

PRODUCTS OF MICHAEL SPACES AND COMPLETELY METRIZABLE SPACES

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ABSTRACT. For disjoint subsets A, C of $[0, 1]$ the Michael space $M(A, C) = A \cup C$ has the topology obtained by isolating the points in C and letting the points in A retain the neighborhoods inherited from $[0, 1]$. We study normality of the product of Michael spaces with complete metric spaces. There is a ZFC example of a Lindelöf Michael space $M(A, C)$, of minimal weight \aleph_1 , with $M(A, C) \times B(\aleph_0)$ Lindelöf but with $M(A, C) \times B(\aleph_1)$ not normal. ($B(\aleph_\alpha)$ denotes the countable product of a discrete space of cardinality \aleph_α .) If $M(A)$ denotes $M(A, [0, 1] \setminus A)$, the normality of $M(A) \times B(\aleph_0)$ implies the normality of $M(A) \times S$ for any complete metric space S (of arbitrary weight). However, the statement “ $M(A, C) \times B(\aleph_1)$ normal implies $M(A, C) \times B(\aleph_2)$ normal” is axiom sensitive.

1. INTRODUCTION

The *Michael space* $M(A, C)$, associated with a pair of disjoint sets A, C in the unit interval $I = [0, 1]$, is the set $A \cup C$ equipped with the topology consisting of all sets $(U \cap (A \cup C)) \cup C'$ where U is open in I and $C' \subseteq C$. That is, isolate the points in C and let the points in A retain the usual neighborhoods from the topology inherited from I . We shall write $M(A) = M(A, I \setminus A)$. The usual Michael Line would be homeomorphic to the space $M(\mathbb{Q} \cap (0, 1))$, where \mathbb{Q} is the set of rational numbers. E. Michael introduced this example in [M1] as a paracompact space whose product with the separable complete metric space of irrational numbers \mathbb{P} is not normal. Using CH, Michael ([M1], [M2]) also introduced a Lindelöf subspace X of the Michael Line having the property that $X \times \mathbb{P}$ is not normal. This prompted him to ask the question, still unanswered today, of whether there exists a ZFC example of a Lindelöf space whose product with \mathbb{P} is not normal. While our results do not address this question directly, it is related to the purpose of the example given in Section 2, where a ZFC example is given of a Lindelöf Michael space $M(A, C)$, of weight \aleph_1 , with $M(A, C) \times B(\aleph_0)$ Lindelöf but with $M(A, C) \times B(\aleph_1)$ not normal. (Notation for the Baire Space $B(\aleph_1)$ is explained in a later paragraph of the Introduction. The issue here is that it is a complete metric space of weight \aleph_1 .)

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The first ZFC example of a Lindelöf space and a completely metrizable space having a product which is nonnormal and of minimal weight \aleph_1 was given by Lawrence [L2]. The example in Section 2, with relatively simple structure, gives a much different approach to the construction. As pointed out by Lawrence, the significance of the minimal weight \aleph_1 is that it is not possible to construct a ZFC example of a Lindelöf space, of weight \aleph_1 , having a nonnormal product with \mathbb{P} ([L1]).

In Section 3 we show that for Michael spaces of type $M(A)$, the property of normality of $M(A) \times S$, for S a complete metric space (of arbitrary weight), is actually implied by the normality of $M(A) \times \mathbb{P}$. This is in contrast with the behavior of the main example $M(A, C)$ given in Section 2. In Section 4 we find that the expectation of normality of $M(A, C) \times B(\aleph_1)$ implying the normality of $M(A, C) \times B(\aleph_2)$ is axiom sensitive. That is, it is consistent that there is a Michael space $M(A, C)$ such that $M(A, C) \times B(\aleph_1)$ is normal but $M(A, C) \times B(\aleph_2)$ is not normal. It is also consistent (assuming a supercompact cardinal) that $M(A, C) \times B(\aleph_1)$ normal implies that $M(A, C) \times S$ is normal for every complete metric space S .

By the *Baire Space* $B(\aleph_\alpha)$ we mean the countable product $B(\aleph_\alpha) = D^{\mathbb{N}}$ where D is the discrete space of cardinality \aleph_α and \mathbb{N} is the set of positive integers. Recall that the space $B(\aleph_0) = \mathbb{N}^{\mathbb{N}}$ is homeomorphic to the space \mathbb{P} of irrational numbers. Any completely metrizable space S of weight \aleph_α is a closed image of $B(\aleph_\alpha)$. It follows that, for every X , the normality of $X \times B(\aleph_\alpha)$ implies the normality of $X \times S$.

Any other topological or set theoretic notation used but not defined in this note can be found in [E] or [Ku].

2. A MICHAEL SPACE $M(A, C)$ WITH $M(A, C) \times B(\aleph_0)$ LINDELÖF BUT $M(A, C) \times B(\aleph_1)$ NOT NORMAL

We begin with the following useful observation concerning Michael spaces.

Remark 2.1. The product $M(A, C) \times S$ of a Michael space and a metrizable space S is normal if and only if for each $F \subseteq C \times S$, closed in $M(A, C) \times S$, the pair F and $A \times S$ can be separated in $M(A, C) \times S$ by disjoint open sets.

Proof. Indeed, to see the nontrivial direction of this remark, let E, F be disjoint closed subsets of $M(A, C) \times S$. If $E' = E \cap (A \times S)$ and $F' = F \cap (A \times S)$, then E', F' are separated with respect to the completely normal metric topology on $(A \cup C) \times S$, so there exist disjoint metric open sets U, V , in $(A \cup C) \times S$, containing E', F' , respectively. Now, $T = (E \setminus U) \cup (F \setminus V)$ is closed in $M(A, C) \times S$ and we are assuming that there exist open W such that $T \subseteq W$ and $\overline{W} \cap (A \times S) = \emptyset$. Then $U' = U \setminus \overline{W}$, $V' = V \setminus \overline{W}$ are disjoint open sets in $M(A, C) \times S$ with $E' \subseteq U', F' \subseteq V'$ and

$$\overline{U'} \cap (F \setminus A \times S) = \emptyset, \overline{V'} \cap (E \setminus A \times S) = \emptyset.$$

Continue by covering the “vertical sections” of $E \cap (C \times S)$ and $F \cap (C \times S)$ in $C \times S$. For this, use normality of S to find, for every $z \in C$, disjoint open sets G_z, H_z in S such that

$$E \cap (\{z\} \times S) \subseteq \{z\} \times G_z \subseteq \{z\} \times S \setminus \overline{V'}$$

and

$$F \cap (\{z\} \times S) \subseteq \{z\} \times H_z \subseteq \{z\} \times S \setminus \overline{U'}.$$

Then, $U' \cup (\bigcup_{z \in C} \{z\} \times G_z)$ and $V' \cup (\bigcup_{z \in C} \{z\} \times H_z)$ are the desired disjoint open sets separating E, F in $M(A, C) \times S$. □

To describe a Michael space with the properties in the title let us represent the irrationals \mathbb{P} in the unit interval I as the countable product $\mathbb{P} = L^{\mathbb{N}}$ with L homeomorphic to \mathbb{P} , and let $T \subseteq L$ be a set of cardinality \aleph_1 without any Cantor subset. Let $B(\aleph_1) = D^{\mathbb{N}}$ with D discrete of cardinality \aleph_1 . Now fix a bijection $\phi : D \rightarrow T$ and let

$$(1) \quad f = \phi \times \phi \times \dots : D^{\mathbb{N}} \rightarrow T^{\mathbb{N}}.$$

Following Stone [S] let us choose a non- σ -discrete set $E = \{e_\alpha : \alpha < \omega_1\} \subseteq B(\aleph_1)$, all separable subsets of which are countable. We set

$$(2) \quad A = \mathbb{P} \setminus T^{\mathbb{N}} \text{ and } C = f(E) \subseteq T^{\mathbb{N}}.$$

Let us check that

$$(3) \quad M(A, C) \times \mathbb{P} \text{ is Lindel\"of.}$$

Since the neighborhoods of the points in $A \times \mathbb{P}$ are Euclidean, and $A \times \mathbb{P}$ is Lindel\"of, it is enough to show that if U is open in $\mathbb{P} \times \mathbb{P}$ and $A \times \mathbb{P} \subseteq U$, then $F = \mathbb{P} \times \mathbb{P} \setminus U$ intersects only countably many sections $\{c\} \times \mathbb{P}$, $c \in C$. Let H be the projection of F onto the first coordinate. Then H is an analytic set in $T^{\mathbb{N}}$ (cf. (2)), and so is the projection H_i of H onto the i th coordinate. Since H_i contains no Cantor set, it is countable. Therefore, $H \subseteq H_1 \times H_2 \times \dots$, with every H_i countable, and by (1), $f^{-1}(H) \subseteq \phi^{-1}(H_1) \times \phi^{-1}(H_2) \times \dots$ is separable in $B(\aleph_1)$. It follows that $f^{-1}(H) \cap E$ is countable, and so is the set $C' = H \cap C$ (cf. (2)). Because F intersects $\{c\} \times \mathbb{P}$ only for $c \in C'$, this completes our verification of (3).

We now demonstrate that

$$(4) \quad M(A, C) \times B(\aleph_1) \text{ is not normal.}$$

The set

$$F = \{(f(z), z) : z \in E\}$$

is the trace of the graph of f on $(A \cup C) \times B(\aleph_1)$ and by the continuity of f the set F is closed in the product (even with $A \cup C$ carrying the topology inherited from I). Let U be an arbitrary open set in $M(A, C) \times B(\aleph_1)$ containing F . To get (4), it is enough to show that

$$(5) \quad \overline{U} \cap (A \times B(\aleph_1)) \neq \emptyset.$$

For each $z \in E$, let $U(z)$ be a neighborhood of z in $B(\aleph_1)$ with $\{f(z)\} \times U(z) \subseteq U$. Since E is not σ -discrete, we show

$$(6) \quad \begin{array}{l} \text{there exists } w \in E \text{ for which the set } K = \{z \in E : w \in U(z)\} \\ \text{is uncountable.} \end{array}$$

To see that (6) is true, assume otherwise. Then $\{U(z) \cap E : z \in E\}$ is index point-countable. For our contradiction it is enough to argue that this shows E is σ -discrete. We may assume that each $U(z)$ is actually a $\frac{1}{n}$ -ball $B(z, \frac{1}{n})$, for some $n \in \mathbb{N}$, with respect to a fixed metric. If $E_n = \{z \in E : U(z) = B(z, \frac{1}{n})\}$, then each $\{U(z) \cap E_n : z \in E_n\}$ index point-countable implies that every $U(z) \cap E_n$, for $z \in E_n$, is countable. This says that $\{U(z) \cap E_n : z \in E_n\}$ witnesses its own local countability. Now, a standard "chaining argument" (cf. [E], 5.3.A) would imply that each E_n is σ -discrete. That shows (6) is true. To complete the

proof that $M(A, C) \times B(\aleph_1)$ is not normal we observe that by (3), $M(A, C)$ is Lindelöf, so the uncountable set $f(K)$ has an accumulation point $a \in A$. Hence $(a, w) \in \overline{f(K) \times \{w\}} \subseteq \overline{U}$, witnessing (5).

We conclude this section with a formal statement of the example just constructed.

Example 2.2. There is a Lindelöf space $M(A, C)$, of weight \aleph_1 , such that $M(A, C) \times B(\aleph_0)$ is Lindelöf but $M(A, C) \times B(\aleph_1)$ is not normal.

3. NORMALITY OF $M(A) \times B(\aleph_0)$ IMPLIES
NORMALITY OF $M(A) \times S$ FOR ANY COMPLETE S

The reasoning in this section will be based on the following lemma.

Lemma 3.1. *Let $f : S \rightarrow E$ be a continuous map from a completely metrizable space S to a separable metrizable space E , and let \mathcal{F} be a collection of closed subsets of E such that the image under f of any separable set is covered by countably many elements of \mathcal{F} . Then, for some $F \in \mathcal{F}$, $f^{-1}(F)$ has nonempty interior.*

Proof. We justify the assertion in two steps.

Step I. Assume that S has weight \aleph_1 , let $\{x_\alpha : \alpha < \omega_1\}$ be dense in S , and let

$$(1) \quad S_\alpha = \overline{\{x_\beta : \beta < \alpha\}}.$$

By the assumption about f , the closed separable set S_α is covered by countably many sets $f^{-1}(F)$, $F \in \mathcal{F}$, and by the Baire Category Theorem there exist $\phi(\alpha) < \alpha$, $\varepsilon(\alpha) > 0$ and $F_\alpha \in \mathcal{F}$ such that

$$B(x_{\phi(\alpha)}, \varepsilon(\alpha)) \cap S_\alpha \subseteq f^{-1}(F_\alpha),$$

where $B(x, \varepsilon)$ is the ε -ball about x , with respect to a fixed metric in S . Since ϕ is regressive, there exists an uncountable set $\Lambda \subseteq \omega_1$ and $r > 0$ with $\phi(\xi) = \alpha$, $\varepsilon(\xi) \geq r$ for all $\xi \in \Lambda$. Letting $a = x_\alpha$, we obtain

$$(2) \quad H_\xi = B(a, r) \cap S_\xi \subseteq f^{-1}(F_\xi) \text{ for } \xi \in \Lambda.$$

Since $\bigcup_{\alpha \in \Lambda} f(H_\alpha)$ is separable, and $H_\alpha \subseteq H_\beta$, for $\alpha < \beta$, there exists $\xi \in \Lambda$ with $f(H_\xi)$ dense in the union, i.e., by (2), $\bigcup_{\alpha \in \Lambda} f(H_\alpha) \subseteq \overline{f(H_\xi)} \subseteq F_\xi$, and hence

$$(3) \quad \bigcup_{\alpha \in \Lambda} H_\alpha \subseteq f^{-1}(F_\xi).$$

But, by (2),

$$\bigcup_{\alpha \in \Lambda} H_\alpha = B(a, r) \cap \bigcup_{\alpha \in \Lambda} S_\alpha = B(a, r),$$

and therefore, $B(a, r) \subseteq f^{-1}(F_\xi)$. This demonstrates the assertion for spaces S of weight \aleph_1 .

Step II. We will now consider the case of arbitrary complete S . Aiming at a contradiction, assume that for every $F \in \mathcal{F}$ the preimage $f^{-1}(F)$ has empty interior. We shall define inductively closed separable sets in S ,

$$(4) \quad S_1 \subseteq S_2 \subseteq \dots \subseteq S_\alpha \subseteq \dots, \quad \alpha < \omega_1,$$

and countable collections $\mathcal{F}_\alpha \subseteq \mathcal{F}$ such that

$$(5) \quad f(S_\alpha) \subseteq \bigcup \mathcal{F}_\alpha, \quad \alpha < \omega_1,$$

and

$$(6) \quad f^{-1}(F) \cap S_\alpha \subseteq \overline{S_{\alpha+1} \setminus f^{-1}(F)}, \quad \text{for } F \in \bigcup_{\beta \leq \alpha} \mathcal{F}_\beta.$$

Let us start from $S_0 = \{\emptyset\}$ and $\mathcal{F}_0 = \{F_0\}$ with any $F_0 \in \mathcal{F}$. Given $S_\alpha, f(S_\alpha)$ being separable, there exists a countable $\mathcal{F}_\alpha \subseteq \mathcal{F}$ satisfying (5). For each $F \in \bigcup_{\beta \leq \alpha} \mathcal{F}_\beta$, $f^{-1}(F)$ has empty interior and since S_α is separable, there is a countable set $C(F)$ disjoint from $f^{-1}(F)$ such that $f^{-1}(F) \cap S_\alpha \subseteq \overline{C(F)}$. Then

$$S_{\alpha+1} = \overline{\bigcup_{\beta \leq \alpha} \{C(F) : F \in \bigcup_{\beta \leq \alpha} \mathcal{F}_\beta\}}$$

satisfies (6). At each limit stage ξ , we set $S_\xi = \overline{\bigcup_{\alpha < \xi} S_\alpha}$.

Having completed the construction, we let

$$(7) \quad X = \bigcup_{\alpha < \omega_1} S_\alpha, \quad \mathcal{F}' = \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha.$$

Then X is closed in S ; hence it is a complete space of weight \aleph_1 . Let us consider $f|X$, the map f restricted to X . By (5), the images of separable subsets of X are covered by countable many elements of \mathcal{F}' . If $F \in \mathcal{F}_\beta$ and $x \in f^{-1}(F) \cap S_\alpha$, with $\alpha \geq \beta$, then by (6) x is the limit of a sequence from $X \setminus f^{-1}(F)$. It follows that all sets $(f|X)^{-1}(F)$, $F \in \mathcal{F}'$, have empty interior in X , contradicting the fact we have already established in Step I. □

Remark 3.2. The reasoning in Step I also yields the following fact. Let S be a union of an increasing sequence of closed separable sets $S_1 \subseteq S_2 \subseteq \dots \subseteq S_\xi \subseteq \dots$, $\alpha < \omega_1$, with each $P_\xi = S_\xi \setminus \bigcup_{\alpha < \xi} S_\alpha$ dense in S_ξ and let \mathcal{F}_ξ be a countable subcollection of \mathcal{F} covering $f(P_\xi)$. Then for all but non-stationary many ξ , there exists $F \in \mathcal{F}_\xi$ with the interior of $f^{-1}(F)$ intersecting P_ξ .

Theorem 3.3. *Let $A \subseteq I$ be such that the product of the Michael space $M(A)$ and the irrationals \mathbb{P} is normal. Then the product $M(A) \times S$ with any completely metrizable space S is normal.*

Proof. Let S be a completely metrizable space, and let $F \subseteq I \times S$ be a closed set in the product (where I carries the Euclidean topology) disjoint from $A \times S$.

By Remark 2.1, it is enough to check that F can be separated from $A \times S$ by sets open in the product $M(A) \times S$. Let

$$(8) \quad \mathcal{E} = \{E \subseteq F : \text{there exists an open } W \supseteq A \text{ with } E \cap (W \times S) = \emptyset\}.$$

We shall check that

$$(9) \quad \mathcal{E} \text{ is an exhaustive cover of } F.$$

That is, for each nonempty closed subset H of F there exists $E \in \mathcal{E}$ with $E \cap H$ nonempty and relatively open in H [M3]. To this end, let $f : H \rightarrow I$ be the projection onto the first axis restricted to H , and let \mathcal{F} be the collection of compact subsets of $I \setminus A$ (with the Euclidean topology). Let L be an arbitrary closed separable subset of H . Then $f(L)$ is an analytic subset of $I \setminus A$ and by the Kechris-Louveau-Woodin dichotomy ([K], Theorem 21.22), either $f(L)$ is covered by countable many elements of \mathcal{F} , or else there exists a Cantor set $K \subseteq f(L) \cup A$ such that $K \cap A$ is countable and dense in K . But then, by the classical theorem

of Michael, the closed subspace $K \times B(\aleph_0)$ of $M(A) \times B(\aleph_0)$ is not normal, contradicting the assumption about $M(A)$. It follows that Lemma 3.1 can be applied to f and \mathcal{F} which provides us with $T \in \mathcal{F}$ such that $f^{-1}(T) \cap H$ has a relatively nonempty interior in H . Denote this interior by E . Then $\overline{f(E)} \cap A = \emptyset$; that is, $E \in \mathcal{E}$.

Having established (9), we can now proceed as follows. By (9), there exist pairwise disjoint elements $E_\alpha \in \mathcal{E}$, $\alpha < \lambda$, such that $F = \bigcup_{\alpha < \lambda} E_\alpha$ and, for $\beta < \lambda$, E_β is open relative to $F \setminus \bigcup_{\alpha < \beta} E_\alpha$. Now using the fact that I has a countable base \mathcal{B} and S has a σ -discrete base $\bigcup_{n=1}^\infty \mathcal{G}_n$, we see there are closed discrete collections $\mathcal{F}_n \subseteq \mathcal{E}$, $n \in \mathbb{N}$, with $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ covering F (let us recall that we consider F as a subspace of $I \times S$). Indeed, for $W \in \mathcal{B}$, $\beta < \lambda$, and $k \in \mathbb{N}$ let

$$Z(W, \beta, k) = \bigcup \{ (W \times G) \cap E_\beta : G \in \mathcal{G}_k \text{ and } F \cap (W \times G) \setminus \bigcup_{\alpha < \beta} E_\alpha \subseteq E_\beta \}.$$

The collection $\{ \overline{Z(W, \beta, k)} : \beta < \lambda \}$ is discrete. Using this type of construction, we can assume in addition that the projection from $I \times S$ onto S takes distinct elements of \mathcal{F}_n to disjoint sets and the projection of \mathcal{F}_n is a collection discrete in S . Since the projection of any $E \in \mathcal{F}_n$ onto the first axis can be separated in $M(A)$ from A by open sets (cf. (8)), one readily defines, for $n \in \mathbb{N}$, open sets W_n in $M(A) \times S$ with $\bigcup \mathcal{F}_n \subseteq W_n$ and $\overline{W_n} \cap (A \times S) = \emptyset$. Also, since F is closed in $I \times S$, the metric subset $A \times S$ can be covered by countably many open sets V_n with $\overline{V_n} \cap F = \emptyset$. Then, a pair of disjoint open sets in $M(A) \times S$ separating F and $A \times S$ can be easily defined in a standard way (cf. [E], Lemma 1.5.15). \square

It may be useful to illustrate Theorem 3.3 with a couple of examples. Michael [M2] was aware that if $A \subseteq I$ such that $I \setminus A$ is an uncountable set containing no Cantor sets (for example, if A was a Bernstein subset of I), then $M(A) \times P$ is normal (in fact, Lindelöf) even though $M(A) \times (I \setminus A)$ is not normal. Hence, for such A , $M(A) \times S$ is normal for all complete metric spaces S .

For a different example, suppose $A \subseteq I$ where, for every countable set $D \subseteq A$, there is a G_δ -set H with $D \subseteq H \subseteq A$. An example of such A would be the union of an increasing ω_1 -sequence of G_δ -sets as described in [Kur] (page 517). To see that $M(A) \times \mathbb{P}$ is normal look at arbitrary closed $F \subseteq (I \setminus A) \times \mathbb{P}$ (and show F can be separated from $A \times \mathbb{P}$). For this set A , a Cantor set $K \subseteq I$ cannot have a countable dense subset D with $K \cap A = D$, so the Kechris-Louveau-Woodin Theorem used in the proof of Theorem 3.3 would say that the projection $p_1(F)$ into I can be covered by countably many compact sets disjoint from A . Say $p_1(F) \subseteq \bigcup_{n=1}^\infty C_n$ where every C_n is compact and $C_n \cap A = \emptyset$. Now, $\{p_1^{-1}(C_n) : n \in \mathbb{N}\}$ is a cover of F consisting of sets which are open and closed in $M(A) \times \mathbb{P}$ and disjoint from $A \times \mathbb{P}$. Also, since we may assume F is closed relative to $I \times \mathbb{P}$, the metric subset $A \times \mathbb{P}$ can be covered by countably many open sets V_n with $\overline{V_n} \cap F = \emptyset$. Then, a pair of disjoint open sets in $M(A) \times \mathbb{P}$ separating F and $A \times \mathbb{P}$ can be defined in a standard way. Now that $M(A) \times \mathbb{P}$ is seen to be normal, Theorem 3.3 says that $M(A) \times S$ is normal for every complete metric space S . The reader may observe here that the above proof that $M(A) \times \mathbb{P}$ is normal is actually a simplified version of the proof of Theorem 3.3 itself.

Although stated in terms of normality, Theorem 3.3 actually gives a result about paracompactness of products $M(A) \times S$, for S completely metrizable, using $M(A) \times \mathbb{P}$ as a test space. Rudin and Starbird [RS] have shown that if X is metrizable and

Y is paracompact, then $X \times Y$ is normal if and only if $X \times Y$ is paracompact. Since any Michael space $M(A, C)$ is clearly paracompact, the proof of the following corollary is immediate.

Corollary 3.4. *For $A \subseteq I$, $M(A) \times \mathbb{P}$ is normal if and only if $M(A) \times S$ is paracompact for every completely metrizable space S .*

4. THE STATEMENT “ $M(A, C) \times B(\aleph_1)$ NORMAL
IMPLIES $M(A, C) \times B(\aleph_2)$ NORMAL” IS AXIOM SENSITIVE

The aim of this section is to substantiate the assertion in the title. We start from the following observation (the notion of an exhaustive collection was recalled in Section 3, following (9)). Recall [E] that a space E is said to be an *absolute G_δ* if E is a G_δ -set in every metric space in which it is embedded.

Lemma 4.1. *Let \mathcal{E} be a disjoint decomposition of a metrizable space X such that each countable union of elements of \mathcal{E} is an absolute G_δ -set. Then the collection \mathcal{E} is exhaustive, and, in particular, X is completely metrizable.*

Proof. Let H be a closed subset of X and let $E \in \mathcal{E}$. Then $E \cap H$ is an F_σ -set in X . Otherwise, $E \cap H$ being an absolute G_δ -set, a theorem of Hurewicz [H] would yield a Cantor set K in H with $K \setminus E$ countable and dense in K . However, $K \setminus E$ is the trace on K of the union of the countable collection $\{G \in \mathcal{E} : G \cap (K \setminus E) \neq \emptyset\}$, i.e., a G_δ -set in K , which contradicts the Baire Category Theorem.

Arriving at a contradiction, suppose that for each $E \in \mathcal{E}$, $E \cap H$ has relatively empty interior in H and so $E \cap H$ has a boundary point in H . Then one can choose inductively, for $i \in \mathbb{N}$, disjoint sets $E_i \in \mathcal{E}$ and points $a_i \in E_i \cap H$ such that the set $Q = \{a_1, a_2, \dots\}$ has no isolated points (cf. [Ko], proof of Theorem 2). We obtain in this way a completely metrizable space $G = \overline{Q} \cap \bigcup_{i=1}^\infty E_i$ which can be split into two disjoint dense G_δ -sets. This, however, is impossible, by the Baire Theorem.

Now that \mathcal{E} is known to be exhaustive, the fact that X must be completely metrizable follows from Theorem 1.3 in [M3]. This is true because, following terminology from [M3], the (constant) sequence $(\mathcal{E})_{n \in \mathbb{N}}$ is a complete sequence of exhaustive covers. □

To show that in some models of set theory the implication is not true we use the following lemma. (Recall [Kur] that a subset T of the unit interval is a λ -set provided all countable subsets of T are relatively G_δ in T .)

Lemma 4.2. *It is consistent with ZFC that, simultaneously,*

- (i) *there exists a λ -set $T \subseteq I$ of cardinality \aleph_2 , and*
- (ii) *there exists a non- σ -discrete set $E \subseteq B(\aleph_2)$, with $|E| = \aleph_2$, such that all subsets of E of cardinality \aleph_1 are σ -discrete.*

Let us indicate some references concerning this lemma: Jensen [Je], assuming $V = L$, shows $E(\omega_2)$ holds. That is, in this model M , there is an ω_2 stationary set $\Lambda \subseteq \omega_2$ of ordinals with countable cofinality, such that all intersections $\Lambda \cap \alpha$, $\alpha < \omega_2$, of cofinality ω_1 , are non-stationary. Now, $E(\omega_2)$ is preserved under ccc extensions [F2] and there is a ccc extension to a model $M[G]$ where Martin’s Axiom MA and $\aleph_2 < 2^{\aleph_0}$ both hold ([Ku]). In this model, if $T \subseteq I$ with $|T| = \omega_2$, then T is actually a Q -set (all subsets are relatively G_δ) so T is certainly a λ -set.

To get (ii) in $M[G]$, let us fix, for each $\alpha \in \Lambda$, a function $x_\alpha : \mathbb{N} \rightarrow \omega_2$ with $x_\alpha(n) \not\prec \alpha$. Then $E = \{x_\alpha : \alpha \in \Lambda\} \subseteq B(\alpha)$ has the required properties (cf. [F1]; [P], Lemma p. 141).

Proposition 4.3. *The statements (i) and (ii) in Lemma 4.2 yield disjoint sets A, C in I with $M(A, C) \times B(\aleph_1)$ normal but $M(A, C) \times B(\aleph_2)$ not normal.*

Proof. We shall modify the construction from Section 2, adopting the notation, but replacing \aleph_1 by \aleph_2 and taking the sets T and E from (i) and (ii) of Lemma 4.2, respectively.

The arguments in Section 2 can be modified appropriately to show that $M(A, C) \times B(\aleph_0)$ is Lindelöf and $M(A, C) \times B(\aleph_2)$ is not normal. To check that $M(A, C) \times B(\aleph_1)$ is normal we shall first verify the following.

Claim. *Let $H \subseteq C$ be a set of cardinality \aleph_1 . Then there is a decomposition $H = \bigcup_{n=1}^{\infty} H_n$ and disjoint collections \mathcal{G}_n , for $n \in \mathbb{N}$, of subsets of $I \setminus A$ such that each $G \in \mathcal{G}_n$ intersects H_n in a singleton and the union of every countable subfamily of \mathcal{G}_n is a G_δ -set in $I \setminus A$.*

To this end, taking into account (1) and (2) in Section 2, let us write $H = f(L)$ with $L \subseteq E$ of cardinality \aleph_1 . The set L is σ -discrete in $B(\aleph_2) = D^{\mathbb{N}}$, D being the discrete space of cardinality \aleph_2 . Therefore, $L = \bigcup_{n=1}^{\infty} L_n$ with each L_n projecting injectively into a finite product $D^{s(n)}$. By (1), $H_n = f(L_n)$ projects injectively onto a subset Z_n of $T^{s(n)}$. Let

$$\mathcal{G}_n = \{\{t\} \times T \times T \times T \times \cdots : t \in Z_n\}.$$

Then each element of \mathcal{G}_n intersects H_n in a singleton. Since T is a λ -set, so is $T^{s(n)}$ and hence for each countable $W \subseteq Z_n$, $W \times T \times T \times \cdots$ is a G_δ -set in $T^{\mathbb{N}} = I \setminus A$ (cf. (2), Section 2). Therefore, countable unions of elements of \mathcal{G}_n are G_δ -sets in $I \setminus A$.

Having established the claim, let us now consider a set $F \subseteq C \times B(\aleph_1)$, closed in $M(A, C) \times B(\aleph_1)$. By Remark 2.1, it is enough to separate F from $A \times B(\aleph_1)$ by open sets in $M(A, C) \times B(\aleph_1)$. Let $p : M(A, C) \times B(\aleph_1) \rightarrow M(A, C)$ be the projection. For any closed separable subset S of $B(\aleph_1)$, the product $M(A, C) \times S$ is Lindelöf and hence $p(F \cap (M(A, C) \times S))$ is a countable subset of C . It follows that

$$(1) \quad H = p(F) \subseteq C \text{ has cardinality } \leq \aleph_1.$$

Let H_n and \mathcal{G}_n be the sets and the families described in the claim. Let Y be the closure of F in the product $I \times B(\aleph_1)$. Then

$$(2) \quad p(Y) \subseteq I \setminus A.$$

Let $\mathcal{E}_n = \{p^{-1}(G) \cap Y : G \in \mathcal{G}_n\}$. Then each countable union of elements of \mathcal{E}_n is of the form $p^{-1}(\bigcup \mathcal{A}) \cap Y$ with a countable $\mathcal{A} \subseteq \mathcal{G}_n$, and since $\bigcup \mathcal{A}$ is a G_δ -set in $I \setminus A$, (2) implies that $p^{-1}(\bigcup \mathcal{A}) \cap Y$ is a G_δ -set in Y ; hence it is an absolute G_δ -set.

From Lemma 4.1 we infer that each \mathcal{E}_n is exhaustive in the space $X_n = p^{-1}(\bigcup \mathcal{G}_n) \cap Y$, with the subspace topology inherited from $I \times B(\aleph_1)$. Since for each $c \in H_n$, $F \cap p^{-1}(c) \subseteq p^{-1}(G)$, where G is the unique element of \mathcal{G}_n containing c , it follows that the collection of fibers $\{F \cap p^{-1}(c) : c \in C\}$ is σ -discretely decomposable in $I \times B(\aleph_1)$.

In effect, a separation of F from $A \times B(\aleph_1)$ in $M(A, C) \times B(\aleph_1)$ can be defined in the same way as in the second part of the proof of Theorem 3.3. \square

Now, in the opposite direction, we will show that Fleissner’s axiom $(SC\omega_2)$ [F1], combined with CH, guarantees the implication in the title. We refer the reader to Fremlin [Fr2] for a discussion of topics related to Fleissner’s axiom.

Proposition 4.4. *Assuming Fleissner’s axiom $(SC\omega_2)$ and CH, if A, C are disjoint sets in I and $M(A, C) \times B(\aleph_1)$ is normal, then $M(A, C) \times S$ is normal for every completely metrizable space S .*

Proof. As was already mentioned in the Introduction the normality of $M(A, C) \times B(\aleph_1)$ implies that

$$(3) \quad M(A, C) \times Y \text{ is normal for completely metrizable spaces of weight } \leq \aleph_1.$$

Let S be an arbitrary completely metrizable space, and let $F \subseteq C \times S$ be closed in $M(A, C) \times S$. We have to separate F from $A \times S$ by open sets in the product $M(A, C) \times S$ (cf. Remark 2.1).

Since we assume CH, $I \setminus A$ is a union of compact sets K_ξ , $\xi < \omega_1$, such that each compact set in $I \setminus A$ is contained in some union $\bigcup_{\alpha < \xi} K_\alpha$. Let

$$Z_\xi = K_\xi \setminus \bigcup_{\alpha < \xi} K_\alpha, \quad \xi < \omega_1.$$

Then

$$(4) \quad \text{the closure of } Z_\xi \text{ in } I \text{ is disjoint from } A, \text{ and}$$

$$(5) \quad \text{each set in } I \text{ intersecting uncountably many } Z_\xi \text{ has an accumulation point in } A.$$

Let $p_1 : I \times S \rightarrow I$ be the projection onto the first coordinate. We will show that

$$(6) \quad \mathcal{E} = \{F \cap p_1^{-1}(Z_\xi) : \xi < \omega_1\} \text{ is } \sigma\text{-discretely decomposable in } I \times S,$$

where I is considered with the Euclidean topology. Now, Fleissner’s axiom reduces (6) to a verification of the following claim (cf. [F1], Diagram 2 on page 315). By a selector T for \mathcal{E} , we mean a set $T \subseteq \bigcup \mathcal{E}$ such that $|T \cap E| = 1$ for every $E \in \mathcal{E}$.

$$(7) \quad \text{Each selector } T \text{ for } \mathcal{E} \text{ is } \sigma\text{-discrete.}$$

Let Y be the closure of the projection of T onto the second axis (by projection p_2). By (3), $X = M(A, C) \times Y$ is normal so we can find an open set U in X containing $F \cap X$ and having no accumulation points in $A \times Y$. For any uncountable $W \subseteq T$, the projection $p_1(W)$ has an accumulation point in A (cf. (5) and (6)). Therefore, for any $y \in Y$, $I \times \{y\}$ hits at most countably many sets $p_1^{-1}(c) \cap U$, with $c \in p_1(T)$. It follows that the vertical sections $U(c) = \{y \in Y : (c, y) \in U\}$ form an indexed point-countable collection $\{U(c) : c \in p_1(T)\}$. Since each $U(c)$ is open in Y an argument similar to that following (6) in Section 2 shows that $p_2(T)$ is σ -discrete. From this it follows that (7) is true. With (6) verified, we can now proceed again as in the second part of the proof of Theorem 3.3 to define a pair of open sets in $M(A, C) \times S$ separating F from $A \times S$. That concludes the proof that $M(A, C) \times S$ is normal. \square

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