^1(\mathbb{R}^n)$, with Fourier transform

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix\cdot\xi}\, dx. \tag{1}$$

We set

$$S_R f(x) = (2\pi)^{-n/2} \int_{|\xi|\le R} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{2}$$

When $n = 1$, the Dirichlet-Jordan test for pointwise convergence of $S_R f(x)$ as $R \to \infty$ states that, if f has bounded variation, then for each $x \in \mathbb{R}$,

$$\lim_{R\to\infty} S_R f(x) = \frac{1}{2} \lim_{\varepsilon \searrow 0} [f(x+\varepsilon) + f(x-\varepsilon)]. \tag{3}$$

This can be established as follows. Pick a function $h(t)$, equal to 0 for $t < 0, 1$ for $0 < t \le 1$, smooth on $(0, \infty)$, and equal to 0 for $t \ge 2$. Set $h(0) = 1/2$. By Riemann's localization principle there is no loss of generality in assuming f has compact support. If f has bounded variation, its distributional derivative $f' = \mu$ is a (signed) measure, and we have

$$f(x) = \int h(x-y)\, d\mu(y) + g(x), \tag{4}$$

with $g \in C_0^\infty(\mathbb{R})$. If $f(x)$ is adjusted to equal the right side of (3) at each point of discontinuity, then (4) holds for all $x \in \mathbb{R}$. Then we have

$$S_R f(x) = \int S_R h(x-y)\, d\mu(y) + S_R g(x). \tag{5}$$

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Obviously $S_R g(x) \to g(x)$ for all x. The Dirichlet-Jordan result can then be proven using the following two properties of $S_R h$:

$$S_R h(x) \to h(x), \quad \text{for every } x \in \mathbb{R} \tag{6}$$

(including $x = 0$), and, for some $C < \infty$, independent of x, R,

$$|S_R h(x)| \le C. \tag{7}$$

To establish (6), one can appeal to the Dini test, or use localization and smoothness for $x \neq 0$, plus a symmetrization argument to cover the case $x = 0$. The bound (7) is a consequence of the analysis of the Gibbs phenomenon for $S_R h$. From this, the Dirichlet-Jordan result can be deduced via Lebesgue's dominated convergence theorem. Let us state an abstract version of this last segment of the argument.

Let $(Y, \mathfrak{B})$ be a set with sigma algebra, let μ be a finite signed measure on $\mathfrak{B}$, and let X be a set. Let $h_R : X \times Y \to \mathbb{C}$ be given, for each $R \in (0, \infty)$. Assume that $h_R(x, \cdot)$ is $\mathfrak{B}$-measurable, for each $x \in X, R \in (0, \infty)$, that

$$|h_R(x, y)| \le C, \quad \forall x \in X, y \in Y, R \in (0, \infty), \tag{8}$$

and that

$$\lim_{R \to \infty} h_R(x, y) = h(x, y), \quad \forall x \in X, y \in Y. \tag{9}$$

Then

$$\lim_{R \to \infty} \int_Y h_R(x, y)\, d\mu(y) = \int_Y h(x, y)\, d\mu(y), \quad \forall x \in X. \tag{10}$$

As mentioned, this is simply a consequence of the dominated convergence theorem. The role played by X here is, in essence, trivial, except for the fact that it arises in nontrivial contexts.

Multidimensional analogues of functions for which (6)–(7) hold arise as follows. Let Σ be a smooth $(n-1)$-dimensional surface in $\mathbb{R}^n$. Let $\mathcal{C}_1(\Sigma)$ denote the set of caustic points of order ≥ 1, in the terminology used in §10 of [PT]. (This follows Definition 5.2.3 of [Dui], in the case where Λ is the Lagrangian flow-out of the unit normal bundle of Σ.) Let $\mathcal{O}_\Sigma$ be an open neighborhood of $\mathcal{C}_1(\Sigma)$. Let $h(x)$ be a piecewise smooth function, with compact support, with simple jump across Σ. For $x \in \Sigma$, set $h(x)$ equal to the mean value of its limits from each side. The fact that

$$S_R h(x) \to h(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}_\Sigma, \tag{11}$$

follows from Proposition 26 in §10 of [PT] (the result for $x \in \Sigma$ holding by the analysis in §11). The fact that, for any compact $K \subset \mathbb{R}^n \setminus \mathcal{O}_\Sigma$,

$$|S_R h(x)| \le C_K, \quad \forall R \in (0, \infty), x \in K, \tag{12}$$

follows from the analysis of the Gibbs phenomenon in §11 of [PT] (cf. also [CV]). We note that $\mathcal{C}_1(\Sigma)$ is empty when $n = 2$. Also, when $n = 3$, $\mathcal{C}_1(\Sigma)$ is empty if Σ is real analytic and not part of a sphere (as noted by [K]).

Now suppose we have a foliation of an open set $\Omega \subset \mathbb{R}^n$ by such surfaces. More precisely, suppose we have smooth functions $u_1, \dots, u_{n-1}, v$ on Ω, producing a diffeomorphism

$$(u_1, \dots, u_{n-1}, v) : \Omega \to Q \subset \mathbb{R}^n, \tag{13}$$

where Q is the open cube $(-\pi,\pi)\times\cdots\times(-\pi,\pi)$. We consider the family of surfaces $\Sigma_c = \{x \in \Omega : v(x) = c\}$. Assume that $\mathcal{O}$ is an open neighborhood of the union of the sets $\mathcal{C}_1(\Sigma_c)$. Fix $\varphi \in C_0^\infty(\Omega)$. Let $h_t : \Omega \to \mathbb{R}$ be given by

$$h_t(x) = \begin{cases} 1 & \text{if } v(x) > t, \\ \frac{1}{2} & \text{if } v(x) = t, \\ 0 & \text{if } v(x) < t. \end{cases}$$

Let K be any compact set in $\mathbb{R}^n \setminus \mathcal{O}$. Then, for each $g \in C^\infty(\Omega)$, we have

$$|S_R(g\varphi h_t)(x)| \le C_K(g), \quad \forall R \in (0,\infty), x \in K, t \in I = (-\pi,\pi). \tag{14}$$

Hence, if we set $\Phi(g)(R,x,t) = S_R(g\varphi h_t)(x)$, we have

$$\Phi : C^\infty(\Omega) \to L^\infty((0,\infty)\times K \times (-\pi,\pi)). \tag{15}$$

Now, if we compose this with the inclusion $\iota : L^\infty((0,\infty)\times K\times(-\pi,\pi)) \to L^\infty_{\mathrm{loc}}((0,\infty)\times K\times(-\pi,\pi))$, it is easy to see that the map

$$\iota \circ \Phi : C^\infty(\Omega) \to L^\infty_{\mathrm{loc}}((0,\infty)\times K\times(-\pi,\pi))$$

is continuous. It follows that the map Φ in (15) has closed graph. Hence, we can apply the closed graph theorem and deduce that

$$\sup_{x\in K, t\in I, R\in(0,\infty)} |S_R(g\varphi h_t)(x)| \le C_K \|g\|_{H^\ell(\Omega)}, \tag{16}$$

for some finite ℓ. This estimate can also be demonstrated by a recollection of what makes geometrical optics constructions work, up to any given finite order, and its implementation for the analysis of the Gibbs phenomenon in [PT]. (It would be of interest to study the optimal value of ℓ, but we will not pursue this here. We will stipulate that $\ell > n/2$.)

Now, if μ is a finite (signed) measure on I we can say that, for each $g \in H^\ell(\Omega)$,

$$f(x) = \int_I g(x)\varphi(x)h_t(x)\,d\mu(t) \Rightarrow S_R f(x) \to f(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}. \tag{17}$$

This class of synthesized functions is somewhat constrained, but it will serve as a starting point for an analysis of a much more natural class of functions, which we will now introduce.

Let $\Omega \subset \mathbb{R}^n$ be open and let $\mathcal{F} = \{\Sigma_c : c \in I\}$ be a foliation of Ω by smooth $(n-1)$-dimensional surfaces. Let $\mathcal{M}(\Omega)$ denote the space of finite (signed) Borel measures on Ω. We say

$$f \in \mathcal{B}(\Omega,\mathcal{F}) \tag{18}$$

if f is a compactly supported element of $L^\infty(\Omega)$ with the property that

$$X_1 \cdots X_k f \in \mathcal{M}(\Omega), \tag{19}$$

for any k, and any smooth vector fields $X_1,\dots,X_k$ on Ω, provided that *at most one* of them is not tangent to $\mathcal{F}$. One would have the same class of functions if one insisted the one exceptional vector field be X_1 (or that it be X_k). The following is our main result.

Theorem 1. *Given $f \in \mathcal{B}(\Omega,\mathcal{F})$, there exists a Borel measurable $\tilde{f}$, equal to f a.e., such that, as $R \to \infty$,*

$$S_R f(x) \to \tilde{f}(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}, \tag{20}$$

where $\mathcal{O}$ is a neighborhood of the union of $\mathcal{C}_1(\Sigma_c)$, $c \in I$.

To begin the proof, we note that $\mathcal{B}(\Omega, \mathcal{F})$ is clearly a module over $C_0^\infty(\Omega)$. Hence, using a partition of unity, we can assume that Ω is as in (13), and $\Sigma_c = \{v = c\}$. Use the inverse of the diffeomorphism in (13) to pull f back to a compactly supported element $g \in L^\infty(Q)$, with the property on $\omega = \partial g/\partial x_n$ that

$$\Delta_T^M \omega \in \mathcal{M}(Q), \qquad M = 0, 1, 2, \dots, \tag{21}$$

where

$$\Delta_T = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}. \tag{22}$$

For the first $n-1$ factors of $(-\pi, \pi)$ in Q, throw in the endpoints and identify them, to regard ω as a compactly supported measure on $\mathbb{T}^{n-1} \times (-\pi, \pi)$. We have, for φ continuous on $[-\pi, \pi]$,

$$|\langle \varphi(t) e^{-ik\cdot x'}, \Delta_T^M \omega\rangle| \le C_M \|\varphi\|_{L^\infty}, \tag{23}$$

with $x' = (x_1, \dots, x_{n-1}) \in \mathbb{T}^{n-1}$, $k \in \mathbb{Z}^{n-1}$, so

$$|\langle \varphi(t) e^{-ik\cdot x'}, \omega\rangle| \le C_M \langle k\rangle^{-M} \|\varphi\|_{L^\infty}. \tag{24}$$

Hence, we have measures μ_k on $(-\pi, \pi)$, supported on $[-a, a]$ for some $a < \pi$, such that

$$\|\mu_k\|_{\mathcal{M}(I)} \le C_M \langle k\rangle^{-2M}, \qquad \omega = \sum_k e^{ik\cdot x'} \mu_k, \tag{25}$$

where the norm denotes the total variation of μ_k. Hence

$$g(x', y) = \int_{-\pi}^{y} \sum_k e^{ik\cdot x'}\, d\mu_k(t), \tag{26}$$

so

$$\begin{aligned} f(x) &= \sum_k e^{ik\cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t) \\ &= \varphi(x) \sum_k e^{ik\cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t) \\ &= \sum_k \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t), \end{aligned} \tag{27}$$

where we choose $\varphi \in C_0^\infty(\Omega)$ equal to 1 on the support of f, and set $g_k(x) = e^{ik\cdot u(x)}$, with $u(x) = (u_1(x), \dots, u_{n-1}(x))$. The estimates done above imply convergence in sup-norm of the infinite series, to a function $\tilde{f}(x)$ equal a.e. to $f(x)$. The analysis done above also shows that, for

$$f_k(x) = \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t), \tag{28}$$

we have

$$S_R f_k(x) \to f_k(x), \quad x \in \mathbb{R}^n \setminus \mathcal{O}, \tag{29}$$

and, for each compact $K \subset \mathbb{R}^n \setminus \mathcal{O}$,

$$\sup_{R\in(0,\infty), x\in K} |S_R f_k(x)| \le C_K \|g_k\|_{H^\ell(\Omega)} \|\mu_k\|_{\mathcal{M}(I)}. \tag{30}$$

Now

$$\|g_k\|_{H^\ell(\Omega)} \le C\langle k\rangle^\ell, \tag{31}$$

so, given N, we can produce $M = M(\ell, N)$ and apply (25) to obtain

$$\sup_{R\in(0,\infty),x\in K} |S_R f_k(x)| \le C_{KN}\langle k\rangle^{-N}. \tag{32}$$

Thus, from (27), we have, for $x \in K$,

$$\lim_{R\to\infty} S_R f(x) = \sum_k f_k(x) = \tilde{f}(x), \tag{33}$$

and the theorem is proven.

It is clear what sort of representative of the class of $f \in \mathcal{B}(\Omega, F)$ the function $\tilde{f}(x)$ is. If $x_0 \in \Sigma_c \subset \Omega$, then $\tilde{f}(x_0)$ is the mean of the limit of $\tilde{f}(x)$ as $x \to x_0$ from within $\{v(x) > c\}$ and as $x \to x_0$ from within $\{v(x) < c\}$. In particular, for each $x_0 \in \Omega$,

$$\tilde{f}(x_0) = \lim_{r\searrow 0} \frac{1}{V_n r^n} \int_{|y|<r} f(x_0 + y)\, dy, \tag{34}$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$.

There are other Riemannian manifolds M besides $\mathbb{R}^n$ for which there are analogues of Theorem 1, with

$$S_R f(x) = \chi_R(\sqrt{-\Delta}) f(x), \tag{35}$$

where Δ is the Laplace-Beltrami operator on M and $\chi_R(\lambda)$ is 1 for $|\lambda| < R$, 0 for $|\lambda| > R$, and 1/2 for $|\lambda| = R$. One class of examples is the class of "strongly scattering manifolds," in the terminology of [PT], §10. Using the "compactification" trick from §6 of [PT], we can extend Theorem 1 to the case where M is a sphere S^n, or other compact rank-one symmetric space. Using results of [BC], we can extend Theorem 1 to compact 2-dimensional manifolds (and then $\mathcal{O}$ is empty). Using Theorem 5.4 of [T], we can extend Theorem 1 to the case $M = \mathbb{T}^3$, as long as all the leaves Σ_c of $\mathcal{F}$ have nonzero Gauss curvature in Ω.

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