

## THE MIXED HODGE STRUCTURE ON THE FUNDAMENTAL GROUP OF A PUNCTURED RIEMANN SURFACE

RAINER H. KAENDERS

(Communicated by Leslie D. Saper)

ABSTRACT. Given a compact Riemann surface  $\bar{X}$  of genus  $g$  and distinct points  $p$  and  $q$  on  $\bar{X}$ , we consider the non-compact Riemann surface  $X := \bar{X} \setminus \{q\}$  with basepoint  $p \in X$ . The extension of mixed Hodge structures associated to the first two steps of  $\pi_1(X, p)$  is studied. We show that it determines the element  $(2gq - 2p - K)$  in  $\text{Pic}^0(\bar{X})$ , where  $K$  represents the canonical divisor of  $\bar{X}$  as well as the corresponding extension associated to  $\pi_1(\bar{X}, p)$ . Finally, we deduce a pointed Torelli theorem for punctured Riemann surfaces.

### INTRODUCTION

Let  $q$  be a point in a compact Riemann surface  $\bar{X}$  of genus  $g$ . In this paper we want to study the complement of  $q$  in  $\bar{X}$ , i.e.  $X := \bar{X} \setminus \{q\}$  with a basepoint  $p \in X$ . We refer to this situation as a *punctured Riemann surface  $X$  with puncture  $q$  and basepoint  $p$* .

For the fundamental group  $\pi_1(\bar{X}, p)$  of the compact Riemann surface  $(\bar{X}, p)$ , Hain and Pulte ([Hai87b], [Pul88]) proved that the extension of mixed Hodge structures associated to the quotient of its group ring by  $W_{-3}$  determines the base point (see Theorem 2.1). From this result and from the classical Torelli theorem they derived a *pointed Torelli theorem* (see Theorem 2.3) as a corollary.

For the fundamental group  $\pi_1(\bar{X} \setminus \{q\}, p)$  of the punctured Riemann surface  $(\bar{X} \setminus \{q\}, p)$ , the corresponding extension of mixed Hodge structures  $w_{pq}$ , i.e. the extension associated to the quotient of its group ring by  $W_{-3}$ , is one dimension bigger than in the compact case. We show that it determines the element

$$(2gq - 2p - K) \text{ in } \text{Pic}^0(\bar{X}),$$

where  $K$  represents the canonical divisor, and the corresponding extension associated to  $\pi_1(\bar{X}, q)$  (see Theorem 1.2). This may have implications on possible normal functions on the moduli space of complex projective curves (cf. [HL97], 7.4).

Finally, we prove that this extension  $w_{pq}$  determines both, the basepoint  $p$  and the puncture  $q$ . This, together with the pointed Torelli theorem of Hain and Pulte yields a *punctured pointed Torelli theorem* (see Theorem 2.8).

---

Received by the editors March 3, 1999 and, in revised form, July 26, 1999.

2000 *Mathematics Subject Classification*. Primary 14H40, 14H30; Secondary 14F35.

The author was partly supported by grant ERBCHICT930403 (HCM) from the European Community and The Netherlands Organisation for Scientific Research (NWO).

1. EXTENSIONS AND THE THETA DIVISOR

For the definition of iterated integrals and of the mixed Hodge structure (MHS) on the fundamental group we refer to the introductory article [Hai87b].

Let  $\bar{X}$  be a compact Riemann surface of genus  $g$  and let  $q$  be a point on  $\bar{X}$ . We consider the pointed space  $(X, p)$ , where  $X := \bar{X} \setminus \{q\}$  and  $p$  is a basepoint on  $X$ . Denote by  $J \subset \mathbb{Z}\pi_1(X, p)$  and  $\bar{J} \subset \mathbb{Z}\pi_1(\bar{X}, p)$  the augmentation ideals in the group rings of the respective fundamental groups. Note that  $J/J^2 = H_1(X)$  resp.  $\bar{J}/\bar{J}^2 = H_1(\bar{X})$ , and since we remove only a single point from  $\bar{X}$ , we have that  $X \hookrightarrow \bar{X}$  induces an isomorphism of pure Hodge structures between  $H_1(X)$  and  $H_1(\bar{X})$ , both of weight  $-1$ . This allows us to identify these two Hodge structures. Similarly, we identify the weight 1 Hodge structures  $H^1(X)$  and  $H^1(\bar{X})$ . We will write just  $H_1$  and  $H^1$ . The MHS on the fundamental group  $\pi_1(X, p)$  resp.  $\pi_1(\bar{X}, p)$  consists by definition of MHS's on the integral lattices  $J/J^{s+1}$  resp.  $\bar{J}/\bar{J}^{s+1}$  for  $s \geq 2$ .

This definition of the MHS's is possible because of Chen's  $\pi_1$ -De Rham-Theorem, telling us that integration of iterated integrals yields isomorphisms

$$(1.1) \quad \begin{aligned} H^0 \bar{B}_s(E^\bullet(\bar{X} \log q), p) &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(J/J^{s+1}, \mathbb{C}) =: (J/J^{s+1})_{\mathbb{C}}^* \quad \text{resp.} \\ H^0 \bar{B}_s(E^\bullet(\bar{X}), p) &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(\bar{J}/\bar{J}^{s+1}, \mathbb{C}) =: (\bar{J}/\bar{J}^{s+1})_{\mathbb{C}}^*. \end{aligned}$$

Here  $E^\bullet(\bar{X} \log q)$  denotes the differential graded algebra (dga) of  $C^\infty$ -forms on  $X = \bar{X} \setminus \{q\}$  with logarithmic singularities at  $q$  and  $E^\bullet(\bar{X})$  denotes the dga of smooth complex valued forms on  $\bar{X}$ . The objects on the left of (1.1) are the complex vector spaces of iterated integrals of length  $\leq s$ , which are homotopy functionals — considered as functions on the fundamental group. These vector spaces can be described purely algebraically in terms of the augmented dga's  $E^\bullet(\bar{X} \log q)$  and  $E^\bullet(\bar{X})$ . This is part of a general construction, *the reduced bar construction*, whence the elaborate notation (cf. [Che76] or [Hai87a]). Here we identify these different descriptions deliberately.

In the two cases under consideration, the weight filtration  $W_\bullet$  is already given on the lattices  $J/J^{s+1}$  resp.  $\bar{J}/\bar{J}^{s+1}$  by the  $J$ -adic filtration, i.e.

$$W_{-l} J/J^{s+1} = J^l/J^{s+1} \quad \text{resp.} \quad W_{-l} \bar{J}/\bar{J}^{s+1} = \bar{J}^l/\bar{J}^{s+1} \quad \text{for } 0 < l \leq s + 1.$$

For  $l = 1$  and  $s = 1$  we recover the pure Hodge structure on homology, i.e.  $W_{-1} J/J^2 = J/J^2 = H_1 = \bar{J}/\bar{J}^2 = W_{-1} \bar{J}/\bar{J}^2$ . The weights  $-1$  and  $-2$  give rise to two extensions of MHSs  $w_{pq}$  and  $w_p$ , related by the following commutative diagram:

$$(1.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & J^2/J^3 & \longrightarrow & J/J^3 & \longrightarrow & J/J^2 \longrightarrow 0; w_{pq} \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & \bar{J}^2/\bar{J}^3 & \longrightarrow & \bar{J}/\bar{J}^3 & \longrightarrow & \bar{J}/\bar{J}^2 \longrightarrow 0; w_p. \end{array}$$

The multiplication in the group rings defines surjective maps  $J/J^2 \otimes J/J^2 \rightarrow J^2/J^3$  and  $\bar{J}/\bar{J}^2 \otimes \bar{J}/\bar{J}^2 \rightarrow \bar{J}^2/\bar{J}^3$  whose dual morphisms are inclusions  $(J^2/J^3)^* \hookrightarrow H^1 \otimes H^1$  and  $(\bar{J}^2/\bar{J}^3)^* \hookrightarrow H^1 \otimes H^1$ . It is well-known (cf. [Hai87b]) that in both cases, the image of the above inclusions coincides with the kernel of the cup-product. Hence the inclusions give isomorphisms  $(J^2/J^3)^* \cong H^1 \otimes H^1$  and  $(\bar{J}^2/\bar{J}^3)^* \cong K$ , where  $K := \ker\{\cup : H^1(\bar{X}) \otimes H^1(\bar{X}) \rightarrow H^2(\bar{X})\}$ . As  $\cup$  is a morphism of Hodge

structures,  $K$  inherits a pure Hodge structure of weight 2 from  $H^1 \otimes H^1$ . Dualizing the diagram (1.2) yields extensions of MHS's  $m_{pq}$  and  $m_p$  — dual to  $w_{pq}$  and  $w_p$

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & (J/J^3)^* & \longrightarrow & H^1 \otimes H^1 \longrightarrow 0; m_{pq} \\ & & \uparrow = & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H^1 & \longrightarrow & (\bar{J}/\bar{J}^3)^* & \longrightarrow & K \longrightarrow 0; m_p. \end{array}$$

Since the exact sequence

$$(1.3) \quad 0 \rightarrow K_{\mathbb{Z}} \rightarrow H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1 \xrightarrow{\cup} H^2(\bar{X}, \mathbb{Z}) \rightarrow 0$$

of Hodge structures of weight 2 splits over  $\mathbb{Q}$  but not over  $\mathbb{Z}$ , let us first clarify the nature of the embedding  $K \hookrightarrow H^1 \otimes H^1$ . Identify  $H^2(\bar{X}, \mathbb{Z})$  with  $\mathbb{Z}$ . There is a bilinear form

$$b : (H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1) \times (H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1) \longrightarrow \mathbb{Z},$$

given by  $b((x_1 \otimes x_2), (y_1 \otimes y_2)) := (x_1 \cup y_2) \cdot (y_1 \cup x_2)$ , which has mixed signature and is nondegenerate. Consider the rank 1 sublattice  $Q_{\mathbb{Z}}$  of  $H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$  orthogonal to the kernel of the cup-product  $K_{\mathbb{Z}} \subset H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$  with respect to  $b$ . The submodule  $Q_{\mathbb{Z}}$  projects to  $2g H^2(\bar{X}, \mathbb{Z})$  under  $\cup$ .

$Q_{\mathbb{Z}}$  is generated by one element  $\mathfrak{X}$  in  $H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$ , which is invariant under complex conjugation. Hence  $H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$  induces on  $Q_{\mathbb{Z}}$  a  $\mathbb{Z}$ -HS, isomorphic to  $H^2(\bar{X}, \mathbb{Z})$  or  $\mathbb{Z}(-1)$ . Since (1.3) splits over the rationals we have  $K_{\mathbb{Q}} \oplus Q_{\mathbb{Q}} = H_{\mathbb{Q}}^1 \otimes H_{\mathbb{Q}}^1$ .

Note that  $m_p$ , the restriction of  $m_{pq}$  to  $K$ , is the extension associated to  $\pi_1(\bar{X}, p)$ , which is used in the pointed Torelli theorem of Hain and Pulte (Theorem 2.3).

**Definition 1.1.** Define  $k_{pq} = [0 \rightarrow H^1 \rightarrow E_{pq} \rightarrow Q \rightarrow 0] \in \text{Ext}_{\text{MHS}}(Q; H^1)$  to be the restriction of the extension  $m_{pq}$  to an extension of  $Q$  by  $H^1$ .

Let  $\Psi : \text{Ext}_{\text{MHS}}(Q; H^1) \xrightarrow{\cong} \text{Pic}^0(\bar{X})$  be the natural isomorphism (see [Car80]). Then the main theorem of this paper is

**Theorem 1.2.** *In  $\text{Pic}^0(\bar{X})$  we have  $\Psi(k_{pq}) = (2gq - 2p - K)$ .*

**1.1. Riemann's constant.** Let  $u : \text{Pic}^0(\bar{X}) \rightarrow \text{Jac}(\bar{X})$  be the Abel-Jacobi map and define the divisor

$$W_{p,g-1} := \left\{ \sum_{j=1}^{g-1} u(q_j - p) \mid \sum_{j=1}^{g-1} q_j \in \bar{X}^{(g-1)} \right\}.$$

Denote the *theta divisor* on  $\text{Jac}(\bar{X})$  by  $\Theta$  and *Riemann's constant* by  $\kappa_p \in \text{Jac}(\bar{X})$ , such that Riemann's classical theorem<sup>1</sup> reads:  $\Theta = W_{p,g-1} + \kappa_p$ .

Using the Riemann-Roch theorem one can prove that Riemann's constant  $\kappa_p$  is related to the canonical divisor by the fact that for any divisor  $K$  of a holomorphic 1-form holds  $u((2g-2)p - K) = 2\kappa_p$  and that the canonical divisor is characterized by this equation (for a proof we refer to [GH78], p. 340). Theorem 1.2 is then a consequence of the following theorem, whose proof will be given in the sequel.

**Theorem 1.3.** *In the Jacobian  $\text{Jac}(\bar{X})$  we have  $u(\Psi(k_{pq}) + 2g(p - q)) = 2\kappa_p$ .*

<sup>1</sup>Proofs of this theorem can be found in [Rie92] (VI, 22., pp. 132-136; XI, pp. 213-224) or [Lan02]. For proofs in modern language we refer to [Lew64], [Mum83] (Theorem 3.1, pp. 149-151) or to [GH78]. In the theory of  $\theta$ -functions it is more convenient to define  $\varkappa_p$  to be an element of  $\mathbb{C}^g$  like in [Rie92], [Lan02], [Lew64], [Fay73] (here Riemann's constant is defined as  $-\varkappa_p$ ) and [Mum83].

The rest of this section is devoted to the proof of Theorem 1.3. First we interpret the right-hand side of the equation by means of an expression for  $\kappa_p$  in terms of iterated integrals, as it was already known to Riemann.

To present this formula we need some more notation. Denote by  $\gamma_1, \dots, \gamma_{2g}$  and  $\delta$  (representing a small loop around  $q$ ) a system of elements in  $\pi_1(X, p)$  having the property, that the fundamental group  $\pi_1(X, p)$  is the quotient of the free group  $F\langle \gamma_1, \dots, \gamma_{2g}, \delta \rangle$  generated by the  $\gamma_i$  and  $\delta$  subject to the commutator relation

$$(1.4) \quad [\gamma_1, \gamma_{g+1}] \cdots [\gamma_g, \gamma_{2g}] = \delta.$$

Let  $dz_1, \dots, dz_g$  be a basis of holomorphic 1-forms on  $\bar{X}$ , such that  $\int_{\gamma_\nu} dz_i = \delta_{i\nu}$ , i.e. the period matrix can be written  $\Omega = (\omega_{i\mu})_{\substack{i=1, \dots, g \\ \mu=1, \dots, 2g}} = (\Omega_1, \Omega_2) = (I, Z)$ . By Riemann's bilinear relations,  $Z$  is a symmetric  $g \times g$ -matrix with positive definite imaginary part. Having made these choices we may represent the Jacobian of  $\bar{X}$  as  $\text{Jac}(\bar{X}) := \mathbb{C}^g / \Omega \mathbb{Z}^{2g}$ . The following expression of  $\kappa_p$  in terms of iterated integrals

$$(1.5) \quad \kappa_p = \left[ - \sum_{\nu=1}^g \int_{\gamma_\nu} dz_i dz_\nu + \frac{1}{2} \int_{\gamma_{g+i}} dz_i \right]_{i=1, \dots, g} \in \text{Jac}(\bar{X})$$

was known to Bernhard Riemann in 1865 (see [Rie92], p. 213, or [Fay73]).

**1.2. Extension data.** According to [Car80] we need two things for the computation of the extension data  $k_{pq} \in \text{Ext}_{\text{MHS}}(Q, H^1)$ : a Hodge filtration preserving section  $s_F : (Q, F^\bullet) \rightarrow (E_{pq}, F^\bullet)$  and an integral retraction  $r_{\mathbb{Z}} : (E_{pq})_{\mathbb{Z}} \rightarrow H_{\mathbb{Z}}^1$ .

Let  $dx_1, \dots, dx_{2g}$  be the real harmonic 1-forms such that  $\int_{\gamma_j} dx_i = \delta_{ij}$ . Then a generator  $\mathfrak{X}$  of  $Q_{\mathbb{Z}}$  is given by  $\mathfrak{X} := \sum_{\nu=1}^g ([dx_\nu] \otimes [dx_{g+\nu}] - [dx_{g+\nu}] \otimes [dx_\nu])$ . Riemann's first bilinear relation tells us that  $\mathfrak{X} \in F^1(H_{\mathbb{C}}^1 \otimes H_{\mathbb{C}}^1)$ . To be more precise, we can write

$$\sum_{\nu=1}^g (dx_\nu \otimes dx_{g+\nu} - dx_{g+\nu} \otimes dx_\nu) = \sum_{j,k=1}^g (a_{jk} dz_j \otimes d\bar{z}_k + \bar{a}_{jk} d\bar{z}_j \otimes dz_k)$$

with  $A = (a_{jk})_{jk} = (\bar{\Omega}_2 \Omega_1^t - \bar{\Omega}_1 \Omega_2^t)^{-1} = (\bar{Z} - Z)^{-1}$ . Observe:  $A^t = -\bar{A}$ .

Define  $\wedge \mathfrak{X} := \sum_{\nu=1}^g (dx_\nu \wedge dx_{g+\nu} - dx_{g+\nu} \wedge dx_\nu) \in F^1 E^2(\bar{X})$ . The strictness of the differential with respect to the Hodge filtration on  $E^\bullet(\bar{X} \log q)$  implies that there is a  $\mu_q \in F^1 E^1(\bar{X} \log q)$  such that  $\wedge \mathfrak{X} + d\mu_q = 0$ . This condition implies that the iterated integral  $\int \mathfrak{X} + \mu_q := \int \sum_{\nu=1}^g (dx_\nu dx_{g+\nu} - dx_{g+\nu} dx_\nu) + \mu_q$  is a homotopy functional.

A Hodge filtration preserving section  $s_F$  is then defined by  $s_F(\mathfrak{X}) = \int \mathfrak{X} + \mu_q$  and an integral retraction  $r_{\mathbb{Z}}$  is given by the map, which sends an iterated integral  $\int I$  of length  $\leq 2$  with values in  $\mathbb{Z}$  to  $r_{\mathbb{Z}}(\int I) := \sum_{j=1}^{2g} (\int_{\gamma_j} I)[dx_j]$ . Again a standard computation shows

$$u \circ \Psi(k_{pq}) = \left( \sum_{\nu=1}^g \left( \int_{\gamma_\nu} dz_i \int_{\gamma_{g+\nu}} \mathfrak{X} + \mu_q - \int_{\gamma_{g+\nu}} dz_i \int_{\gamma_\nu} \mathfrak{X} + \mu_q \right) \right)_{i=1, \dots, g} \in \text{Jac} \bar{X}.$$

**1.3. A higher reciprocity law.** Generally for functions  $F, G : \pi_1(X, p) \rightarrow \mathbb{C}$  we introduce  $\Pi(F; G) := \sum_{\nu=1}^g (F(\gamma_\nu)G(\gamma_{g+\nu}) - F(\gamma_{g+\nu})G(\gamma_\nu))$ . For instance, Riemann's classical period relation reads  $\Pi(\int dz_i; \int dz_j) = 0$ . With this notation we can state a higher reciprocity law.

**Theorem 1.4.** *For any holomorphic 1-form  $\omega$  on  $\bar{X}$  we have modulo periods of  $\omega$*

$$\sum_{\nu=1}^g \left\{ \int_{\gamma_\nu} \omega \int_{\gamma_{g+\nu}} \mathfrak{X} + \mu_q - \int_{\gamma_{g+\nu}} \omega \int_{\gamma_\nu} \mathfrak{X} + \mu_q \right\} \equiv 2g \int_p^q \omega + \sum_{j,k=1}^g a_{jk} \left\{ \Pi \left( \int \omega; \int dz_j \int d\bar{z}_k \right) + 2 \Pi \left( \int \omega \int d\bar{z}_k; \int dz_j \right) - 2 \Pi \left( \int \omega dz_j; \int d\bar{z}_k \right) \right\}.$$

1.3.1. *Observation.* The proof of the *higher reciprocity law* in Theorem 1.4 and also later the proof of the *higher period relation* of Theorem 1.6 are direct generalizations of Riemann’s bilinear relations as they are proved in [Che77] or [Gun69]. We use the following procedure.

Let  $c_i := (\gamma_i - 1)$  and  $d := (\delta - 1)$  denote the elements in  $J$  corresponding to  $\gamma_i$  and  $\delta$  in  $\pi_1(X, p)$ . If we interpret relation (1.4) in  $\mathbb{Z}\pi_1(X, p)$  modulo  $J^4$ , we obtain

$$(1.6) \quad \sum_{\nu=1}^g \{ c_\nu c_{g+\nu} - c_{g+\nu} c_\nu + (c_{g+\nu} c_\nu c_{g+\nu} - c_\nu c_{g+\nu} c_\nu) - (c_\nu c_{g+\nu} c_{g+\nu} - c_{g+\nu} c_\nu c_\nu) \} \equiv d \pmod{J^4}.$$

When the linear extension of a homotopy functional  $F : \pi_1(X, p) \rightarrow \mathbb{C}$  to  $\mathbb{Z}\pi_1(X, p)$  satisfies  $F(J^4) = 0$ , then it has to respect relation (1.6). For instance iterated integrals of length  $\leq 3$ , which are homotopy functionals, are examples of such  $F$ .

*Remark 1.5.* Also in [PY96] the above described procedure is employed to derive *higher period relations for iterated integrals*. In subsection 1.4 we will apply the method to one specific iterated integral. Since the homotopy functionals in [PY96] do not take the polarization or likewise a puncture into account, there are no *higher reciprocity laws for iterated integrals*.

*Proof of Theorem 1.4.* Use the fact that for any closed path  $\alpha$ ,

$$\int_\alpha d\bar{z}_j dz_k + \int_\alpha dz_k d\bar{z}_j = \int_\alpha d\bar{z}_j \int_\alpha dz_k$$

to prove that the left-hand side of the equation in Theorem 1.4 equals

$$\Pi \left( \int \omega; \int \sum_{j,k=1}^g 2a_{jk} dz_j d\bar{z}_k + \mu_q \right) - \Pi \left( \int \omega; \sum_{j,k=1}^g a_{jk} \int dz_j \int d\bar{z}_k \right).$$

Note that  $\int I := \int \sum_{j,k=1}^g 2a_{jk} \omega dz_j d\bar{z}_k + \omega \mu_q$  is a homotopy functional, so its values on both sides of (1.6) coincide. Recall that for 1-forms  $\varphi, \psi, \chi$  and closed paths  $\alpha, \beta$  with  $a = (\alpha - 1)$ ,  $b = (\beta - 1)$  and  $ab = (\alpha\beta - \alpha - \beta + 1)$  holds:  $\int_{ab} \varphi\psi\chi = \int_a \varphi \int_b \psi\chi + \int_a \varphi\psi \int_b \chi$ . Using this rule, a direct computation shows that the value of  $\int I$  on the left-hand side of relation (1.6) takes the value:

$$\Pi \left( \int \omega; \int I \right) + \sum_{j,k=1}^g 2a_{jk} \left\{ \Pi \left( \int \omega dz_j; \int d\bar{z}_k \right) - \Pi \left( \int \omega \int d\bar{z}_k; \int dz_j \right) - \Pi \left( \int \omega; \int dz_j \int d\bar{z}_k \right) \right\}.$$

According to our observation 1.3.1 this has to be equal to the value of the homotopy functional  $\int I$  applied to the right-hand side of (1.6). We compute

this value as follows. From  $\wedge \mathfrak{X} + d\mu_q = 0$  we can determine the shape of  $\mu_q$ . Using Stokes' theorem, a standard argument shows that there is a simply connected holomorphic coordinate plot  $(U, z)$  on  $\bar{X}$  containing  $q$  and all of a representing path for  $\delta \in \pi_1(X, p)$  such that on  $U$  we may write  $\mu_q = \frac{2g}{2\pi i} \frac{dz}{z} + \varphi$ , where  $\varphi$  is a smooth (non-closed) 1-form in  $E^1(U)$ . Since this representative of  $\delta$  is 0-homotopic in  $U$ , the homotopy functional  $\sum_{j,k=1}^g 2a_{jk} \int \omega dz_j d\bar{z}_k + \omega\varphi$  vanishes on it. Consequently  $\int_\delta I = \int_\delta \omega(\frac{2g}{2\pi i} \frac{dz}{z}) = 2g \int_p^q \omega$ . Putting all ingredients together provides the proof.  $\square$

**1.4. A higher period relation.** Recall that our period matrix  $\Omega$  is of the form  $(I, Z)$  where  $Z$  is symmetric and has positive imaginary part. Like before set  $A = (\bar{Z} - Z)^{-1}$ . Define for  $i = 1, \dots, g$  the  $g \times g$ -matrices

$$I_1^i := \left( \int_{c_\nu} dz_i dz_j \right)_{\nu,j} \quad \text{and} \quad I_2^i := \left( \int_{c_{g+\nu}} dz_i dz_j \right)_{\nu,j} \in \text{Mat}(g \times g; \mathbb{C}).$$

Then we define the following two vectors with entries in  $\text{Mat}(g \times g; \mathbb{C})$ :

$$I_1 = \begin{pmatrix} I_1^1 \\ \vdots \\ I_1^g \end{pmatrix}, \quad I_2 = \begin{pmatrix} I_2^1 \\ \vdots \\ I_2^g \end{pmatrix} \in \text{Mat}(g \times 1; \text{Mat}(g \times g)).$$

For a matrix  $M$ , denote by  $\text{tr } M$  the trace of  $M$  and by  $\text{diag } M$  its diagonal. Define the trace of a vector of matrices to be the vector consisting of the traces of its components. The following theorem is the announced higher period relation.

**Theorem 1.6.** *With the above notation, we have*

$$\begin{aligned} & (2 \text{tr}(I_2 A) - 2 \text{tr}(I_1 A Z)) + (\text{diag}(Z A Z) - Z \text{diag}(A Z)) \\ & + (\text{diag}(Z A) - Z \text{diag}(A)) + (\text{diag}(A Z) - Z \text{diag}(Z A)) \equiv 0 \pmod{(I, Z)\mathbb{Z}^{2g}}. \end{aligned}$$

*Proof.* Apply the homotopy functional  $\sum_{j,k=1}^g a_{jk} \int dz_j dz_i dz_k$  to (1.6).  $\square$

We use this higher period relation to continue our computation of the extension  $k_{pq}$ . After Theorem 1.4 it makes sense to speak of  $k_{pp}$ ; we have

$$\Psi(k_{pq}) = 2g(q - p) + \Psi(k_{pp}).$$

In the above introduced notation, Theorem 1.4 tells us that  $u \circ \Psi(k_{pp}) \in \mathbb{C}^g / \Omega \mathbb{Z}^{2g}$  can be written as

$$\begin{aligned} u \circ \Psi(k_{pp}) &= \text{diag}(Z A \bar{Z}) - Z \text{diag}(A) \\ &+ 2 \text{diag}(Z A) - 2Z \text{diag}(A \bar{Z}) - 2 \text{tr}(I_1 A \bar{Z}) + 2 \text{tr}(I_2 A). \end{aligned}$$

Transform this expression such that it only contains (iterated) integrals over holomorphic forms. Observe  $\text{diag}(Z A \bar{Z}) = \text{diag}(Z(\bar{Z} - Z)^{-1}(\bar{Z} - Z)) + \text{diag}(Z A Z) = \text{diag}(Z) + \text{diag}(Z A Z)$  and similarly  $2Z \text{diag}(A \bar{Z}) \equiv 2Z \text{diag}(A Z) \pmod{(I, Z)\mathbb{Z}^{2g}}$  and  $2 \text{tr}(I_1 A \bar{Z}) = 2 \text{tr}(I_1) + 2 \text{tr}(I_1 A Z)$ . Using these identities we continue

$$\begin{aligned} u \circ \Psi(k_{pp}) &\equiv \text{diag}(Z) + \text{diag}(Z A Z) - Z \text{diag}(A) \\ &+ 2 \text{diag}(Z A) - 2Z \text{diag}(A Z) \\ &- 2 \text{tr}(I_1) - 2 \text{tr}(I_1 A Z) + 2 \text{tr}(I_2 A) \pmod{(I, Z)\mathbb{Z}^{2g}}. \end{aligned}$$

Notice:  $\text{diag}(ZA) - \text{diag}(AZ) = 0$ . When we apply Theorem 1.6, we finally get

$$u \circ \Psi(k_{pp}) \equiv \text{diag } Z - 2 \text{tr}(I_1) \pmod{(I, Z)\mathbb{Z}^{2g}}.$$

Writing this out, we find  $u \circ \Psi(k_{pp}) \equiv 2\kappa_p \pmod{(I, Z)\mathbb{Z}^{2g}}$  by virtue of formula (1.5). This is the proof of Theorem 1.3.

2. A POINTED TORELLI THEOREM FOR PUNCTURED RIEMANN SURFACES

Here we want to show that the extension  $w_{pq}$  or respectively  $m_{pq}$  determines  $p$  and  $q$ . Finally, we will combine this with results of Hain and Pulte [Hai87b], [Pul88], which we briefly sketch first.

**2.1. The pointed Torelli theorem.** The pointed Torelli theorem of Hain and Pulte is based on the following.

**Theorem 2.1** (Hain, Pulte). *The map from  $\text{Pic}^0 \bar{X}$  to  $\text{Ext}_{\text{MHS}}(K; H^1)$  which maps  $(p - p')$  to  $m_p - m_{p'}$  is well-defined and injective.*

We write  $(\bar{X}, p) \cong (\bar{X}, p')$  if there is an automorphism  $\phi : \bar{X} \rightarrow \bar{X}$  that maps  $p$  to  $p'$ . For a point  $p$  on  $\bar{X}$  we define the set of alternatives for  $p$  as

$$a_{\bar{X}}(p) := \{p\} \cup \{p' \in \bar{X} \mid m_{p'} = -m_p \text{ in } \text{Ext}_{\text{MHS}}(K; H^1) \text{ and } (\bar{X}, p) \not\cong (\bar{X}, p')\}.$$

The following is a consequence of Theorem 2.1. Let us give a short proof of it.

**Corollary 2.2.** *For a pointed compact Riemann surface  $(\bar{X}, p)$ , the set  $a_{\bar{X}}(p)$  consists of at most two points. Up to automorphism of  $\bar{X}$ , there cannot be more than one pair of different points  $\{p, p'\}$  on  $\bar{X}$  such that  $a_{\bar{X}}(p) = \{p, p'\} = a_{\bar{X}}(p')$ .*

*Proof.* The first assertion is an obvious consequence of 2.1. To prove the second assertion, assume that  $\tilde{p}$  and  $\tilde{p}'$  is another such pair with  $a_{\bar{X}}(\tilde{p}) = \{\tilde{p}, \tilde{p}'\} = a_{\bar{X}}(\tilde{p})$ . Then by 2.1, the divisors  $p + p' = \tilde{p} + \tilde{p}'$  are linearly equivalent. It follows that either  $\{p, p'\} = \{\tilde{p}, \tilde{p}'\}$  or  $\bar{X}$  is hyperelliptic and the hyperelliptic involution maps  $p$  to  $p'$  and  $q$  to  $q'$ , which contradicts the assumptions on  $p, p'$  and  $q, q'$ .  $\square$

Together with the classical Torelli theorem, Hain and Pulte used Theorem 2.1 to prove the following *pointed Torelli theorem*. For a pointed compact Riemann surface  $(\bar{Z}, z_0)$  denote by  $J_{z_0}(\bar{Z})$  the augmentation ideal in  $\mathbb{Z}\pi_1(\bar{Z}, z_0)$ .

**Theorem 2.3** (Hain, Pulte). *Suppose that  $(\bar{X}, p)$  and  $(\bar{Y}, r)$  are two pointed compact Riemann surfaces. If there is a ring homomorphism*

$$\mathbb{Z}\pi_1(\bar{X}, p) / J_p(\bar{X})^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(\bar{Y}, r) / J_r(\bar{Y})^3$$

*which induces an isomorphism of MHS's, then there is an isomorphism  $f : \bar{X} \rightarrow \bar{Y}$  with  $f(p) \in a_{\bar{Y}}(r)$ .*

*Remark 2.4.* As far as the author knows, still no example is known of a pointed compact Riemann surface  $(\bar{X}, p)$  with  $|a_{\bar{X}}(p)| = 2$ . M. Pulte [Pul88] has shown that such an  $(\bar{X}, p)$  with  $|a_{\bar{X}}(p)| = 2$  must have zero harmonic volume. B. Harris [Har83] proved that a generic smooth projective complex curve has nonzero harmonic volume. Moreover, Pulte showed (loc. cit.) that, if there are two points  $p, p'$  with  $a_{\bar{X}}(p) = \{p, p'\} = a_{\bar{X}}(p')$ , then  $(g - 1)(p + p') - K = 0 \in \text{Pic}^0 \bar{X}$ , where  $K$  is the canonical divisor. For pointed hyperelliptic curves  $(\bar{X}, p)$  always holds:  $a_{\bar{X}}(p) = \{p\}$ , since here  $m_p = -m_{p'}$  implies  $(\bar{X}, p) \cong (\bar{X}, p')$  by the hyperelliptic involution.

**2.2. A punctured pointed Torelli theorem.** The following theorem will follow directly from Lemma 2.9, which we prove at the end of this section.

**Theorem 2.5.** *For all  $p \in \bar{X}$ , the map  $\text{Pic}^0 \bar{X} \rightarrow \text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$  which maps  $(q - q')$  to  $m_{pq} - m_{pq'}$  is well-defined and injective.*

Let  $\Delta$  be the diagonal in  $\bar{X} \times \bar{X}$ . Combining Theorem 2.5 with the results of Hain and Pulte we find

**Proposition 2.6.** *The map from  $(\bar{X} \times \bar{X}) \setminus \Delta$  to  $\text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$  given by  $(p, q) \mapsto m_{pq}$  is well-defined, extends to the diagonal  $\Delta$  and is injective.*

*Proof of 2.6.* Note that the map of complex tori

$$\text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \rightarrow \text{Ext}_{\text{MHS}}(K \oplus Q; H^1)$$

is a covering map, since  $\text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}} \hookrightarrow \text{Hom}(K \oplus Q; H^1)_{\mathbb{Z}}$ . Moreover, we have the commutative diagram:

$$\begin{CD} (\bar{X} \times \bar{X}) \setminus \Delta @>\tilde{\varphi}>> \text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \\ @VVV @VV\text{covering map}V \\ \bar{X} \times \bar{X} @>\varphi>> \text{Ext}_{\text{MHS}}(K; H^1) \oplus \text{Pic}^0 \bar{X} \\ (p, q) @>>> (m_p, (2gq - 2p - K)). \end{CD}$$

The map  $\varphi$  is continuous ( $m_p$  is — in a coordinate system — an expression of iterated integrals over paths with basepoint  $p$ ). As the map  $\tilde{\varphi}(p, q) = m_{pq}$  is a lifting of  $\varphi$ , we see that  $\tilde{\varphi}$  is continuous too. The fact that the map  $m_{pq} \mapsto (m_p, k_{pq})$  is a covering map tells us moreover that we may extend  $\tilde{\varphi}$  to the diagonal  $\Delta$ . Since the extension  $m_{pq}$  determines  $m_p$  it determines by Theorem 2.1 of Hain and Pulte also  $p$ . By virtue of Theorem 2.5 it determines  $q$ . □

Pulling back the intersection form  $H_1(\bar{X}, \mathbb{Z}) \otimes H_1(\bar{X}, \mathbb{Z}) \rightarrow \mathbb{Z}$  along the natural isomorphism  $J/J^2 \xrightarrow{\cong} \bar{J}/\bar{J}^2$  induces a polarization on  $\text{Gr}_{-1}^W(J/J^3) = H^1(X)$ . We can also put a polarization on  $\text{Gr}_{-2}^W(J/J^3) = J^2/J^3 \cong J/J^2 \otimes J/J^2 = \text{Gr}_{-1}^W(J/J^3) \otimes \text{Gr}_{-1}^W(J/J^3)$ , by taking the tensor product of the polarized Hodge structure  $H^1(X)$  in the category of polarized Hodge structures. In that sense,  $J/J^3$  becomes a *graded polarized MHS*, i.e. each  $\text{Gr}_l^W$  is a polarized Hodge structure.

For points  $p$  and  $q$  on  $\bar{X}$  we define

$$A_{\bar{X}}(p, q) := \{(p, q)\} \cup \left\{ (p', q') \in \bar{X} \times \bar{X} \left| \begin{array}{l} m_{p'q'} = -m_{pq} \text{ in } \text{Ext}_{\text{MHS}}((H^1)^{\otimes 2}; H^1) \\ \text{and } (\bar{X} \setminus \{q\}, p) \not\cong (\bar{X} \setminus \{q'\}, p') \end{array} \right. \right\}.$$

The following is then a consequence of Proposition 2.6.

**Corollary 2.7.**  *$A_{\bar{X}}(p, q)$  consists of at most two elements.* □

Our results lead to the following *punctured pointed Torelli theorem*.

**Theorem 2.8.** *Suppose that  $(\bar{X} \setminus \{q\}, p)$  and  $(\bar{Y} \setminus \{s\}, r)$  are two punctured compact Riemann surfaces with basepoint. If there is a ring isomorphism*

$$\mathbb{Z}\pi_1(\bar{X} \setminus \{q\}, p) / J_p(\bar{X} \setminus \{q\})^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(\bar{Y} \setminus \{s\}, r) / J_r(\bar{Y} \setminus \{s\})^3,$$

*which induces an isomorphism of graded polarized MHS's, then there is a biholomorphism  $f : \bar{X} \rightarrow \bar{Y}$  with  $(f(p), f(q)) \in A_{\bar{Y}}(r, s)$ .*

*Proof of 2.8.* The proof goes along the lines of the proof of the pointed Torelli theorem in [Pul88] and [Hai87b]. Let  $J_{pq} = J_p(\bar{X} \setminus \{q\})$  and  $J_{rs} = J_r(\bar{Y} \setminus \{s\})$ . We have an isomorphism of MHS's,  $\lambda : J_{pq}/J_{pq}^3 \xrightarrow{\cong} J_{rs}/J_{rs}^3$  and in particular,  $\lambda$  induces an isomorphism of polarized Hodge structures

$$\lambda^* : H^1(\bar{Y}) = W_1(J_{rs}/J_{rs}^3)^* \rightarrow W_1(J_{pq}/J_{pq}^3)^* = H^1(\bar{X}).$$

By the classical Torelli theorem (cf. for instance [Mar63]) we know that there is a biholomorphism  $f : \bar{X} \rightarrow \bar{Y}$  such that  $f^* : H^1(\bar{Y}) \rightarrow H^1(\bar{X})$  is  $\pm\lambda^*$ . Since  $\lambda$  respects the ring structure, the  $\lambda$  induced map  $(J_{rs}^2/J_{rs}^3)^* \rightarrow (J_{pq}^2/J_{pq}^3)^*$  is determined by  $\lambda^* : H^1(\bar{Y}) \rightarrow H^1(\bar{X})$  and hence,

$$f^* : (J_{rs}^2/J_{rs}^3)^* = H^1(\bar{Y}) \otimes H^1(\bar{Y}) \rightarrow H^1(\bar{X}) \otimes H^1(\bar{X}) = (J_{pq}^2/J_{pq}^3)^*$$

is equal to  $\lambda^* \otimes \lambda^*$ . Without loss of generality, we may therefore assume that  $(\bar{Y} \setminus \{s\}, r) = (\bar{X} \setminus \{q'\}, p')$  for two points  $p'$  and  $q'$  in  $\bar{X}$  and that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & (J_{pq}/J_{pq}^3)^* & \longrightarrow & H^1 \otimes H^1 \longrightarrow 0 \\ & & \pm id \downarrow & & \downarrow \lambda^* & & \downarrow id \\ 0 & \longrightarrow & H^1 & \longrightarrow & (J_{p'q'}/J_{p'q'}^3)^* & \longrightarrow & H^1 \otimes H^1 \longrightarrow 0. \end{array}$$

It follows that  $m_{pq} = \pm m_{p'q'}$ . This means that there either is an automorphism  $\phi : (\bar{X} \setminus \{q\}, p) \rightarrow (\bar{X} \setminus \{q'\}, p')$  or  $A_{\bar{X}}(p, q) = \{(p, q); (p', q')\} = A_{\bar{X}}(p', q')$ . In both cases, the identity map is the map with the desired properties.  $\square$

**2.3. A technical lemma.** Theorem 2.5 is a consequence of the following:

**Lemma 2.9.** *For each element  $\sum_i (q_i - q'_i) \in \text{Pic}^0 \bar{X}$ , we have*

$$\sum_i (q_i - q'_i) = 0 \in \text{Pic}^0 \bar{X} \Leftrightarrow \sum_i (m_{pq_i} - m_{pq'_i}) = 0 \in \text{Ext}_{\text{MHS}}((H^1)^{\otimes 2}; H^1).$$

*Proof.* Consider the isomorphism (cf. [Car80]) from  $\text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$  to

$$\text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} / (F^0 \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} + \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}}).$$

The image of an extension  $m_{pq}$  is  $[\phi_{pq}]$  for a certain  $\phi_{pq} \in \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}}$ , which we now explain. On an element  $[\varphi] \otimes [\psi] \in H^1 \otimes H^1$ , the homomorphism  $\phi_{pq}$  has the following property. There is an  $\eta_q \in F^1 E^1(X \log q)$  such that  $\varphi \wedge \psi + d\eta_q = 0$  and  $\phi_{pq}([\varphi] \otimes [\psi]) = \sum_{j=1}^{2g} (\int_{\gamma_j} \varphi \psi + \mu_q) [dx_j]$ . If  $[\varphi] \otimes [\psi] \in K$ , then  $\eta_q$  can be chosen in  $F^1 E^1(X)$  and does not depend on  $q$ , which shows that  $(\phi_{pq} - \phi_{p'q'})$  is zero on  $K$ . Therefore it is determined by its value on one element of  $(H^1 \otimes H^1) \setminus K$ ; for instance on  $[dx_1] \otimes [dx_{g+1}]$ .

Given a divisor  $D = \sum_i (q_i - q'_i)$  define the homomorphism  $\Phi_D := \sum_i (\phi_{pq_i} - \phi_{p'q'_i}) : H^1 \otimes H^1 \rightarrow H^1$ . We will derive a series of equivalences. First, we have:

$$\begin{aligned} \sum_i (m_{pq_i} - m_{p'q'_i}) = 0 \in \text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \\ \Leftrightarrow \Phi_D \in F^0 \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} + \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}}. \end{aligned}$$

Now let  $\mathbf{w} \in H^{0,1} \otimes H^{0,1}$  be such that  $[dx_1] \otimes [dx_{g+1}] - \mathbf{w} \in F^1(H^1 \otimes H^1) = H^{1,0} \otimes H^1 + H^1 \otimes H^{1,0}$ . Note that  $H^{0,1} \otimes H^{0,1} \subset K$  and hence  $\Phi_D(\mathbf{w}) = 0$ . Moreover,

$H^{1,0} \otimes H^{1,0} \subset K$  and  $\Phi_D (H^{1,0} \otimes H^{1,0}) = 0$ . Therefore, we may continue the series of equivalences by

$$\Leftrightarrow \Phi_D ([dx_1] \otimes [dx_{g+1}] - \mathbf{w}) \in H^{1,0} + H_{\mathbb{Z}}^1 \Leftrightarrow \Phi_D ([dx_1] \otimes [dx_{g+1}]) \in H^{1,0} + H_{\mathbb{Z}}^1.$$

Let  $\eta_{q_i} \in F^1 E^1(X \log q_i)$  and  $\eta_{q'_i} \in F^1 E^1(X \log q'_i)$  be such that  $dx_1 \wedge dx_{g+1} + d\eta_{q_i} = 0$  and  $dx_1 \wedge dx_{g+1} + d\eta_{q'_i} = 0$ . Note that this implies  $\text{Res}_{q_i} \eta_{q_i} = \frac{1}{2\pi i} = \text{Res}_{q'_i} \eta_{q'_i}$ . Then a direct computation shows that we may go on:

$$\begin{aligned} &\Leftrightarrow \sum_{j=1}^{2g} \sum_i \left( \int_{\gamma_j} \mu_{q_i} - \mu_{q'_i} \right) [dx_j] \in H^{1,0} + H_{\mathbb{Z}}^1 \\ &\Leftrightarrow \left( \Pi \left( \int dz_{\nu}; \int (\mu_{q_i} - \mu_{q'_i}) \right) \right)_{\nu} \equiv 0 \pmod{\Omega \mathbb{Z}^{2g}}. \end{aligned}$$

By the reciprocity law for differentials of the third kind (cf. [GH78]), we find as  $(\mu_{q_i} - \mu_{q'_i})$  is meromorphic with simple poles  $\Leftrightarrow \sum_i (q_i - q'_i) = 0 \in \text{Pic}^0 \bar{X}$ . That proves the lemma.  $\square$

#### ACKNOWLEDGEMENTS

The contents of this paper were part of my Ph.D. thesis which was accepted at the Catholic University of Nijmegen, NL in October 1997. I am indebted to my advisor Joseph Steenbrink, who posed the question for the above extension of mixed Hodge structures, and to Richard Hain and Eduard Looijenga for interesting discussions on the subject.

#### REFERENCES

- [Car80] J. A. Carlson. Extensions of mixed Hodge structures. In *Journées de Géométrie Algébrique d'Angers*, pages 107–128, Alphen aan den Rijn, 1980. Sijthoff and Noordhoff. MR **82g**:14013
- [Che76] K. T. Chen. Reduced bar constructions on de Rham complexes. In A. Heller and (eds.) M. Tierny, editors, *Algebra, Topology and Category Theory*, pages 19–32, New York, 1976. Academic Press. MR **54**:1272
- [Che77] K. T. Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83:831–879, 1977. MR **56**:13210
- [Fay73] J. D. Fay. *Theta Functions on Riemann Surfaces*. Number 352 in Lecture Notes in Math. Springer Verlag, Berlin, 1973. MR **49**:569
- [GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, Inc., USA, 1978. MR **80b**:14001
- [Gun69] R. C. Gunning. Quadratic periods of hyperelliptic abelian integrals. In *Problems in Analysis*, pages 239–247, New Jersey, 1969. Princeton University Press. MR **50**:7511
- [Hai87a] R. M. Hain. The de Rham homotopy theory of complex algebraic varieties I. *K-Theory*, 1:271–324, 1987. MR **88h**:14029
- [Hai87b] R. M. Hain. The geometry of the mixed Hodge structure on the fundamental group. *Algebraic Geometry, Bowdoin, PSPM*, 46:247–28, 1987. MR **89g**:14010
- [Har83] B. Harris. Harmonic volumes. *Acta Math.*, 150(1–2):91–123, 1983. MR **84k**:32032
- [HL97] R. M. Hain and E. J. N. Looijenga. Mapping class groups and moduli spaces of curves. *Proc. Symp. Pure Math.*, 62(pt.II):97–142, 1997. Algebraic geometry, Santa Cruz. MR **99a**:14032
- [Jab86] E. R. Jablow. Quadratic vector classes on Riemann surfaces. *Duke Math. J.*, 53(1):221–232, 1986. MR **87g**:32026
- [Lan02] E. Landfriedt. *Thetafunktionen und hyperelliptische Funktionen*. Göschen'sche Verlagshandlung, Leipzig, 1902.
- [Lew64] J. Lewittes. Riemann surfaces and the theta function. *Acta Math.*, 111:37–61, 1964. MR **28**:206
- [Mar63] H. H. Martens. A new proof of Torelli's theorem. *Annals of Math.*, 78(1):107–111, 1963. MR **27**:2506

- [Mum83] D. Mumford. *Tata Lectures on Theta I*. Number 28 in Progress in Math. Birkhäuser, Boston, 1983. MR **85h**:14026
- [Pul88] M. Pulte. The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles. *Duke Math. J.*, 57(3):721–760, 1988. MR **89m**:32048
- [PY96] C. Poor and D. S. Yuen. Relations on the period mapping giving extensions of mixed Hodge structures on compact Riemann surfaces. *Geometriae Dedicata*, 59:243–291, 1996. MR **98g**:32033
- [Rie92] B. Riemann. *Bernhard Riemann's gesammelte mathematische Werke*. B. G. Teubner, Leipzig, 1892.

MATHEMATISCHES INSTITUT, HEINRICH-HEINE-UNIVERSITÄT DÜSSELDORF, UNIVERSITÄTS-  
STRASSE, 40225 DÜSSELDORF, GERMANY

*E-mail address:* [kaenders@cs.uni-duesseldorf.de](mailto:kaenders@cs.uni-duesseldorf.de)