

THE EXPECTED L_p NORM OF RANDOM POLYNOMIALS

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ABSTRACT. The results of this paper concern the expected L_p norm of random polynomials on the boundary of the unit disc (equivalently of random trigonometric polynomials on the interval $[0, 2\pi]$). Specifically, for a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

let

$$\|q_n\|_p^p = \int_0^{2\pi} |q_n(\theta)|^p d\theta / (2\pi).$$

Assume the random variables $X_k, k \geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$, a finite p^{th} moment $E(|X_k|^p)$. Then

$$\frac{E(\|q_n\|_p^p)}{n^{p/2}} \rightarrow \Gamma(1 + p/2)$$

and

$$\frac{E(\|q_n^{(r)}\|_p^p)}{n^{(2r+1)p/2}} \rightarrow (2r+1)^{-p/2} \Gamma(1 + p/2)$$

as $n \rightarrow \infty$.

In particular if the polynomials in question have coefficients in the set $\{+1, -1\}$ (a much studied class of polynomials), then we can compute the expected L_p norms of the polynomials and their derivatives

$$\frac{E(\|q_n\|_p)}{n^{1/2}} \rightarrow (\Gamma(1 + p/2))^{1/p}$$

and

$$\frac{E(\|q_n^{(r)}\|_p)}{n^{(2r+1)/2}} \rightarrow (2r+1)^{-1/2} (\Gamma(1 + p/2))^{1/p}.$$

This complements results of Fielding in the $p := 0$ case, Newman and Byrnes in the $p := 4$ case, and Littlewood et al. in the $p = \infty$ case.

1. INTRODUCTION

There are a number of difficult old conjectures that concern the possible rates of growth of polynomials with all coefficients in the set $\{+1, -1\}$. Since many of these were raised by Littlewood we denote the set of such polynomials by \mathcal{L}_n and

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refer to them as Littlewood polynomials. Specifically

$$\mathcal{L}_n := \left\{ p : p(x) = \sum_{j=0}^n a_j x^j, \quad a_j \in \{-1, 1\} \right\}.$$

In [Li-66] Littlewood conjectures that it is possible to find $p_n \in \mathcal{L}_n$ so that

$$C_1 \sqrt{n+1} \leq |p_n(z)| \leq C_2 \sqrt{n+1}$$

for all complex z of modulus 1. Such polynomials are often called “flat”. Because the L_2 norm of a polynomial from \mathcal{L}_n is exactly $\sqrt{n+1}$ the constants must satisfy $C_1 \leq 1$ and $C_2 \geq 1$. This is discussed in some detail in problem 19 of Littlewood’s delightful monograph [Li-68]. A sequence of polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. No sequence is known that satisfies the lower bound.

This conjecture is complemented by a conjecture of Erdős [Er-62] that the constant C_2 is bounded away from 1 (independently of n). This is also still open. Although a remarkable result of Kahane [Kah-85, Kah-80] shows that if the polynomials are allowed to have complex coefficients of modulus 1, then “flat” polynomials exist and indeed that it is possible to make C_1 and C_2 asymptotically arbitrarily close to 1. Equally remarkable is a result of Beck [Bec-91] who proves that “flat” polynomials exist from the class of polynomials of degree n whose coefficients are 400th roots of unity.

The relationship between these problems and Barker polynomials is discussed in [Sa-90]. The most famous problem concerning polynomials with coefficients in the set $\{0, -1, +1\}$ is the celebrated, now resolved, Littlewood conjecture [Ko-80]. It asserts that the L_1 norm of a polynomial $\sum_{j=0}^n \pm x^{k_j}$ must grow at least like $\log(n)$. Here and in what follows the L_p norms are on the boundary of the unit disk in the complex plane.

In [Sa-54] it is shown that for all but $o(2^n)$ Littlewood polynomials the supremum on the unit disc lies between $c_1 \sqrt{n \log n}$ and $c_2 \sqrt{n \log n}$. In fact, Halász [Ha-73] shows that $\lim \|q_n / \sqrt{n \log n}\|_\infty = 1$ almost surely. See also [An-83].

The expected L_4 to the fourth power of a Littlewood polynomial of degree n is computed by Newman and Byrnes [Ne-90]. They show that

$$E(\|p\|_4^4) = 2(n+1)^2 - (n+1)$$

where p is a random element of \mathcal{L}_n .

In the L_0 case Fielding [Fi-70] computes the expected norm (which in this case is the Mahler Measure) over the polynomials with *complex* coefficients of modulus 1. He proves that

$$E(\|p\|_0) \geq \exp(-\gamma/2) \sqrt{n} (1 + O(n^{-1/2+\delta}))$$

where γ is Euler’s constant. See also [Ul-88].

Our principal aim in this paper is to compute the expected L_p norms of Random Littlewood polynomials. The complete results are stated in the next sections. For random Littlewood polynomials, $q_n \in \mathcal{L}_n$, we have

$$\frac{E(\|q_n\|_p)}{n^{1/2}} \rightarrow (\Gamma(1 + p/2))^{1/p}$$

and for their derivatives

$$\frac{\mathbb{E}(\|q_n^{(r)}\|_p)}{n^{(2r+1)/2}} \rightarrow (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

From this and the inequality [Bor-95, p. 406]

$$\frac{\|q'_n\|_p}{n\|q_n\|_p} \leq 1$$

we can deduce an expected Bernstein Inequality for Littlewood polynomials, namely

$$\mathbb{E}\left(\frac{\|q'_n\|_p}{n\|q_n\|_p}\right) \rightarrow \frac{1}{\sqrt{3}}.$$

This should be compared to interesting results of Nazarov and Queffélec and Saffari [Qe-96] which say that

$$\max_{q_n \in \mathcal{L}_n} \frac{\|q'_n\|_p}{n\|q_n\|_p} \rightarrow 1$$

for all $p > 1$ except $p = 2$ where the lim sup is $1/\sqrt{3}$.

Of course, because of the monotonicity of the L_p norms it is relevant to rephrase Littlewood's conjecture in other norms. It has been conjectured that

$$\|p\|_4^4 \geq (7-\delta)n^2/6$$

for $p \in \mathcal{L}_n$ and n sufficiently large. This would be best possible and would imply Erdős' conjecture above. See [Bor-98] for a discussion of this.

Random polynomials have been much looked at, particularly the location of their roots. See for example [Bh-86], [Bri-75] and [Kac-48].

2. RESULTS

Consider a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

for $0 \leq \theta \leq 2\pi$. We study the p^{th} power of the L_p norm of q_n ; that is,

$$\|q_n\|_p^p = \int_0^{2\pi} |q_n(\theta)|^p d\theta/2\pi.$$

Theorem 1. *Fix $0 < p < \infty$. Assume that the random variables $X_k, k \geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$, a finite p^{th} moment $\mathbb{E}(|X_k|^p)$. Then*

$$\frac{\mathbb{E}(\|q_n\|_p^p)}{n^{p/2}} \rightarrow \Gamma(1+p/2)$$

as $n \rightarrow \infty$. If, in addition, $\mathbb{E}(|X_k|^{2p}) < \infty$, then

$$\frac{\|q_n\|_p}{n^{1/2}} \rightarrow \Gamma(1+p/2)^{1/p}$$

in probability and

$$\frac{\mathbb{E}(\|q_n\|_p)}{n^{1/2}} \rightarrow (\Gamma(1+p/2))^{1/p}.$$

For ± 1 coefficients randomly chosen the moment conditions are trivially satisfied for all finite p . Numerical confirmation of the principle results based on computations up to degree 24 is presented in Figures 1 and 2. The calculations are courtesy of Lesley Robinson [Ro-97]. Figure 1 shows the average L_3 and L_4 norms (normalized by division by $\sqrt{n+1}$) for the Littlewood polynomials up to degree 24. Figure 2 is a similar graph for L_3^3 and L_4^4 (normalized by division by $(n+1)^{3/2}$ and $(n+1)^2$ respectively).

Proof. In what follows all unlabelled sums run from 0 through $n-1$. Define

$$\sigma_{n,c}^2(\theta) = \sum \cos^2(j\theta)$$

and

$$\sigma_{n,s}^2(\theta) = \sum \sin^2(j\theta)$$

and write

$$a_{k,n}(\theta) = \frac{\cos(k\theta)}{\sigma_{n,c}(\theta)}$$

and, for θ not a multiple of π ,

$$b_{k,n}(\theta) = \frac{\sin(k\theta)}{\sigma_{n,s}(\theta)}.$$

Then

Lemma 2.1. *There is a constant M , free of n , k and θ such that*

$$|a_{k,n}(\theta)| + |b_{k,n}(\theta)| \leq \frac{M}{\sqrt{n}}.$$

Postpone the proof of the lemma and consider an arbitrary sequence θ_n . The lemma permits application of the Lindeberg central limit theorem to show that for an arbitrary sequence θ_n

$$C_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \cos(k\theta_n)}{\sigma_{n,c}(\theta_n)} = \sum_0^{n-1} a_{k,n}(\theta_n) X_k$$

converges in distribution to standard normal and, provided no θ_n is an integer multiple of π ,

$$S_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \sin(k\theta_n)}{\sigma_{n,s}(\theta_n)}$$

converges in distribution to standard normal. Moreover, for any fixed θ not an integer multiple of π , the elementary convergences $\sigma_{n,c}^2/n \rightarrow 1/2$, $\sigma_{n,s}^2/n \rightarrow 1/2$ and $\sum \cos(k\theta) \sin(k\theta)/n \rightarrow 0$ show that the covariance matrix of $(C_n(\theta), S_n(\theta))$ converges to the 2-by-2 identity so that $(C_n(\theta), S_n(\theta))$ converges in distribution to (Z_1, Z_2) where the Z_i are independent standard normals. It follows for such θ that $|q_n(\theta)|^2/n$ converges in distribution to $(Z_1^2 + Z_2^2)/2$ which has a standard exponential distribution.

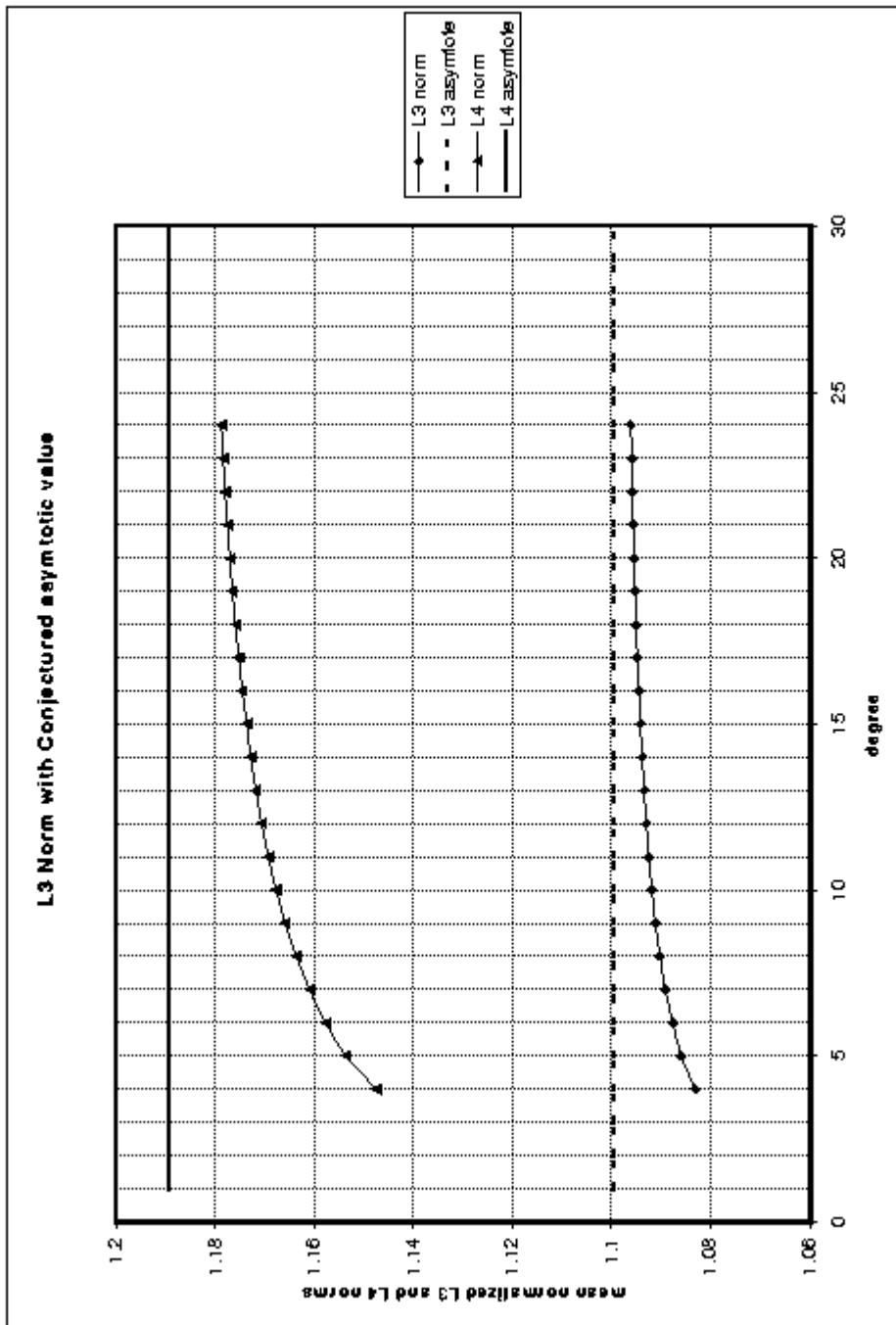


FIGURE 1.

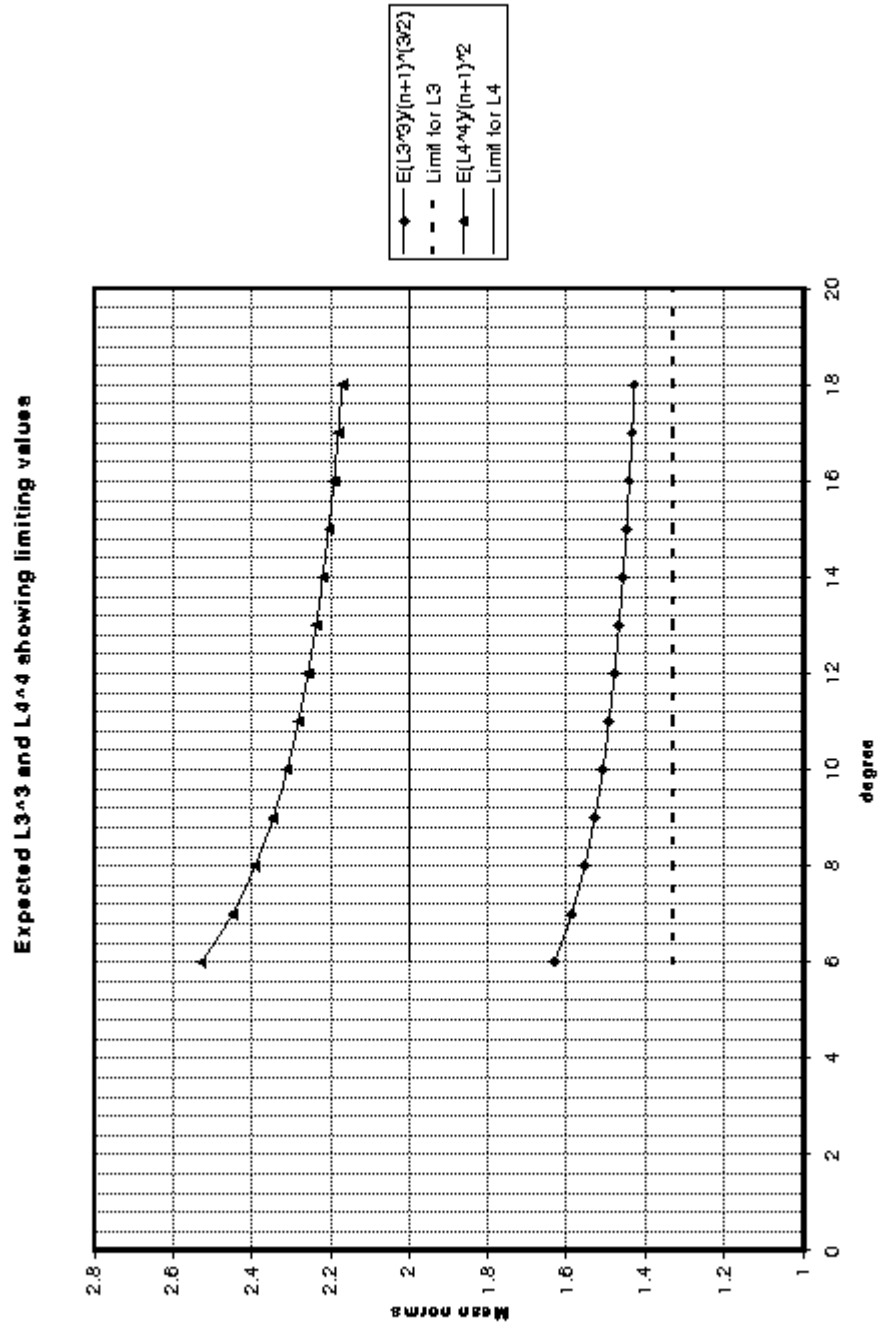


FIGURE 2.

For $p > 2$, a theorem of Bernstein ([Be-39], [Br-70]) asserts that, in the central limit theorem, p^{th} moments converge to the p^{th} moment of the normal distribution provided an analogue of the Lindeberg condition holds. In particular,

$$E(|C_n(\theta_n)|^p) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u|^p \exp(-u^2/2) du$$

provided

$$\sum E(|a_{k,n}|^p |X_k|^p 1(a_{k,n}^2 X_k^2 > \eta)) \rightarrow 0$$

for each positive η (and analogously for S_n replacing $a_{k,n}$ by $b_{k,n}$). (For $p \leq 2$ we could proceed straight from the central limit theorem.) The quantity in question is bounded by

$$\sum |a_{k,n}|^p E(|X_0|^p 1(X_0^2 > n\eta/M^2)) \leq \sum |a_{k,n}|^p E(|X_0|^p 1(X_0^2 > n\eta/M^2)).$$

But $\sum |a_{k,n}|^p$ is, for $p \geq 2$, bounded by $\sum a_{k,n}^2 = 1$ and X_0 has a finite p^{th} moment so the bound converges to 0.

Now note

$$\begin{aligned} |q_n(\theta)|^p/n^{p/2} &= \left(\frac{\sigma_{n,c}^2(\theta)C_n^2(\theta) + \sigma_{n,s}^2(\theta)S_n^2(\theta)}{n} \right)^{p/2} \\ &\leq 2^{p/2-1} \left[\left(\frac{\sigma_{n,c}^2(\theta)}{n} \right)^{p/2} |C_n(\theta)|^p + \left(\frac{\sigma_{n,s}^2(\theta)}{n} \right)^{p/2} |S_n(\theta)|^p \right] \\ &\leq 2^{p/2-1} [|C_n(\theta)|^p + |S_n(\theta)|^p]. \end{aligned}$$

Putting $g_n(\theta) = E(|q_n(\theta)|^p/n^{p/2})$ and $h_n(\theta) = 2^{p/2-1}E(|C_n(\theta)|^p + |S_n(\theta)|^p)$ we see that $0 \leq g_n \leq h_n$ almost everywhere and that h_n converges uniformly on $(0, \pi) \cup (\pi, 2\pi)$ to $2^{p/2}E(|Z|^p)$ where Z is standard normal. If we establish that g_n converges almost everywhere to $\Gamma(1 + p/2)$, then the theorem will follow by dominated convergence.

To do so we can apply the following result. Suppose U_n and V_n are random variables with U_n converging in distribution to some U , V_n converging in distribution to some V and $0 \leq U_n \leq V_n$. If $E(V_n) \rightarrow E(V) < \infty$, then $E(U_n) \rightarrow E(U)$. (This is the Dominated Convergence Theorem for convergence in distribution.)

Use the result with $U_n = |q_n(\theta)|^p/n^{p/2}$, which converges in distribution, for θ not an integer multiple of π , to $U = ((Z_1^2 + Z_2^2)/2)^{p/2}$, and

$$V_n = 2^{p/2-1} [|C_n(\theta_n)|^p + |S_n(\theta_n)|^p]$$

which converges in distribution to $V = 2^{p/2-1} (|Z_1|^p + |Z_2|^p)$. Since $E(U) = \Gamma(1 + p/2)$ we are done.

To establish the convergence in probability we compute the variance of $\|q_n\|_p^p/n^{p/2}$ and show that this converges to zero. It suffices to show that

$$\frac{E(\|q_n\|_p^{2p})}{n^p} \rightarrow \Gamma^2(1 + p/2).$$

However,

$$\frac{E(\|q_n\|_p^{2p})}{n^p} = \int_0^{2\pi} \int_0^{2\pi} g_n^{(*)}(\theta_1, \theta_2) d\theta_1 d\theta_2,$$

where $g_n^{(*)}(\theta_1, \theta_2) = E(|q_n(\theta_1)q_n(\theta_2)|^p)/n^p$. For fixed θ_1 and θ_2 , neither an integer multiple of π , it is easily checked that the variance covariance matrix of $(C_n(\theta_1), S_n(\theta_1), C_n(\theta_2), S_n(\theta_2))$ converges to the 4-by-4 identity matrix. In view of the Lindeberg condition already checked the random vector converges in distribution to the standard normal and $|q_n(\theta_1)q_n(\theta_2)|^p/n^p$ converges in distribution to $(Z_1^2 + Z_2^2)^p(Z_3^2 + Z_4^2)^p/2^{2p}$ where Z_1, \dots, Z_4 are independent standard normal. Since

$$g_n^{(*)}(\theta_1, \theta_2) \leq h_n^{(*)}(\theta_1, \theta_2) \equiv \left(\frac{E(|q_n(\theta_1)|^{2p})}{n^p} + \frac{E(|q_n(\theta_2)|^{2p})}{n^p} \right) / 2$$

we may apply the first part of the theorem and the dominated convergence theorem to conclude that $g_n^{(*)}$ converges almost everywhere to $\Gamma^2(1 + p/2)$. Moreover we have already checked that $h_n^{(2)}$ converges almost everywhere to $\Gamma(1 + p)$ and that

$$\int_0^{2\pi} \int_0^{2\pi} h_n^{(*)}(\theta_1, \theta_2) d\theta_1 d\theta_2 / (4\pi^2) \rightarrow \Gamma(1 + p).$$

Hence

$$\int_0^{2\pi} \int_0^{2\pi} g_n^{(*)}(\theta_1, \theta_2) d\theta_1 d\theta_2 / (4\pi^2) \rightarrow \Gamma^2(1 + p/2)$$

by dominated convergence. This establishes the convergence in probability.

The final statement of the theorem is simply dominated convergence applied via the elementary inequality $\|q_n/n^{1/2}\|_p \leq 1 + \|q_n/n^{1/2}\|_p^p$. □

Proof of Lemma. For $\theta \leq \pi/(2n)$ and $1 \leq j \leq n - 1$ we have $2j\theta/\pi \leq \sin(j\theta) \leq j\theta$ whence

$$|b_{k,n}| \leq \frac{\pi k \theta}{2\sqrt{\sum_{j=1}^{n-1} j^2 \theta^2}} = \frac{\sqrt{6}\pi k}{2\sqrt{(n-1)n(2n-1)}} \leq \frac{\sqrt{6}\pi/2}{\sqrt{n}}.$$

Moreover, on $[0, \pi/(2n)]$ the function $\cos(k\theta)$ is monotone decreasing for all $k \leq n$. Hence

$$\begin{aligned} |a_{k,n}(\theta)| &\leq \frac{1}{\sigma_{n,c}(\theta)} \\ &\leq \frac{1}{\sigma_{n,c}(\pi/(2n))} \\ &= \sqrt{\frac{2}{n+1}} \\ &\leq \frac{\sqrt{2}}{\sqrt{n}}. \end{aligned}$$

On the other hand if $\theta > \pi/(2n)$, then

$$\sum \{ \cos^2(k\theta) - \sin^2(k\theta) \} = \sin^2(n\theta) + \sin(n\theta) \cos(\theta) \cos(n\theta) / \sin(\theta).$$

Since

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \geq \theta$$

we see that for $\pi/(2n) \leq \theta \leq \pi/2$ we have

$$\left| \sum \{ \cos^2(k\theta) - \sin^2(k\theta) \} \right| \leq 1 + 1/\tan(\theta) \leq 1 + 2n/\pi \leq \delta_0 n$$

for a $\delta_0 < 1$ and all $n \geq 3$. It follows that

$$\sum \cos^2(k\theta) \geq \frac{1 - \delta_0}{2}n$$

and

$$\sum \sin^2(k\theta) \geq \frac{1 - \delta_0}{2}n$$

from which the lemma follows for $0 \leq \theta \leq \pi/2$. Use easy symmetries of the weights $a_{k,n}$ and $b_{k,n}$ to get all values of θ . \square

Similar techniques permit the extension of our main result to the derivative $q_n^{(r)}$ of order r .

Theorem 2. Fix $0 < p < \infty$. Assume that the random variables $X_k, k \geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$, a finite p^{th} moment $E(|X_k|^p)$. Then

$$\frac{E(\|q_n^{(r)}\|_p^p)}{n^{(2r+1)p/2}} \rightarrow (2r+1)^{-p/2}\Gamma(1+p/2)$$

as $n \rightarrow \infty$. If, in addition, $E(|X_k|^{2p}) < \infty$, then

$$\frac{\|q_n^{(r)}\|_p}{n^{(2r+1)/2}} \rightarrow (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}$$

in probability and

$$\frac{E(\|q_n^{(r)}\|_p^p)}{n^{(2r+1)/2}} \rightarrow (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

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