

A HELSON-LOWDENSLAGER-DE BRANGES THEOREM IN L^2

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ABSTRACT. This paper presents a generalization of the invariant subspace theorem of Helson and Lowdenslager along the lines of de Branges' generalization of Beurling's theorem.

1. INTRODUCTION

The Helson-Lowdenslager invariant subspace theorem is a significant generalization, as well as an extension to the Lebesgue space L^2 of the unit circle, of Beurling's famous invariant subspace theorem. Their theorem characterizes the class of all simply invariant subspaces of L^2 . See section 2 for details as well as [2], [3] and [4]. The theorem and its proof are important from several points of view. One of these is the fact that the proof is essentially free from the function theory of the disc. This enables the theorem to be stated and proved in very general situations. Equally important, the theorem enables a great deal of function theory to be derived in these very general situations. See [2], [3] and [4].

In recent times, L. de Branges has also generalized Beurling's theorem as well as its vector valued versions due to Lax and Halmos [5]. In the scalar context de Branges' theorem characterizes the class of all Hilbert spaces that are contractively contained in the Hardy space H^2 and on which multiplication by the coordinate function acts as an isometric operator. This theorem is the starting point of the theory of complementary spaces developed by de Branges [1], and furthered by Sarason [6], [7] and numerous other contributors.

This note is concerned with the presentation of a theorem which generalizes the Helson-Lowdenslager theorem along the lines of De Branges' generalization of Beurling's theorem. As an immediate corollary we deduce the Helson-Lowdenslager theorem. We also present some examples which show that a straightforward generalization of De Branges theorem to the L^2 case is impossible.

2. TERMINOLOGY AND A PRELIMINARY RESULT

The unit circle in the complex plane will be denoted by T and the open unit disk by D . The familiar Lebesgue and Hardy spaces on T will be denoted by L^p and H^p , respectively. It is well known that H^p can also be viewed as a space of analytic functions on D . Recall that L^2 , and consequently also H^2 , are Hilbert spaces under

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the usual inner product. A Hilbert space \mathcal{M} is said to be *boundedly contained* in a Hilbert space \mathcal{H} if \mathcal{M} is a vector subspace of \mathcal{H} (in the algebraic sense) and if the inclusion map is bounded, i.e. $\|x\|_{\mathcal{H}} \leq C \|x\|_{\mathcal{M}}$ for all x in \mathcal{M} and some constant C . When $C = 1$, we say that \mathcal{H} is *contractively contained* in \mathcal{M} . For further details on all the above we refer to [2], [3], [4] and [5].

The operator of multiplication by the coordinate function z will be denoted by S . It is easy to see that S is an isometry on L^2 and hence also on H^2 . A closed subspace \mathcal{M} of L^2 is said to be *simply invariant* under S if $S(\mathcal{M})$ is a subset of \mathcal{M} and $S(\mathcal{M}) \neq \mathcal{M}$, which we denote $S(\mathcal{M}) \subset \mathcal{M}$. For our purposes, a Hilbert space \mathcal{M} which is boundedly contained in L^2 will be said to be simply invariant if $S(\mathcal{M}) \subset \mathcal{M}$.

If T is an operator on some Hilbert space \mathcal{K} , then we let $\mathcal{R}(T)$ denote the range of T , which is a (not necessarily closed) subspace of \mathcal{K} . However, if we endow $\mathcal{R}(T)$ with the norm, $\|h\|_{\mathcal{R}(T)} = \inf\{\|k\|_{\mathcal{K}} : Tk = h\}$, then $\mathcal{R}(T)$ is a Hilbert space in this norm called the *range space of T* and it is boundedly contained in \mathcal{K} . When T is a contraction then the range space of T is contractively contained in \mathcal{K} .

We use M_ϕ to denote the operator of multiplication by ϕ on L^2 .

We record the following result for later use:

Theorem 2.1. *Let \mathcal{H} be boundedly contained in L^2 . Then S acts unitarily on \mathcal{H} if and only if there exists a function ϕ in L^∞ such that $\mathcal{H} = \mathcal{R}(M_\phi)$ isometrically, i.e., $\|h\|_{\mathcal{H}} = \|h\|_{\mathcal{R}(M_\phi)}$ for all h in \mathcal{H} . When \mathcal{H} is contractively contained in L^2 we have $\|\phi\|_\infty \leq 1$.*

Proof. Assume that S acts unitarily on \mathcal{H} and let $C : \mathcal{H} \rightarrow L^2$ denote the bounded containment. Then there exists a unitary operator $W : \mathcal{H} \rightarrow \mathcal{H}$ such that $SC = CW$. This means that $SCC^*S^* = CC^*$, because W is unitary. Hence, $CC^* = M_\psi$ for some ψ in L^∞ with $\psi \geq 0$ and so $\mathcal{H} = \mathcal{R}(C) = \mathcal{R}(M_\phi)$ where $\phi = \psi^{1/2}$ by Douglas' factorization theorem. Moreover, $\|h\|_{\mathcal{H}} = \|h\|_{\mathcal{R}(C)} = \|h\|_{\mathcal{R}(M_\phi)}$.

Conversely, if $\mathcal{H} = \mathcal{R}(M_\phi)$ isometrically for some ϕ in L^∞ , then $h \in \mathcal{H}$ implies $e^{-i\theta}h, e^{+i\theta}h$ are both in \mathcal{H} and hence S is one-to-one and onto. Finally,

$$\|h\|_{\mathcal{H}} = \inf\{\|g\|_{L^2} : \phi g = h\} = \inf\{\|g\|_{L^2} : \phi g = e^{+i\phi}h\} = \|e^{i\phi}h\|_{\mathcal{H}},$$

from which it follows that S is an isometry on \mathcal{H} , and hence unitary. \square

We have the following immediate consequence:

Corollary 2.2. *Let \mathcal{H} be boundedly contained in L^2 and let S act unitarily on \mathcal{H} . Then $\mathcal{H} \cap L^\infty \neq \{0\}$ if and only if $\mathcal{H} \neq \{0\}$.*

Proof. The result is obvious since, by Theorem 2.1, $\mathcal{H} = \mathcal{R}(M_\phi)$ and so $\{\phi\} \subseteq \mathcal{H} \cap L^\infty$. \square

3. THE FAILURE OF DE BRANGES THEOREM IN L^2

The theorem of de Branges (scalar version) [1] states: Let \mathcal{M} be a Hilbert space such that:

- (i) \mathcal{M} is contractively contained in H^2 ,
- (ii) $S(\mathcal{M}) \subseteq \mathcal{M}$ and S is an isometry on \mathcal{M} .

Then there is a b in the unit ball of H^∞ such that

$$\mathcal{M} = bH^2$$

and

$$\|bf\|_{\mathcal{M}} = \|f\|_{H^2}$$

for all f in H^2 , i.e., $\mathcal{M} = \mathcal{R}(M_b)$ isometrically.

This clearly generalizes the theorem of Beurling, which says that if \mathcal{M} is a closed subspace of H^2 invariant under the action of S , then there is an inner function b (i.e., $|b| = 1$ a.e. on T) such that $\mathcal{M} = bH^2$.

The Helson-Lowdenslager theorem is a generalization and extension of the above theorem of Beurling to the space L^2 and states: Let \mathcal{M} be a closed subspace of L^2 such that $S(\mathcal{M}) \subset \mathcal{M}$. Then there is a unimodular function ϕ ($|\phi| = 1$ a.e. on T) in L^2 such that $\mathcal{M} = \phi H^2$.

A straightforward generalization on L^2 of the Helson-Lowdenslager theorem along the lines of de Branges' generalization of Beurling's theorem is not possible. In other words, the following question has an answer in the negative: Let \mathcal{M} be a Hilbert space such that:

- (i) \mathcal{M} is a vector subspace of L^2 .
- (ii) $S(\mathcal{M}) \subset \mathcal{M}$ and S acts isometrically on \mathcal{M} .
- (iii) $\|f\|_{\mathcal{M}} \leq \|f\|_{L^2}$ for all f in \mathcal{M} .

Then does there exist a b in L^∞ not vanishing on any set of positive measure and such that $\mathcal{M} = bH^2$ with $\|bf\|_{\mathcal{M}} = \|f\|_{L^2}$ for all f in H^2 ?

The following examples illustrate why the situation can be as bad as possible:

(A) Let E_1, E_2, E_3 be mutually disjoint measurable subsets of T with $m(E_i) > 0$, $i = 1, 2, 3$. Then

$$\begin{aligned} \mathcal{X}_{E_i}H^2 \cap \mathcal{X}_{E_j}H^2 &= \{0\}, \\ \mathcal{X}_{E_i}H^2 \cap \mathcal{X}_{E_j}L^2 &= \{0\}, \quad i, j = 1, 2, 3, i \neq j. \end{aligned}$$

Hence, $\mathcal{M} = \mathcal{X}_{E_1}H^2 + \mathcal{X}_{E_2}H^2 + \mathcal{X}_{E_3}L^2$ is a vector subspace of L^2 , algebraically isomorphic to the direct sum.

Define

$$\|X_{E_1}g_1 + X_{E_2}g_2 + X_{E_3}g_3\|_{\mathcal{M}}^2 = \|g_1\|_2^2 + \|g_2\|_2^2 + \|g_3\|_2^2.$$

Clearly $\|f\|_{\mathcal{M}} \geq \|f\|_{L^2}$ for all f in \mathcal{M} . Hence \mathcal{M} is contractively embedded in L^2 and S acts isometrically on \mathcal{M} with $S(\mathcal{M}) \subset \mathcal{M}$.

However, in the \mathcal{M} norm, we find that $\mathcal{X}_{E_1}H^2 \perp \mathcal{X}_{E_2}H^2 \perp \mathcal{X}_{E_3}L^2$. If \mathcal{M} was equal to bH^2 for any b in L^∞ such that $\|bf\|_{\mathcal{M}} = \|f\|_{H^2}$, then S would be a shift of multiplicity one on \mathcal{M} , as is the case in the theorem of de Branges and in the Helson-Lowdenslager theorem. However, we find that S has a unitary part given by its restriction to $\mathcal{X}_{E_3}L^2$ and it has a shift part of multiplicity 2 given by its restriction to the subspace $\mathcal{X}_{E_1}H^2 + \mathcal{X}_{E_2}H^2$.

(B) Let E_3 be as above and let $\mathcal{M} = \mathcal{X}_{E_3}L^2 + H^2$ with

$$\|\mathcal{X}_{E_3}g + h\|_{\mathcal{M}}^2 = 2(\|g\|_2^2 + \|h\|_2^2).$$

Clearly \mathcal{M} is contractively contained in L^2 and S acts as an isometry on \mathcal{M} with $S(\mathcal{M}) \subset \mathcal{M}$. However S has a unitary part and a shift part of multiplicity one. Hence, arguing as above $\mathcal{M} \neq bH^2$ for any b in L^∞ .

4. THE HELSON-LOWDENSLAGER-DE BRANGES THEOREM

We are now in a position to state and prove the following:

Theorem 4.1. *Let $\mathcal{M} \neq \{0\}$ be a simply invariant Hilbert space contractively contained in L^2 and on which S acts isometrically. Further, suppose that there is a p , $2 \leq p \leq \infty$ and $\delta > 0$ such that*

$$(4.1) \quad \|f\|_{\mathcal{M}} \leq \delta \|f\|_p \quad (\text{for all } f \text{ in } \mathcal{M} \cap L^p).$$

Then there exists a unique b (up to a scalar multiple of modulus 1) in the unit ball of L^∞ , which is non-zero a.e. such that

$$(1) \quad \mathcal{M} = bH^2 \text{ with } \|bf\|_{\mathcal{M}} = \|f\|_{H^2} \text{ for all } f \text{ in } H^2,$$

$$(2) \quad b^{-1} \in L^s \text{ where } s = \begin{cases} \infty, & p = 2, \\ \frac{2p}{p-2}, & 2 < p < \infty \text{ and } \|b^{-1}\|_s \leq \delta, \\ 2, & p = \infty. \end{cases}$$

(Note: We do not assume in (4.1) that $\mathcal{M} \cap L^p \neq \{0\}$ for $p > 2$.)

Proof. By the Wold-Halmos decomposition, [5],

$$(4.2) \quad \mathcal{M} = \mathcal{H} \oplus \mathcal{N} \oplus S(\mathcal{N}) \oplus \cdots$$

where S is unitary on \mathcal{H} and $\mathcal{N} = \mathcal{M} \ominus S(\mathcal{M})$ and the summands are orthogonal in \mathcal{M} . Let $\mathcal{M}_1 = \mathcal{M} \ominus \mathcal{H}$. We shall first analyze \mathcal{M}_1 . We assume that $\mathcal{M}_1 \neq \{0\}$ and hence $\mathcal{N} \neq \{0\}$, a fact we shall justify later. Let b be an arbitrarily chosen non-zero element of unit norm in \mathcal{M} . Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be any fixed element of H^2 and put $f_n(z) = \sum_{k=0}^n \alpha_k z^k$. Then $\|f_n - f\|_{L^2} \rightarrow 0$ as $n \rightarrow \infty$. Now observe that by the decomposition (4.2)

$$(4.3) \quad \|bf_n\|_{\mathcal{M}} = \|f_n\|_{L^2}$$

and so $\{bf_n\}$ is a Cauchy sequence in \mathcal{M}_1 . Since \mathcal{M}_1 is a Hilbert space there is a g in \mathcal{M}_1 such that $\|bf_n - g\|_{\mathcal{M}} \rightarrow 0$ as $n \rightarrow \infty$. This implies by the contractivity condition that $\|bf_n - g\|_{L^2} \rightarrow 0$ and so $bf_n \rightarrow g$ almost everywhere (at least a subsequence does). Since we already know that $f_n \rightarrow f$ almost everywhere we conclude that $bf_n \rightarrow bf$ almost everywhere and so $bf = g$. Since g is in L^2 and f is an arbitrarily chosen element of H^2 we infer that b multiplies H^2 into \mathcal{M} and hence into L^2 and so b is in L^∞ . This forces us to deduce that $\mathcal{N} \subset L^\infty$ since b was any element of \mathcal{N} . We can also see from (4.2) that

$$\|bf\|_{\mathcal{M}} = \|f\|_{L^2}$$

for all f in H^2 and for any non-zero b in \mathcal{N} .

We now show that no element of \mathcal{N} can vanish on any set of positive measure unless it is zero. Choose any nonzero d in \mathcal{N} . Assume that d is identically zero on a set of positive measure, say E . Let

$$k_n = \begin{cases} n & \text{on } E, \\ 1 & \text{on } E^c. \end{cases}$$

Then $k_n \in L^\infty$ for each n . Put

$$h_n = \exp(k_n + i\tilde{k}_n)$$

where \tilde{k} denotes the harmonic conjugate of k , so that $h_n \in H^\infty$ for all n . Hence $dh_n \in \mathcal{M}$ for all n by the above argument, replacing b by d . Thus,

$$\|h_n\|_{L^2} = \|dh_n\|_{\mathcal{M}} \leq \delta \|dh_n\|_{L^p}.$$

By the definition of h_n , the right hand side of the above inequality is bounded by a fixed constant which is independent of n and the left hand side is unbounded. This contradiction forces us to conclude that d cannot vanish on a set of positive measure.

Next we show that \mathcal{N} is one dimensional. Suppose not. Then, by the condition $S(\mathcal{M}) \subset \mathcal{M}$ and the assumption that $\mathcal{N} \neq \{0\}$ we can fix a b and b_1 both of unit norm in \mathcal{N} such that $b_1 \perp b$ in \mathcal{M} . Then in view of (4.2) it is easy to see that $bH^2 \perp b_1H^2$ in \mathcal{M} . Since b is in L^∞ and in \mathcal{M}_1 we see that $\{b\} \subseteq \mathcal{M}_1 \cap L^p$. Further, $z(\mathcal{M}_1 \cap L^p) \subseteq \mathcal{M}_1 \cap L^p$. Let $A(\mathcal{M}_1 \cap L^p)$ denote the annihilator of $\mathcal{M}_1 \cap L^p$. This is a closed subspace of $L^{\frac{p}{p-1}}$ and $zA(\mathcal{M}_1 \cap L^p) \subseteq A(\mathcal{M}_1 \cap L^p)$. Choose any f in $A(\mathcal{M}_1 \cap L^p)$. Let $\{p_n(z)\}$ be a sequence of analytic polynomials that converges boundedly and pointwise to $\exp(-(|f| + i|\tilde{f}|))$. Clearly then $p_n(z)f$ converges to $\exp(-(|f| + i|\tilde{f}|))f$ in $L^{\frac{p}{p-1}}$ so that $[\exp(-(|f| + i|\tilde{f}|))]f$ is in $A(\mathcal{M}_1 \cap L^p) \cap L^\infty$. Hence, we can and do choose an f in $A(\mathcal{M}_1 \cap L^p)$, which is in L^∞ . Then

$$\int_T fbz^n d\theta = 0, \quad n = 0, 1, 2, \dots$$

and

$$\int_T fb_1z^n d\theta = 0, \quad n = 0, 1, 2, \dots$$

so that

$$fb = k$$

and

$$fb_1 = k_1$$

where, k, k_1 are in H^∞ . Since b and k do not vanish on any set of positive measure, we conclude that f does not vanish on any set of positive measure. Thus

$$b = \frac{k}{f}$$

and

$$b_1 = \frac{k_1}{f}.$$

Consider the function $\frac{kk_1}{f}$ which does not vanish on any set of positive measure. This is equal to bk_1 and it is in bH^2 . It is also equal to b_1k and hence it is in b_1H^2 . However, we have observed earlier that $bH^2 \cap b_1H^2 = \{0\}$. Thus, $\frac{kk_1}{f} = 0$ which contradicts the fact that this function does not vanish on any set of positive measure. Therefore, \mathcal{N} is one dimensional.

We now show that the space $\mathcal{H} = \{0\}$, which justifies our assumption that $\mathcal{M}_1 \neq \{0\}$. We first observe by Corollary 2.2 that if $\mathcal{H} \neq \{0\}$, then $\mathcal{H} \cap L^p$ has a non-zero element of L^∞ . Let this element be φ . Let f be a non-zero element in the annihilator of $\mathcal{M} \cap L^p$. Certainly f is in the annihilator of $\mathcal{M}_1 \cap L^p$ and so by reasoning employed above in dealing with the situation concerning k and k_1 we

can assume that f does not vanish on any set of positive measure. Now φz^n is in $\mathcal{H} \cap L^p$ for all $n = 0, \pm 1, \pm 2, \dots$. Therefore, we conclude that

$$\int_T f \varphi e^{in\theta} d\theta = 0$$

for all integers n . This means that $f\varphi = 0$ and hence $\varphi = 0$. This contradiction obviously stems from our assumption that $\mathcal{H} \neq \{0\}$.

We have thus far shown that in the decomposition (4.2) $\mathcal{N} = \{\alpha b : \alpha \in \mathbb{C}\}$ and $\mathcal{H} = \{0\}$. This gives us $\mathcal{M} = bH^2$, $\|bf\|_{\mathcal{M}} = \|f\|_{H^2}$ for all f in H^2 and b does not vanish on any set of positive measure.

To complete the proof of the theorem we need to establish (2). Suppose $2 < p < \infty$. Let, for each positive integer n , $E_n = \{e^{i\theta} : |b(e^{i\theta})| > \frac{1}{n}\}$ and let for a fixed real r ,

$$k_n = \begin{cases} r \log |b| & \text{on } E_n, \\ 0 & \text{on } E_n^c, \end{cases}$$

$$h_n = \exp(k_n + i\tilde{k}_n).$$

We then obtain the inequality

$$\begin{aligned} \left(\frac{1}{2\pi} \int_{E_n} |b|^{2r} d\theta\right)^{1/2} &\leq \|h_n\|_2 \\ &= \|bh_n\|_{\mathcal{M}} \leq \delta \left(\frac{1}{2\pi} \int_T |b|^p |h_n|^p d\theta\right)^{1/p}. \end{aligned}$$

Letting $n \rightarrow \infty$ in the above set of inequalities we obtain

$$\left(\frac{1}{2\pi} \int_T |b|^{2r} d\theta\right)^{\frac{1}{2}} \leq \delta \left(\frac{1}{2\pi} \int_T |b|^{p(1+r)} d\theta\right)^{1/p}.$$

Choosing $r = \frac{p}{2-p} < 0$ we have $2r = p(1+r) = -s$. Hence the above inequality yields

$$\left(\frac{1}{2\pi} \int_T |b^{-1}|^s\right)^{\frac{1}{2} - \frac{1}{p}} \leq \delta,$$

that is, $\|b^{-1}\|_s \leq \delta$.

When $p = 2$, we need only look at the inequality

$$\|h\|_{L^2} = \|bh\|_{\mathcal{M}} \leq \delta \|bh\|_{L^2}$$

to deduce that

$$\frac{1}{2\pi} \int_T (\delta^2 |b|^2 - 1) |h|^2 d\theta$$

for all trigonometric polynomials h , from which it easily follows that $\|b^{-1}\|_{\infty} \leq \delta$.

Finally, when $p = +\infty$, $r = -1$ we find, upon taking limits in the inequalities $\left(\frac{1}{2\pi} \int_{E_n} |b|^{2r} d\theta\right)^{1/2} \leq \|h_n\|_2 = \|bh_n\|_{\mathcal{M}} \leq \delta \|bh_n\|_{\infty}$ and by using the easily deduced fact that $\|bh_n\|_{\infty} \rightarrow \delta$, that

$$\|b^{-1}\|_2 \leq \delta.$$

□

5. SOME CONSEQUENCES OF THE THEOREM

In this section we discuss some immediate consequences of the theorem. The first is the theorem of Helson and Lowdenslager.

Corollary 5.1 (Helson-Lowdenslager, [2], [3]). *Let \mathcal{M} be a simply invariant subspace of L^2 . Then there exists a b in L^∞ , $|b| = 1$ a.e. such that*

$$\mathcal{M} = bH^2.$$

Proof. It is rather straightforward to check that \mathcal{M} satisfies all the conditions of our theorem with $p = 2$ and $\delta = 1$. Hence there is a b in the unit ball of L^∞ such that $\mathcal{M} = bH^2$ and $\|b^{-1}\|_\infty \leq 1$. Thus, $|b| = 1$ almost everywhere. \square

The second consequence asserts that there are no non-trivial simply invariant subspaces contractively contained in L^r for some $r > 2$ on whom S acts isometrically and which satisfy the condition (4.1) of our theorem.

Corollary 5.2. *Let \mathcal{M} be a simply invariant Hilbert space contractively contained in L^r for some $r > 2$. Suppose that S acts isometrically on \mathcal{M} and that \mathcal{M} satisfies the condition (4.1) of Theorem 4.1 for some $p > 2$. Then $\mathcal{M} = \{0\}$.*

Proof. Assume $\mathcal{M} \neq \{0\}$. By the facts that $L^r \subset L^2$ and the L^2 norm is dominated by the L^r norm (for functions in L^r) we conclude that \mathcal{M} satisfies the conditions of our theorem. Hence there is a b in L^∞ such that $\mathcal{M} = bH^2$. However, this is not possible as for each b in L^∞ there is an f in H^2 such that $bf \notin L^r$. This contradiction shows that $\mathcal{M} = \{0\}$. \square

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