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FS-PROPERTY FOR C^*-ALGEBRAS

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ABSTRACT. A C^*-algebra A is said to *have the FS-property* if the set of all self-adjoint elements in A has a dense subset of elements with finite spectrum. We shall show that this property is not stable under taking the minimal C^*-tensor products even in case of separable nuclear C^*-algebras.

§1. INTRODUCTION

A C^*-algebra A is said to *have the FS-property* if the set of all self-adjoint elements in A $(= A_{sa})$ has a dense subset of elements with finite spectrum. In [3], Brown and Pedersen provided the non-commutative real rank for A $(= RR(A))$ and showed that the FS-property is equivalent to real rank zero, i.e. A_{sa} has a dense subset of invertible elements. $RR(A)$ is the least integer n such that $\{(a_0, a_1, \cdots, a_n) \in A_{sa}^{n+1} : \sum_{k=0}^{n} Aa_k = A\}$ is dense in A_{sa}^{n+1} which is the analogue of the dimension of a topological space $X(= \dim X)$. If A is non-unital, its real rank is defined by $RR(\tilde{A})$, where $\tilde{A}$ is the C^*-algebra obtained from A by adjoining a unit. From this definition it is obvious that $\dim X = RR(C(X))$ for a compact Hausdorff space X.

In the previous note [9], we showed that there are unital C^*-algebras A and C with $RR(A) = RR(C) = 0$ such that $RR(A \otimes C) \neq 0$. To specify let B be one of the Bunce-Deddens algebras and $\mathbf{B}(H)$ the C^*-algebra of bounded operators on a countably infinite dimensional Hilbert space H. Then by Blackadar and Kumjian [1], $RR(B) = 0$ and it is well-known that $RR(\mathbf{B}(H)) = 0$. In [9], we showed that $RR(B \otimes \mathbf{B}(H)) \neq 0$. This means that the FS-property for C^*-algebras is not stable under taking the minimal C^*-tensor products. But $\mathbf{B}(H)$ is not separable. In this note, we shall give two examples:

Example 1. There are separable unital C^*-algebras A and C with $RR(A) = RR(C) = 0$ such that A and C are nuclear and that $RR(A \otimes C) \neq 0$.

Example 2. There are separable unital C^*-algebras A and C with $RR(A) = RR(C) = 0$ such that A is not nuclear but exact, C is nuclear, and that $RR(A \otimes C) \neq 0$.

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Throughout this note, tensor products of C^*-algebras mean the minimal C^*-tensor products.

§2. Result

First, we shall give a well-known lemma.

Lemma 1 ([10]). *Let A be a C^*-algebra and $\mathbf{K}$ the C^*-algebra of compact operators on some Hilbert space. If $0 \to \mathbf{K} \to A \to A/\mathbf{K} \to 0$ is an exact sequence, then $RR(A) = RR(A/\mathbf{K})$.*

Proof. We shall give an elementary proof by Prof. Nagisa. Since it is clear that $RR(A/\mathbf{K}) \le RR(A)$, we show the reverse inequality. Let $n = RR(A/\mathbf{K})$ and π be the canonical homomorphism from A onto $A/\mathbf{K}$. Let ϵ be any positive real number and $x_0, x_1, \cdots, x_n$ any elements in A_{sa}. By the assumption there exist $(n+1)$ elements $y_0, y_1, \cdots, y_n$ in A_{sa} and a positive number k such that $\|x_i - y_i\| < \epsilon$ and $\pi(z) \ge k1$, where we put $z = \sum_{i=0}^n y_i^2$. If z is invertible, we can get the conclusion. Thus we need only consider the case where 0 is in the spectrum of z $(= sp(z))$. Let f be a continuous function on $sp(z)$ defined by

$$f(x) = \begin{cases} x, & x \ge \frac{k}{2}, \\ \dfrac{k}{2}, & x < \frac{k}{2}. \end{cases}$$

Since $\pi(f(z)) = f(\pi(z)) = \pi(z)$, $f(z) - z \in \mathbf{K}$. From the spectral decomposition of the positive compact operator $f(z) - z$, we know that there exists a positive real number η such that $sp(z) \subset \{0\} \cup [\eta, \|z\|]$. Let g be a continuous function on $sp(z)$ defined by

$$g(x) = \begin{cases} 1, & x \le \frac{\eta}{2}, \\ 0, & x > \frac{\eta}{2}. \end{cases}$$

Then, $p = g(z)$ is a kernel projection of z. We put $\tilde{y_0} = y_0 + \sqrt{\epsilon} p$, $\tilde{y_i} = y_i$ $(1 \le i \le n)$. We have, then, $\|x_i - \tilde{y_i}\| < \epsilon + \sqrt{\epsilon}$ and $\sum_{i=0}^n \tilde{y_i}^2 = \epsilon p + z$ is invertible. □

Let n be a positive integer with $n \ge 2$ and O_n the corresponding Cuntz algebra generated by n elements $s_1, s_2, \cdots, s_n$ such that $s_i^* s_i = 1$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^n s_i s_i^* = 1$. We consider the C^*-subalgebra E_n generated by $s_1, s_2, \cdots, s_n$ $(= C^*(s_1, s_2, \cdots, s_n))$ of $O_{n+1} = C^*(s_1, s_2, \cdots, s_n, s_{n+1})$. Let $e = s_{n+1} s_{n+1}^*$ and I_n be the closed two-sided ideal of E_n generated by e. Then by Cuntz [4] I_n is proper and isomorphic to $\mathbf{K}$ where $\mathbf{K}$ denotes the C^*-algebra of compact operators on a countably infinite dimensional Hilbert space. Hence there exists an exact sequence

$$0 \to \mathbf{K} \xrightarrow{j} E_n \xrightarrow{\pi} O_n \to 0.$$

Lemma 2. *With the above notation, $RR(E_n) = 0$.*

Proof. Since $RR(O_n) = 0$, from Lemma 1 we know that $RR(E_n) = 0$. □

Lemma 3. *For $i = 0, 1$, $K_i(O_n \otimes O_n) \cong \mathbf{Z}_{n-1}$.*

Proof. This is immediate from Cuntz [6, Theorem 1.5 and Remark 2.1]. □

Let A be a simple unital C^*-algebra. Then, there exists the following exact sequence [6]:

$$0 \to A \otimes \mathbf{K} \xrightarrow{\tilde{j}} A \otimes E_n \xrightarrow{\tilde{\pi}} A \otimes O_n \to 0,$$

where $\tilde{j} = id \otimes j$ and $\tilde{\pi} = id \otimes \pi$. By the above exact sequence, we obtain the following six-term exact sequence:

(1)
$$\begin{tikzcd}
K_0(A \otimes \mathbf{K}) \arrow[r, "\tilde{j}_*^{(0)}"] & K_0(A \otimes E_n) \arrow[r, "\tilde{\pi}_*^{(0)}"] & K_0(A \otimes O_n) \arrow[d, "\tilde{\partial}"] \\
K_1(A \otimes O_n) \arrow[u, "\tilde{\delta}"] & K_1(A \otimes E_n) \arrow[l, "\tilde{\pi}_*^{(1)}"'] & K_1(A \otimes \mathbf{K}). \arrow[l, "\tilde{j}_*^{(1)}"']
\end{tikzcd}$$

From exact sequence (1), Cuntz in [6] obtained the following six-term exact sequence:

(2)
$$\begin{tikzcd}
K_0(A) \arrow[r, "(n-1)^{(0)}"] & K_0(A) \arrow[r, "i_*^{(0)}"] & K_0(A \otimes O_n) \arrow[d, "\partial"] \\
K_1(A \otimes O_n) \arrow[u, "\delta"] & K_1(A) \arrow[l, "i_*^{(1)}"'] & K_1(A), \arrow[l, "(n-1)^{(1)}"']
\end{tikzcd}$$

where $(n-1)^{(j)}$ denotes multiplication by $n-1$ on $K_j(A)$ for $j = 0, 1$.

Theorem 4. *With the above notation, let A be a simple unital C^*-algebra with $RR(A) = 0$. Then $Ker(n-1)^{(1)} = 0$ if and only if $RR(A \otimes E_n) = 0$.*

Proof. We suppose that $Ker(n-1)^{(1)} = 0$. Then there exists a commutative diagram

$$\begin{tikzcd}
K_1(A \otimes \mathbf{K}) \arrow[r, "\tilde{j}_*^{(1)}"] \arrow[d] & K_1(A \otimes E_n) \arrow[d] \\
K_1(A) \arrow[r, "(n-1)^{(1)}"] & K_1(A),
\end{tikzcd}$$

where the vertical arrows are isomorphisms. Thus $Ker\tilde{j}_*^{(1)} = 0$. By exact sequence (1), we can see that $Im\tilde{\partial} = 0$. Hence $\tilde{\partial}$ is the 0-map. Thus $Ker\tilde{\partial} = K_0(A \otimes O_n)$. Since $Ker\tilde{\partial} = Im\tilde{\pi}_*^{(0)}$, $Im\tilde{\pi}_*^{(0)} = K_0(A \otimes O_n)$. Thus $\tilde{\pi}_*^{(0)}$ is surjective. Since $RR(A \otimes \mathbf{K}) = 0$ and $A \otimes O_n$ is a purely infinite simple C^*-algebra [11], [12], by Brown and Pedersen [3, Theorem 3.14 and Proposition 3.15], $RR(A \otimes E_n) = 0$.

Next we suppose that $Ker(n-1)^{(1)} \neq 0$. Then in the same way as in the above, $Ker\tilde{j}_*^{(1)} \neq 0$. By exact sequence (1), we can see that $Im\tilde{\partial} \neq 0$. Since $A \otimes O_n$ is a purely infinite simple C^*-algebra, there is a projection $p \in A \otimes O_n$ such that $\tilde{\partial}([p]) \neq 0$. We suppose that every projection in $A \otimes O_n$ can be lifted to a projection in $A \otimes E_n$. Then there is a projection $q \in A \otimes E_n$ such that $\tilde{\pi}(q) = p$. Hence $(\tilde{\partial} \circ \tilde{\pi}_*^{(0)})([q]) = 0$ by exact sequence (1). On the other hand $(\tilde{\partial} \circ \tilde{\pi}_*^{(0)})([q]) = \tilde{\partial}([p]) \neq 0$. This is a contradiction. Therefore, there is a projection in $A \otimes O_n$ which cannot be lifted to a projection in $A \otimes E_n$. Hence by Brown and Pedersen [3, Theorem 3.14], $RR(A \otimes E_n) \neq 0$. $\square$

Now we shall give Example 1.

Example 1. Let n be an integer with $n \geq 3$ and $A = O_n \otimes O_n$, $C = E_n$. Then A and C are separable nuclear C^*-algebras. By Lemma 2 $RR(C) = 0$. By Blackadar, Kumjian and Rørdam [2], Phillips [11], and Rørdam [12] we know that A is a purely infinite simple C^*-algebra. Hence $RR(A) = 0$ by Zhang [15]. Furthermore, by Lemma 3 $Ker(n-1)^{(1)} \neq 0$. Hence by Theorem 4 $RR(A \otimes C) \neq 0$.

Before we show Example 2, we shall give a lemma.

Let k be a positive integer with $k \geq 2$ and F_k the free group with k generators. Let $C^*_\gamma(F_k)$ be the reduced group C^*-algebra of F_k.

Lemma 5. *With the above notation,*

$$K_0(C^*_\gamma(F_k) \otimes O_n) \cong \mathbf{Z}_{n-1},$$
$$K_1(C^*_\gamma(F_k) \otimes O_n) \cong \mathbf{Z}^k_{n-1}.$$

Proof. These are immediate from Cuntz [6] and Schochet [13]. □

Example 2. Let n be an integer with $n \geq 3$ and $A = C^*_\gamma(F_k) \otimes O_n$, $C = E_n$. Then A and C are separable. Since by De Canniére and Haagerup [7] and Kirchberg [8] $C^*_\gamma(F_k)$ is not nuclear but exact, A is not nuclear but exact. From the same observation in Example 1 we know that A is a purely infinite simple C^*-algebra. Hence $RR(A) = 0$ by Zhang [15]. Furthermore, by Lemma 5 $Ker(n-1)^{(1)} \neq 0$. Hence by Theorem 4 $RR(A \otimes C) \neq 0$.

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