PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 4, Pages 1021–1029
S 0002-9939(00)05714-2
Article electronically published on October 11, 2000

SINGLE ELEMENTS OF FINITE CSL ALGEBRAS

W. E. LONGSTAFF AND ORESTE PANAIA

(Communicated by David R. Larson)

Abstract. An element s of an (abstract) algebra $\mathcal{A}$ is a *single element* of $\mathcal{A}$ if $asb = 0$ and $a, b \in \mathcal{A}$ imply that $as = 0$ or $sb = 0$. Let X be a real or complex reflexive Banach space, and let $\mathcal{B}$ be a finite atomic Boolean subspace lattice on X, with the property that the vector sum $K + L$ is closed, for every $K, L \in \mathcal{B}$. For any subspace lattice $\mathcal{D} \subseteq \mathcal{B}$ the single elements of Alg $\mathcal{D}$ are characterised in terms of a coordinatisation of $\mathcal{D}$ involving $\mathcal{B}$. (On separable complex Hilbert space the finite distributive subspace lattices $\mathcal{D}$ which arise in this way are precisely those which are similar to finite commutative subspace lattices. Every distributive subspace lattice on complex, finite-dimensional Hilbert space is of this type.) The result uses a characterisation of the single elements of matrix incidence algebras, recently obtained by the authors.

Introduction and preliminaries

Throughout, X will denote a non-zero real or complex reflexive Banach space, and H will denote a non-zero complex separable Hilbert space. Also, $\mathbb{F}$ will denote the real or complex field. By a *subspace* of X we mean a norm-closed linear manifold and by an *operator* on X we mean a bounded linear transformation acting on X. The set of operators on X is denoted by $\mathcal{B}(X)$. By a *subspace lattice on* X we mean a family $\mathcal{L}$ of subspaces of X satisfying (i) $(0), X \in \mathcal{L}$ and (ii) for every family $\{L_\gamma\}_\Gamma$ of elements of $\mathcal{L}$, $\cap_\Gamma L_\gamma \in \mathcal{L}, \vee_\Gamma L_\gamma \in \mathcal{L}$ (where '$\vee$' denotes 'closed linear span'). A subspace lattice on a Hilbert space is *commutative* if the (orthogonal) projections onto any two of its members commute. (As usual P_L will denote the (orthogonal) projection onto the subspace L.) The abbreviation 'CSL' will be used for 'commutative subspace lattice'. A subspace lattice on X is *Boolean* if it is complemented and distributive, and *atomic* if each of its elements is the (closed linear) span of the atoms that it contains. The abbreviation 'ABSL' will be used for 'atomic Boolean subspace lattice'. In any subspace lattice $\mathcal{L}$ the 'minus operation' is the self-map defined by

$$L_- = \bigvee\{M \in \mathcal{L} : L \not\subseteq M\},$$

for any element $L \in \mathcal{L}$ (so that $L \not\subseteq M \Rightarrow M \subseteq L_-$). The annihilator $\mathcal{S}^\perp$ of a subset $\mathcal{S} \subseteq X$ is, as usual, given by $\mathcal{S}^\perp = \{x^* \in X^* : x^*(x) = 0, \text{ for every } x \in \mathcal{S}\}$, where X^* denotes the topological dual of X. For any vectors $f \in X, e^* \in X^*$ the

Received by the editors June 20, 1999.
2000 *Mathematics Subject Classification.* Primary 47L35; Secondary 47C05.

operator $e^* \otimes f \in \mathcal{B}(X)$ is defined by $e^* \otimes f(x) = e^*(x)f$, for every $x \in X$. For any family $\mathcal{L}$ of subspaces of X, Alg $\mathcal{L}$ is the operator algebra given by

$$\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(X) : T(L) \subseteq L, \text{ for every } L \in \mathcal{L}\}.$$

The results from abstract lattice theory that we use are to be found in [2].

Let $n \in \mathbb{Z}^+$ and let $\preceq$ be a partial order on $\{1, 2, \ldots, n\}$ *consistent with the natural order* in the sense that $i \preceq j \Rightarrow i \leq j$. Call a subset $\mathcal{F}$ of $\{1, 2, \ldots, n\}$ $\preceq$-*hereditary* (or simply *hereditary* if no confusion can arise) if $i \preceq j \in \mathcal{F} \Rightarrow i \in \mathcal{F}$. The union and the intersection of hereditary subsets are again hereditary, so the set of hereditary subsets is a finite distributive lattice under the inclusion ordering. We will denote this lattice by $\mathcal{D}_n(\preceq)$. The non-zero join-irreducible elements of $\mathcal{D}_n(\preceq)$ are the subsets $\{i : 1 \leq i \leq n \text{ and } i \preceq j\}, 1 \leq j \leq n$. In particular, $\mathcal{D}_n(\preceq)$ has height n.

If $\{H_i : 1 \leq i \leq n\}$ is a pairwise orthogonal family of non-zero subspaces of the complex separable Hilbert space H satisfying $H = \bigoplus_{i=1}^n H_i$, the set of subspaces $\{\bigoplus_{i \in \mathcal{F}} H_i : \mathcal{F} \in \mathcal{D}_n(\preceq)\}$ is a finite CSL on the Hilbert space H, lattice-isomorphic to $\mathcal{D}_n(\preceq)$ by the map $\mathcal{F} \mapsto \bigoplus_{i \in \mathcal{F}} H_i$. All finite CSL's on separable Hilbert space arise in this way. Indeed, let $\mathcal{L}$ be a finite CSL on the Hilbert space H. Then, as is well known, $\mathcal{L}$ is distributive. Let $\{K_j : 1 \leq j \leq n\}$ be the set of non-zero join-irreducible elements of $\mathcal{L}$, enumerated so that $K_i \subseteq K_j \Rightarrow i \leq j$, and define the partial order $\preceq$ on $\{1, 2, \ldots, n\}$ by $i \preceq j$ if $K_i \subseteq K_j$. Then $\preceq$ is consistent with the natural order. For each $1 \leq i \leq n, K_i \cap (K_i)_-$ is the unique element of $\mathcal{L}$ covered by K_i. Put $H_i = K_i \ominus (K_i \cap (K_i)_-), 1 \leq i \leq n$. Then $H_i \perp H_j$ if $i \neq j$. For, if $i \neq j$ and $K_i \subseteq K_j$, then $K_i \subseteq K_j \cap (K_j)_-$ so $H_j \subseteq (K_j \cap (K_j)_-)^\perp \subseteq K_i^\perp \subseteq H_i^\perp$. On the other hand, if $K_i \not\subseteq K_j$, then $K_j \subseteq (K_i)_-$ so $H_i \cap H_j \subseteq K_i \cap K_j \subseteq K_i \cap (K_i)_-$. Thus $H_i \cap H_j \subseteq (K_i \cap (K_i)_-)^\perp \cap (K_i \cap (K_i)_-)$ so $H_i \cap H_j = (0)$. Since the projections P_{H_i} and P_{H_j} commute, it follows that $H_i \perp H_j$. Continuing, we also have $K_j = \bigoplus\{H_i : H_i \subseteq K_j\}$, for every $1 \leq j \leq n$. This can be proved by induction on the height of K_j in $\mathcal{L}$. Note that if the height of K_j is 1, then K_j is an atom of $\mathcal{L}$ and $K_j \cap (K_j)_- = (0)$, so $K_j = H_j$. For every $1 \leq j \leq n, K_j = H_j \oplus (K_j \cap (K_j)_-)$, where the height of $K_j \cap (K_j)_-$ is strictly less than the height of K_j. Since $K_j \cap (K_j)_-$ is the closed linear span of non-zero join-irreducible elements each of height strictly less than that of K_j, it follows by the induction assumption that $K_j = \bigoplus\{H_i : H_i \subseteq K_j\}$. Since $H = \bigvee_{j=1}^n K_j$, we have $H = \bigoplus_{j=1}^n H_j$. Also, since $H_i \subseteq K_j$ if and only if $K_i \subseteq K_j$ ($H_i \subseteq K_j$ and $K_i \not\subseteq K_j$ give $K_j \subseteq (K_i)_-$ and so $H_i \subseteq K_i \cap (K_i)_-$), $K_j = \bigoplus\{H_i : K_i \subseteq K_j\}$, for every $1 \leq j \leq n$. More generally, since $L = \bigvee\{K_j : K_j \subseteq L\}$, for every $L \in \mathcal{L}$, we have $L = \bigoplus\{H_j : K_j \subseteq L\}$. Clearly $\{j : K_j \subseteq L\}$ is $\preceq$-hereditary. Finally, note that for every $\preceq$-hereditary subset $\mathcal{F}$ of $\{1, 2, \ldots, n\}$ we have $\bigoplus\{H_j : j \in \mathcal{F}\} = \bigvee\{K_j : j \in \mathcal{F}\}$ (if $j \in \mathcal{F}$, then $K_j = \bigoplus\{H_i : K_i \subseteq K_j\}$ and $K_i \subseteq K_j$ implies that $i \in \mathcal{F}$). Thus $\bigoplus\{H_j : j \in \mathcal{F}\} \in \mathcal{L}$.

In this way we see that the 'building blocks' of finite CSL's are

(i) an orthogonal decomposition $H = \bigoplus_{i=1}^n H_i$ and
(ii) a partial order $\preceq$ on $\{1, 2, \ldots, n\}$, consistent with the natural order.

With notation as in the preceding paragraph, the non-zero join-irreducible elements of $\mathcal{L}$ are the elements $\bigoplus\{H_i : i \preceq j\}, 1 \leq j \leq n$. An operator $T \in \mathcal{B}(H)$ belongs to Alg $\mathcal{L}$ if and only if it leaves each of the subspaces $\bigoplus\{H_i : i \preceq j\}$

invariant; equivalently, the $n \times n$ (operator) matrix $(T_{i,j})$ of T relative to the decomposition $H = \bigoplus_{i=1}^{n} H_i$ satisfies $T_{i,j} = 0$ whenever $i \not\preceq j$. Thus we may write

$$\operatorname{Alg}\mathcal{L} = \{(T_{i,j}) \in \mathcal{B}(H) : T_{i,j} \in \mathcal{B}(H_j, H_i) \text{ and } T_{i,j} = 0 \text{ if } i \not\preceq j\}.$$

The requirement that $\preceq$ be consistent with the natural order can be dropped from the preceding three paragraphs. The reason that we have included it, and will continue to do so, is that, with notation as in the preceding paragraph, Alg $\mathcal{L}$ becomes an algebra of *upper-triangular* operator-entried matrices when $\preceq$ is consistent with the natural order.

An element s of an (abstract) algebra $\mathcal{A}$ is a *single element* of $\mathcal{A}$ if $asb = 0$ and $a, b \in \mathcal{A}$ imply that $as = 0$ or $sb = 0$. (This notion is useful, for example, in the representation theory of normed algebras [3].) If $\{u_1, u_2, \ldots, u_n\}$ is the usual orthonormal basis for $\mathbb{F}^n$ and $\preceq$ is a partial order on $\{1, 2, \ldots, n\}$ consistent with the natural order, then

$$\mathcal{L}_n(\preceq) = \{\bigvee\{u_i : i \in \mathcal{F}\} : \mathcal{F} \text{ is a } \preceq\text{-hereditary subset of } \{1, 2, \ldots, n\}\}$$

is a finite CSL on $\mathbb{F}^n$ and Alg $\mathcal{L}_n(\preceq)$ can be identified with the matrix incidence algebra

$$\mathcal{A}_n(\preceq) = \{(a_{i,j}) \in M_n(\mathbb{F}) : a_{i,j} = 0 \text{ if } i \not\preceq j\}.$$

The single elements of $\mathcal{A}_n(\preceq)$ have been described, in terms of the partial order $\preceq$, in [7] (see Theorem 1 below). We show below how this leads to a description of the single elements of Alg $\mathcal{L}$, for any finite CSL $\mathcal{L}$. Actually, our main result applies in a slightly wider context. Let $\mathcal{D}$ be a finite distributive subspace lattice on H with the property that the vector sum $K + L$ is closed, for every $K, L \in \mathcal{D}$. (This would be the case if, for example, dim $H < \infty$.) Let the set of non-zero join-irreducible elements of $\mathcal{D}$ be $\{K_j : 1 \le j \le n\}$, enumerated so that $K_i \subseteq K_j \Rightarrow i \le j$. Then, as in the commutative case, we can define $H_i = K_i \ominus (K_i \cap (K_i)_-)$ and it turns out that (see the proof of [4, Theorem 8]) $H = \sum_{i=1}^{n} H_i$ where the H_i's are independent (but not necessarily pairwise orthogonal) in the sense that $H_i \cap (\sum_{j \ne i} H_j) = (0)$. Moreover,

$$(*) \qquad \mathcal{D} = \{\sum_{j \in \mathcal{F}} H_j : \mathcal{F} \text{ is a } \preceq\text{-hereditary subset of } \{1, 2, \ldots, n\}\},$$

where again $\preceq$ is defined by $i \preceq j$ if $K_i \subseteq K_j$. Conversely, every such independent decomposition $H = \sum_{i=1}^{n} H_i$ of H and consistent partial order $\preceq$ on $\{1, 2, \ldots, n\}$ gives rise to a finite distributive subspace lattice $\mathcal{D}$ on H, defined by $(*)$ with the property that $K + L$ is closed, for every $K, L \in \mathcal{D}$. Of course, there is a natural matrix representation of Alg $\mathcal{D}$ for such a subspace lattice $\mathcal{D}$, as in the commutative case. We will not pursue this further as there is nothing essentially different here from the commutative case. Indeed, by [4, Corollary 7.1, Theorem 8] such a finite distributive subspace lattice is similar to a finite CSL.

Given a finite distributive subspace lattice $\mathcal{D}$ on a real or complex Banach space, with the property that the vector sum $K + L$ is closed, for every $K, L \in \mathcal{D}$, the expression $K_i \ominus (K_i \cap (K_i)_-)$ will usually make no sense (with $(K_i)_-$ calculated in $\mathcal{D}$, as earlier). So the subspace 'blocks' with which the representation of $\mathcal{D}$ may be built are not immediately apparent. However, note that if the underlying space is a Hilbert space, and $H_i = K_i \ominus (K_i \cap (K_i)_-), 1 \le i \le n$, then $\{H_i : 1 \le i \le n\}$ is the set of atoms of an ABSL $\mathcal{B}$ on H satisfying $\mathcal{D} \subseteq \mathcal{B}$. Moreover,

$H_i = K_i \cap (K_i \cap (K_i)_-)'$ where $'$ denotes Boolean complement (taken in $\mathcal{B}$). For, $H_i \cap (K_i \cap (K_i)_-) = (0)$ so $H_i \subseteq (K_i \cap (K_i)_-)'$. On the other hand, if $H_j \subseteq K_i \cap (K_i \cap (K_i)_-)'$, then $H_j \nsubseteq K_i \cap (K_i)_-$ and $H_j \subseteq K_i$, so $H_j \nsubseteq (K_i)_-$. Thus $K_j \nsubseteq (K_i)_-$ so $K_i \subseteq K_j$. Also, $K_j \subseteq K_i$ (since otherwise $H_j \subseteq K_i \subseteq (K_j)_-$) so $i = j$.

This suggests the class of finite distributive subspace lattices with which the present note is primarily concerned. Namely, those finite distributive subspace lattices $\mathcal{D}$ on X for which there exists a finite ABSL $\mathcal{B}$ on X with the property that the vector sum $K + L$ is closed, for every $K, L \in \mathcal{B}$, satisfying $\mathcal{D} \subseteq \mathcal{B}$. It is for such distributive subspace lattices that we describe the single elements of Alg $\mathcal{D}$ below (Theorem 2). If the underlying space is a separable complex Hilbert space H, such finite distributive subspace lattices are precisely those that are similar to a finite CSL [4, Corollary 7.1]. If dim $H < \infty$, every distributive subspace lattice on H is of this type [4, Corollaries 7.1, 8.1].

Main theorem

Throughout the remainder of this note we let $\mathcal{D}$ be a finite distributive subspace lattice on X and let $\mathcal{B}$ be a finite ABSL on X such that $\mathcal{D} \subseteq \mathcal{B}$ and such that the vector sum $K + L$ is closed, for every $K, L \in \mathcal{B}$. Let $\{L_1, L_2, \dots, L_m\}$ be the set of atoms of $\mathcal{B}$. Let $\{K_1, K_2, \dots, K_n\}$ be the set of non-zero join-irreducible elements of $\mathcal{D}$ enumerated so that $K_i \subseteq K_j$ implies that $i \leq j$. Let $\preceq$ be the partial order on $\{1, 2, \dots, n\}$ defined by $i \preceq j$ if $K_i \subseteq K_j$. For each $1 \leq i \leq n$ put $W_i = K_i \cap (K_i \cap (K_i)_-)'$, where $'$ denotes Boolean complementation in $\mathcal{B}$, and $(K_i)_-$ is calculated in $\mathcal{D}$. Then, $W_i \in \mathcal{B}, 1 \leq i \leq n$. Our main theorem is to be used in conjunction with the following description of the single elements of $\mathcal{A}_n(\preceq)$, which we include for the convenience of the reader.

Theorem 1 ([7])**.** *Let $\mathcal{A}_n(\preceq)$ be a matrix incidence algebra over a field K. The matrix $S \in \mathcal{A}_n(\preceq)$ is a single element of $\mathcal{A}_n(\preceq)$ if and only if*

(i) $r_i \neq 0$ and $c_j \neq 0 \quad (1 \leq i, j \leq n) \Rightarrow s_{i,j} \neq 0$,

(ii) $i \preceq j_1$ and $i \preceq j_2$, for some $1 \leq i \leq n \Rightarrow r_{j_1}$ and r_{j_2} are linearly dependent,

(iii) $i_1 \preceq j$ and $i_2 \preceq j$, for some $1 \leq j \leq n \Rightarrow c_{i_1}$ and c_{i_2} are linearly dependent.

Here r_i and c_j denote the i-th row and the j-th column of S, respectively.

Before proving our main theorem it is convenient to make some observations.

Observations. 1. From the fact that $K + L$ is closed, for every $K, L \in \mathcal{B}$ it readily follows that, for every finite set $\{M_k : 1 \leq k \leq p\}$ of elements of $\mathcal{B}$, $\sum_{k=1}^{p} M_k$ is a closed vector sum (equal to $\bigvee_{k=1}^{p} M_k$).

2. Each W_i is the vector sum of a unique subset of $\{L_k : 1 \leq k \leq m\}$. This subset is non-empty since $W_i \neq (0)$. (Observe that $W_i \vee (K_i \cap (K_i)_-) = K_i$ and $W_i \cap (K_i \cap (K_i)_-) = (0)$.)

3. If $i \neq j$, the subsets of $\{L_k : 1 \leq k \leq m\}$ comprising W_i and W_j are disjoint. This is because $W_i \cap W_j = (0)$. For, if $i \neq j$, then $K_i \neq K_j$ so, without loss of generality, $K_i \nsubseteq K_j$. Then $K_j \subseteq (K_i)_-$, so

$$W_i \cap W_j \subseteq (K_i \cap K_j) \cap (K_i \cap (K_i)_-)' \subseteq (K_i \cap (K_i)_-) \cap (K_i \cap (K_i)_-)' = (0).$$

4. For every $1 \le j \le n$, $K_j = \bigvee\{W_i : W_i \subseteq K_j\}$. This can be proved by induction on the height of K_j in $\mathcal{D}$. If the height of K_j is 1, then K_j is an atom of $\mathcal{D}$. Then $K_j \cap (K_j)_- = (0)$, so $K_j = W_j$. Since $W_i \cap W_j = (0)$ if $i \ne j$, $K_j = \bigvee\{W_i : W_i \subseteq K_j\}$. To employ the induction assumption observe that, for every $1 \le j \le n, K_j = W_j \vee (K_j \cap (K_j)_-)$ where the height of $K_j \cap (K_j)_-$ is strictly less than that of K_j.

5. Since $\bigvee_{i=1}^n K_i = X$ we have $\bigvee_{i=1}^n W_i = X$. It follows that the subsets of $\{L_1, L_2, \dots, L_m\}$ which comprise the W_i's form a partition of $\{L_1, L_2, \dots, L_m\}$. Thus (see [1, Example 2.7(1)(ii)]) $\{W_1, W_2, \dots, W_n\}$ is the set of atoms of a finite ABSL on $X, \mathcal{C}$ say, satisfying $\mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{B}$. It is also clear that the vector sum $K+L$ is closed, for every $K, L \in \mathcal{C}$, and that the Boolean complement appearing in the definition of W_i can be considered to be taken in $\mathcal{C}$. In short, we may suppose at the outset that $m = n$ (that is, that $\mathcal{C} = \mathcal{B}$). We will assume this in what follows.

6. For every $1 \le j \le n, K_j = \bigvee_{i \preceq j} W_i$. To prove this it is enough, in view of observation 4 above, to show that $W_i \subseteq K_j \Leftrightarrow K_i \subseteq K_j$. The backward implication is obvious. Suppose that $W_i \subseteq K_j$ and $K_i \not\subseteq K_j$. Then $K_j \subseteq (K_i)_-$, so

$$W_i \subseteq (K_i \cap (K_i)_-) \cap (K_i \cap (K_i)_-)' = (0),$$

which contradicts $W_i \ne (0)$. Thus $W_i \subseteq K_j \Rightarrow K_i \subseteq K_j$.

7. For every $\preceq$-hereditary subset $\mathcal{F}$ of $\{1, 2, \dots, n\}$ we have $\bigvee\{W_j : j \in \mathcal{F}\} = \bigvee\{K_j : j \in \mathcal{F}\}$. For, if $j \in \mathcal{F}$, then $K_j \subseteq \bigvee\{W_i : i \in \mathcal{F}\}$ using the preceding observation. Hence $\bigvee\{K_j : j \in \mathcal{F}\} \subseteq \bigvee\{W_i : i \in \mathcal{F}\}$, and the reverse inclusion is obvious. For every $L \in \mathcal{D}, L = \bigvee\{K_j : K_j \subseteq L\}$, where $\{j : K_j \subseteq L\}$ is hereditary. Thus $L = \bigvee\{W_j : K_j \subseteq L\}$. This shows that the map $\phi : \mathcal{D}_n(\preceq) \to \mathcal{D}$ defined by $\phi(\mathcal{F}) = \bigvee\{W_j : j \in \mathcal{F}\}$ is surjective. If $\mathcal{F}, \mathcal{G}$ are hereditary subsets of $\{1, 2, \dots, n\}$, then $\mathcal{F} \subseteq \mathcal{G} \Leftrightarrow \phi(\mathcal{F}) \subseteq \phi(\mathcal{G})$ (the reverse implication follows easily from the fact that $\{W_1, W_2, \dots, W_n\}$ is the set of atoms of an ABSL). Thus ϕ is a lattice-isomorphism. In particular, $(K_j)_- = \bigvee_{j \not\preceq i} W_i = (\bigvee_{j \preceq i} W_i)' = \bigcap_{j \preceq i} W_i'$.

8. By [1, Theorem 2.8], $\mathcal{B}^\perp = \{M^\perp : M \in \mathcal{B}\}$ is an ABSL on X^* with set of atoms $\{(W_1')^\perp, (W_2')^\perp, \dots, (W_n')^\perp\}$. The vector sum $K+L$ is closed, for every $K, L \in \mathcal{B}^\perp$. For, each $x \in X$ has a unique decomposition $x = \sum_{i=1}^n x_i$ with $x_i \in W_i, 1 \le i \le n$, and the map $P_i : X \to X$ defined by $P_i x = x_i$ is linear and closed, and so is bounded by the Closed Graph Theorem. Clearly $(W_j')^\perp \subseteq \ker P_i^*$ if $i \ne j$ and $\mathcal{R}(P_i^*) \subseteq (W_i')^\perp$, for $1 \le i \le n$ (where, for any operator $T, \mathcal{R}(T)$ denotes the range of T). In fact $\mathcal{R}(P_i^*) = (W_i')^\perp$, since $P_i^* e^* = e^*$, for every $e^* \in (W_i')^\perp$. Thus $P_i^*(\sum_{j=1}^n e_j^*) = e_i^*$ whenever $e_j^* \in (W_j')^\perp, 1 \le j \le n$. The continuity of the P_i^* now shows that $\bigvee_{j \in \mathcal{E}} (W_j')^\perp = \sum_{j \in \mathcal{E}} (W_j')^\perp$, for every subset $\mathcal{E}$ of $\{1, 2, \dots, n\}$ and it follows that $K+L$ is closed, for every $K, L \in \mathcal{B}^\perp$. Hence each $y^* \in X^*$ has a unique decomposition $y^* = \sum_{j=1}^n y_j^*$ with $y_j^* \in (W_j')^\perp, 1 \le j \le n$.

9. By [5, Lemmas 2.1, 3.1] it readily follows that an element $S \in \operatorname{Alg} \mathcal{D}$ is a single element if and only if whenever $g^* \in X^*, h \in X$ are vectors for which elements $J, K \in \mathcal{D}$ exist, satisfying $h \in J, g^* \in (K_-)^\perp, J_- \ne X, K \ne (0)$, and $g^*(Sh) = 0$, then $Sh = 0$ or $S^* g^* = 0$.

Theorem 2. *The operator $S \in \mathcal{B}(X)$ is a single element of Alg $\mathcal{D}$ if and only if there exist non-zero vectors $f_i \in W_i, e_j^* \in (W_j')^\perp, 1 \le i, j \le n$, and a single element $(s_{i,j})$ of $\mathcal{A}_n(\preceq)$ such that $S = \sum_{i=1}^n \sum_{j=1}^n s_{i,j}(e_j^* \otimes f_i)$. Moreover, S and $(s_{i,j})$ have the same rank.*

Proof. Let $(s_{i,j}) \in \mathcal{A}_n(\preceq)$ and let $f_i \in W_i, e_j^* \in (W_j')^\perp, 1 \le i,j \le n$, be arbitrary vectors. We show that the operator

$$S = \sum_{i=1}^n \sum_{j=1}^n s_{i,j}(e_j^* \otimes f_i)$$

belongs to Alg $\mathcal{D}$. For this it is enough to show that $SK_k \subseteq K_k$, for every $1 \le k \le n$. If $j \not\preceq k$, then $K_k = \bigvee_{l \preceq k} W_l \subseteq W_j'$, so $e_j^* \in (K_k)^\perp$. Thus, for every $x \in K_k$,

$$Sx = \sum_{i=1}^n (\sum_{j \preceq k} s_{i,j} e_j^*(x)) f_i.$$

But $s_{i,j} = 0$ if $i \not\preceq j$, so

$$Sx = \sum_{i \preceq k}(\sum_{j \preceq k} s_{i,j} e_j^*(x)) f_i \in \bigvee_{i \preceq k} W_i = K_k.$$

Thus $S \in$ Alg $\mathcal{D}$.

Now, additionally, let $(s_{i,j})$ be a single element of $\mathcal{A}_n(\preceq)$ and let $f_i, e_j^*, 1 \le i,j \le n$, be non-zero vectors. We show that S is a single element of Alg $\mathcal{D}$. By observation 9 above it is enough to show that if $h \in X, g^* \in X^*$ and $g^*(Sh) = 0$ and there exist elements $J, K \in \mathcal{D}$ such that $h \in J, g^* \in (K_-)^\perp$ and $J_- \ne X, K \ne (0)$, then $Sh = 0$ or $S^*g^* = 0$. Let $g^*(Sh) = 0$. Now $g^*(Sh) = \sum_{i,j=1}^n s_{i,j} e_j^*(h) g^*(f_i)$. Put $\tilde{g} = \sum_{i=1}^n \overline{g^*(f_i)} u_i$ and $\tilde{h} = \sum_{j=1}^n e_j^*(h) u_j$ (where $\{u_1, u_2, \dots, u_n\}$ is the usual orthonormal basis for $\mathbb{F}^n$). Then $\tilde{g}, \tilde{h} \in \mathbb{F}^n$ and $g^*(Sh) = (\tilde{S}\tilde{h}|\tilde{g})$, where $\tilde{S} = (s_{i,j})$ is regarded as an operator on $\mathbb{F}^n$ (and $(\cdot|\cdot)$ denotes the usual inner-product on $\mathbb{F}^n$). Now $K_- = \bigvee_{k \in \mathcal{E}} W_k$ for some hereditary subset $\mathcal{E}$ of $\{1, 2, \dots, n\}$. It follows that there exists $\tilde{K} \in \mathcal{L}_n(\preceq)$, with $\tilde{K} \ne (0)$, such that $(\tilde{K})_- = \bigvee_{k \in \mathcal{E}} u_k$ (the map $\bigvee_{k \in \mathcal{F}} W_k \mapsto \bigvee_{k \in \mathcal{F}} u_k$, with $\mathcal{F}$ hereditary, is a lattice-isomorphism of $\mathcal{D}$ onto $\mathcal{L}_n(\preceq)$). If $k \in \mathcal{E}$, then $W_k \subseteq K_-$, so $f_k \in K_-$ and $g^*(f_k) = 0 = (\tilde{g}|u_k)$. Hence $\tilde{g} \in ((\tilde{K})_-)^\perp$. Similarly, there exists $\tilde{J} \in \mathcal{L}_n(\preceq)$, with $(\tilde{J})_- \ne \mathbb{F}^n$, such that $\tilde{h} \in \tilde{J}$. So, again by observation 9 above, since $\tilde{S}$ is single and $(\tilde{S}\tilde{h}|\tilde{g}) = 0$, we have $\tilde{S}\tilde{h} = 0$ or $(\tilde{S})^*\tilde{g} = 0$. In the former case $Sh = 0$ and in the latter $S^*g^* = 0$. (Note that in this part of the proof we did not use the fact that all of the vectors $f_i \in W_i, e_j^* \in (W_j')^\perp, 1 \le i,j \le n$, are non-zero.)

Conversely, let S be a single element of Alg $\mathcal{D}$. For every $1 \le j \le n, W_j \subseteq K_j$ and, by [5, Lemma 3.2], SK_j is at most one-dimensional. Hence SW_j is at most one-dimensional. Let $SW_j = \langle g_j \rangle$ with $Sc_j = g_j, c_j \in W_j$ and let $g_j = \sum_{k=1}^n f_{k,j}$ with $f_{k,j} \in W_k, 1 \le k \le n$, be the unique decomposition of g_j. (Here, and in what follows, we let $\langle x \rangle$ denote the linear span of $\{x\}$, for every vector x.) Similarly, since $(W_i')^\perp \subseteq ((K_i)_-)^\perp$, for every $1 \le i \le n$ (see observation 7), and $S^*((K_i)_-)^\perp$ is at most one-dimensional [5, Lemma 3.3], $S^*(W_i')^\perp = \langle h_i^* \rangle$ with $S^* d_i^* = h_i^*, d_i^* \in (W_i')^\perp$ and $h_i^* = \sum_{l=1}^n e_{i,l}^*, e_{i,l}^* \in (W_l')^\perp, 1 \le l \le n$, the unique decomposition of h_i^* (see observation 8).

First we show that if $f_{i,j} \ne 0$ and $f_{i,k} \ne 0$, then $\langle f_{i,j} \rangle = \langle f_{i,k} \rangle$. Now $f_{i,j} \ne 0$ gives $g_j \notin W_i'$ so there exists $e^* \in (W_i')^\perp$ such that $e^*(g_j) \ne 0$. Then $S^*e^* \ne 0$ since $S^*e^*(c_j) = e^*(g_j)$. Thus $S^*(W_i')^\perp \ne (0)$ and $h_i^* \ne 0$. For every $x^* \in (W_i')^\perp$ there exists a unique scalar λ such that $S^*x^* = \lambda h_i^*$. Then

$$x^*(g_j) = x^*(f_{i,j}) = x^*(Sc_j) = \lambda h_i^*(c_j) = \lambda e_{i,j}^*(c_j).$$

Taking x^* equal to e^* shows that $e^*_{i,j}(c_j) \neq 0$. Similarly, $e^*_{i,k}(c_k) \neq 0$. If $f_{i,j}$ and $f_{i,k}$ were linearly independent there would exist $y^* \in (W_i^{'})^\perp$ such that $y^*(f_{i,j}) = 0$ and $y^*(f_{i,k}) \neq 0$ (there would exist a bounded linear functional y^* on W_i such that $y^*(f_{i,j}) = 0$ and $y^*(f_{i,k}) \neq 0$. Now extend y^* to a bounded linear functional on X by defining $y^*(z) = y^*(P_i z), z \in X$; here P_i is as in observation 8). If $S^* y^* = \mu h_i^*$, then $y^*(f_{i,j}) = \mu e^*_{i,j}(c_j) = 0$ and $y^*(f_{i,k}) = \mu e^*_{i,k}(c_k) \neq 0$, which is clearly a contradiction. Thus $\langle f_{i,j} \rangle = \langle f_{i,k} \rangle$.

A similar proof shows that $e^*_{i,j} \neq 0$ and $e^*_{k,j} \neq 0$ imply that $\langle e^*_{i,j} \rangle = \langle e^*_{k,j} \rangle$. (Note that $e^*_{i,j} \neq 0$ implies that $h_i^* \notin W_j^\perp$ so $SW_j \neq (0)$ and $g_j \neq 0$. For every $z \in W_j$ there exists γ such that $e^*_{i,j}(z) = \gamma d_i^*(f_{i,j})$ and $e^*_{k,j}(z) = \gamma d_k^*(f_{k,j})$ with $d_i^*(f_{i,j}) \neq 0$ and $d_k^*(f_{k,j}) \neq 0$. If $e^*_{i,j}$ and $e^*_{k,j}$ were linearly independent, there would exist $z \in W_j$ such that $e^*_{i,j}(z) = 0$ and $e^*_{k,j}(z) \neq 0$. Contradiction.)

By considering the arrays of vectors $(f_{i,j}), (e^*_{i,j})$ it now follows that there exist non-zero vectors $f_i, e_j^*, 1 \leq i,j \leq n$, with $f_i \in W_i, e_j^* \in (W_j^{'})^\perp$ and scalars $\lambda_{i,j}, \mu_{i,j}, 1 \leq i,j \leq n$, such that $f_{i,j} = \lambda_{i,j} f_i, e^*_{i,j} = \mu_{i,j} e_j^*$, for all i and j. Note that since $d_i^*(g_j) = h_i^*(c_j)$, for every i and j, $d_i^*(g_j) \neq 0$ implies that $g_j \neq 0$ and $h_i^* \neq 0$. The reverse implication also holds since S is single. (Note that $c_j \in K_j$ and $d_i^* \in ((K_i)_-)^\perp$. If $d_i^*(g_j) = 0$, then $d_i^*(Sc_j) = 0$, whence by singleness, and observation 9 above, $Sc_j = 0$ or $S^* d_i^* = 0$.)

Define the $n \times n$ scalar matrix $(s_{i,j})$ by

$$s_{i,j} = \begin{cases} \mu_{i,j}\lambda_{i,j}/d_i^*(g_j) & \text{if } g_j \neq 0 \text{ and } h_i^* \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We show that $S = \sum_{i=1}^n \sum_{j=1}^n s_{i,j}(e_j^* \otimes f_i)$. Let $1 \leq i,j \leq n$ and $x \in X$ be arbitrary. Then x has a unique decomposition $x = \sum_{k=1}^n x_k$, with $x_k \in W_k, 1 \leq k \leq n$, and there exist scalars $\alpha_k, 1 \leq k \leq n$, such that $Sx_k = \alpha_k g_k$. Now

$$h_i^*(x_j) = d_i^*(Sx_j) = \alpha_j d_i^*(g_j) = (\sum_{l=1}^n \mu_{i,l} e_l^*)(x_j) = \mu_{i,j} e_j^*(x_j) = \mu_{i,j} e_j^*(x);$$

that is, $\alpha_j d_i^*(g_j) = \mu_{i,j} e_j^*(x)$. Also,

$$\sum_{i=1}^n \sum_{j=1}^n s_{i,j}(e_j^* \otimes f_i)(x) = \sum_{i=1}^n \sum_{j=1}^n s_{i,j} e_j^*(x) f_i$$

and

$$Sx = \sum_{j=1}^n Sx_j = \sum_{j=1}^n \sum_{i=1}^n \alpha_j f_{i,j} = \sum_{i=1}^n \sum_{j=1}^n \alpha_j \lambda_{i,j} f_i.$$

However, $s_{i,j} e_j^*(x) = \alpha_j \lambda_{i,j}$. For, if $g_j \neq 0$ and $h_i^* \neq 0$, then

$$s_{i,j} e_j^*(x) = \frac{\mu_{i,j}\lambda_{i,j}}{d_i^*(g_j)} e_j^*(x) = \alpha_j \lambda_{i,j}.$$

On the other hand, if $g_j = 0$ or $h_i^* = 0$, then $\lambda_{i,j} = 0$ so once again $s_{i,j} e_j^*(x) = \alpha_j \lambda_{i,j}$. (Note that $g_j = \sum_{i=1}^n \lambda_{i,j} f_i$ where $\{f_1, f_2, \ldots, f_n\}$ is linearly independent. Also, if $h_i^* = 0$, then $S^*(W_i^{'})^\perp = (0)$ so $\mathcal{R}(S) \subseteq W_i^{'}$, $g_j \in W_i^{'}$ and $\lambda_{i,j} = 0$.) Thus $S = \sum_{i=1}^n \sum_{j=1}^n s_{i,j}(e_j^* \otimes f_i)$.

Suppose that $s_{i,j} \neq 0$. Then $\lambda_{i,j} \neq 0$ so $g_j \notin W_i^{'}$ and hence (since $g_j \in K_j$ because $S \in \text{Alg } \mathcal{D}$) $K_j \not\subseteq (K_i)_- = \bigcap_{i \preceq k} W_k^{'}$ (see observation 7) so $K_i \subseteq K_j$; that is, $i \preceq j$. This shows that $(s_{i,j}) \in \mathcal{A}_n(\preceq)$.

Next we show that $(s_{i,j})$ is a single element of $\mathcal{A}_n(\preceq)$. Now, for every $1 \leq i \leq n, f_i \notin W_i^{'}$ so there exists $E_i^* \in (W_i^{'})^{\perp}$ such that $E_i^*(f_i) = 1$. Also, since $e_j^* \notin W_j^{\perp}$, there exists $F_j \in W_j$ such that $e_j^*(F_j) = 1$, for every $1 \leq j \leq n$. Clearly $E_i^*(f_j) = e_j^*(F_i) = \delta_{i,j}$. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be elements of $\mathcal{A}_n(\preceq)$ with $A(s_{i,j})B = 0$. Put $\tilde{A} = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}(E_j^* \otimes f_i)$ and $\tilde{B} = \sum_{i=1}^n \sum_{j=1}^n b_{i,j}(e_j^* \otimes F_i)$. Then $\tilde{A}, \tilde{B} \in \text{Alg } \mathcal{D}$ (see the first paragraph of the proof of the theorem). We have

$$\tilde{A}S = \sum_{i,j=1}^n (\sum_{k=1}^n a_{i,k}s_{k,j})(e_j^* \otimes f_i), \quad S\tilde{B} = \sum_{i,j=1}^n (\sum_{l=1}^n s_{i,l}b_{l,j})(e_j^* \otimes f_i)$$

and

$$\tilde{A}S\tilde{B} = \sum_{i,j=1}^n (\sum_{k,l=1}^n a_{i,k}s_{k,l}b_{l,j})(e_j^* \otimes f_i).$$

Since $\tilde{A}S\tilde{B} = 0$ (note that $\sum_{k,l=1}^n a_{i,k}s_{k,l}b_{l,j}$ is the i,j-th entry of $A(s_{i,j})B$), and S is single, $\tilde{A}S = 0$ or $S\tilde{B} = 0$. Since $\{e_j^* \otimes f_i : 1 \leq i,j \leq n\}$ is a linearly independent family of rank one operators it follows that $A(s_{i,j}) = 0$ or $(s_{i,j})B = 0$. Hence $(s_{i,j})$ is a single element of $\mathcal{A}_n(\preceq)$.

Finally we show that S and $\tilde{S} = (s_{i,j})$ have the same rank. Note that, if $x \in X$ and $x = \sum_{k=1}^n x_k$ is the unique decomposition of x with $x_k \in W_k, 1 \leq k \leq n$, then $e_j^* \otimes f_i(x) = e_j^*(x_j)f_i$ and so $Sx = \sum_{i=1}^n \lambda_i f_i$ where $(\lambda_1, \lambda_2, \ldots, \lambda_n)^T = \tilde{S}(e_1^*(x_1), e_2^*(x_2), \ldots, e_n^*(x_n))^T$. Clearly then $\sum_{i=1}^n \mu_i f_i \in \mathcal{R}(S)$ implies that $(\mu_1, \mu_2, \ldots, \mu_n)^T \in \mathcal{R}(\tilde{S})$. The converse is also true. For, with $F_j \in W_j$ satisfying $e_j^*(F_j) = 1$, if $(\mu_1, \mu_2, \ldots, \mu_n)^T = \tilde{S}(\gamma_1, \gamma_2, \ldots, \gamma_n)^T$, then $S\sum_{k=1}^n \gamma_k F_k = \sum_{i=1}^n \mu_i f_i \in \mathcal{R}(S)$. The map $T : \mathcal{R}(\tilde{S}) \to \mathcal{R}(S)$ defined by $T(\mu_1, \mu_2, \ldots, \mu_n)^T = \sum_{i=1}^n \mu_i f_i$ is a linear bijection. It follows that S and $\tilde{S}$ have the same rank.

This completes the proof of the theorem. □

The preceding theorem shows that, for the type of subspace lattice $\mathcal{D}$ considered, the structure of the single elements of Alg $\mathcal{D}$ is completely determined by the poset of non-zero join-irreducible elements of $\mathcal{D}$, partially ordered by inclusion. This is consistent with [6, Theorem 3], which shows that the maximum rank that a single element can have is determined by the relationship between the set of maximal elements and the set of minimal elements of this poset. (It is not difficult to show that the non-zero join-irreducible elements occcuring in the unique irredundant representation of X are precisely the maximal ones.)

The preceding theorem can be interpreted matricially. For, by virtue of the decomposition $X = \sum_{i=1}^n W_i$, $\mathcal{B}(X)$ may be identified with the set of $n \times n$ operator-entried matrices $(T_{i,j})$, with $T_{i,j} \in \mathcal{B}(W_j, W_i), 1 \leq i,j \leq n$, and

$$\text{Alg}\,\mathcal{D} = \{(T_{i,j}) \in \mathcal{B}(X) : T_{i,j} \in \mathcal{B}(W_j, W_i) \text{ and } T_{i,j} = 0 \text{ if } i \not\preceq j\}.$$

If $f_i \in W_i$ and $e_j^* \in (W_j^{'})^{\perp}, 1 \leq i,j \leq n$, are non-zero vectors, then the (operator) matrix $(e_j^* \otimes f_i)$ belongs to $\mathcal{B}(X)$. The theorem says that the single elements of Alg $\mathcal{D}$ are obtained by taking the Hadamard (that is, entry-wise) product of such (operator) matrices with those (scalar) matrices $(s_{i,j})$ which are single elements of $\mathcal{A}_n(\preceq)$, the rank of the resulting operator being equal to the rank of the latter.

References

[1] S. Argyros, M. Lambrou and W. E. Longstaff, *Atomic Boolean subspace lattices and applications to the theory of bases*, Memoirs Amer. Math. Soc. 91 (1991). MR **92m:**46022

[2] T. Donnellan, *Lattice Theory*, Pergamon Press, New York, (1968). MR **38:**2059

[3] J. A. Erdos, S. Giotopoulos and M. S. Lambrou, *Rank one elements of Banach algebras*, Mathematika 24 (1977), 178-181. MR **57:**7176

[4] K. J. Harrison and W. E. Longstaff, *Automorphic images of commutative sub-space lattices*, Trans. Amer. Math. Soc. (1) 296 (1986), 217-228. MR **87g:**46040

[5] M. S. Lambrou, *On the rank of operators in reflexive algebras*, Lin. Alg. & Applic. 142 (1990), 211-235. MR **91k:**47104

[6] W. E. Longstaff and Oreste Panaia, *On the ranks of single elements of reflexive operator algebras*, Proc. Amer. Math. Soc. (10) 125 (1997), 2875-2882. MR **97m:**47061

[7] W. E. Longstaff and Oreste Panaia, *Single elements of matrix incidence algebras*, (manuscript).

Department of Mathematics, The University of Western Australia, Nedlands, Western Australia 6907, Australia

E-mail address: `longstaf@maths.uwa.edu.au`

Department of Mathematics, The University of Western Australia, Nedlands, Western Australia 6907, Australia

E-mail address: `oreste@maths.uwa.edu.au`