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THE ACTION OF $SL(2,\mathbb{Z})$ ON THE SUBSETS OF $\mathbb{Z}^2$

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ABSTRACT. We prove that the orbit equivalence relation of the canonical action of $SL(2,\mathbb{Z})$ on the subsets of $\mathbb{Z}^2$ is a universal countable Borel equivalence relation.

1. INTRODUCTION

The study of Borel equivalence relations with respect to Borel reducibility provides a scale to precisely measure the difficulty of various classification problems (see, e.g., [HK]). In this study classification problems are usually identified, via some coding, with Borel equivalence relations on some Polish spaces. Then different Borel equivalence relations are compared to each other in the partial order of Borel reducibility. For Borel equivalence relations E on X and F on Y, where X and Y are Polish spaces, we say that E is *Borel reducible to* F, denoted by $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y), \text{ for } x, y \in X.$$

Thus, if E is Borel reducible to F, then we have concrete evidence of the fact that the classification problem related to E is easier than that related to F, because any method of assigning complete invariants to the F-classes also gives rise to a classification of the E-classes.

Countable Borel equivalence relations are Borel equivalence relations all of whose classes are countable. An equivalence relation is *essentially countable* if it is Borel reducible to a countable Borel equivalence relation. Essentially countable relations are very common in mathematics. For example, all Borel actions of second countable locally compact groups give rise to essentially countable relations ([Ke1]). [JKL] contains a lot of information about the Borel reducibility among countable Borel equivalence relations. One of the most noticeable facts (which can also be found in [DJK]) is that there is a *universal* countable Borel equivalence relation, i.e., there is a relation E_∞ such that for any countable Borel equivalence relation E, we have $E \leq_B E_\infty$.

One realization of the universal countable equivalence relation E_∞ is the orbit equivalence relation of the shift action of F_2 on the space of its subsets, where F_2 is the free group with 2 generators (see [DJK]). In this paper we give another

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realization, namely the orbit equivalence relation of the canonical action of $SL(2,\mathbb{Z})$ on the subsets of $\mathbb{Z}^2$.

The question whether this relation is universal countable was raised by A. S. Kechris. The original hope was probably to relate this equivalence relation with the classification problem for torsion-free abelian groups of rank 2 (as discussed in [Hj] and [Ke2]).

2. Preliminaries about $SL(2,\mathbb{Z})$

Recall that elements of $SL(2,\mathbb{Z})$ can be represented by square matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } ad-bc=1.$$

For each such matrix A, let us associate a transformation $\lambda(A)$ from $\mathbb{C}\cup\{\infty\}$ to $\mathbb{C}\cup\{\infty\}$ by

$$\lambda(A)=\lambda(a,b,c,d)=\frac{az+b}{cz+d}, \text{ for } z\in\mathbb{C}\cup\{\infty\}.$$

Then the map $\lambda : A\mapsto\lambda(A)$ is a homomorphism from the multiplicative group of $SL(2,\mathbb{Z})$ to the group of all transformations $\Lambda=\{\lambda(a,b,c,d)\,|\,ad-bc=1\}$ with composition as its group operation.

Among the elements of $SL(2,\mathbb{Z})$ we identify some important matrices:

$$U=\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},\ V=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ T=\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},$$

and, in addition, I for the identity matrix. We let H be the subgroup of $SL(2,\mathbb{Z})$ generated by U and V.

Lemma 1. *H is a free group with generators U and V.*

This is a well-known fact. Let us recall the proof here.

Proof. By the above observation on the homomorphism from $SL(2,\mathbb{Z})$ to Λ it is enough to verify that

$$\lambda(U)=\frac{z}{2z+1} \text{ and } \lambda(V)=z+2$$

generate a free subgroup of Λ.

For this, assume a nontrivial composition $\dots\lambda(U)^j\lambda(V)^i$ is equal to the identity in Λ, in particular, it is the identity map from $\mathbb{C}$ to $\mathbb{C}$. We can assume $i\neq 0$ by circulation, if necessary. Then note that

$$\lambda(V)^i(z)=z+2i,\ i\in\mathbb{Z}, \text{ and } \lambda(U)^j(z)=\frac{z}{2jz+1},\ j\in\mathbb{Z};$$

moreover, $\lambda(V)^i$ maps the interior of the disc $|z|\leq 1$ to the exterior, and $\lambda(U)^j$ maps the exterior of $|z|\leq 1$ to the interior of $|z|\leq 1$ without the origin. It follows that the above product cannot map 0 to 0 unless the product is trivial; therefore it cannot be the identity map. □

Note that $\lambda(H)$ is the subgroup of Λ generated by $\lambda(U)$ and $\lambda(V)$. We next use the idea of the above proof to establish some more facts.

Lemma 2. *No elements of $\lambda(H)$ can map 0 to ∞. No elements of $\lambda(H)$ can map 2 to ∞.*

Proof. We consider the first assertion. Let w be an arbitrary element of $\lambda(H)$. We consider several cases.

Case 1. $w = \lambda(U)^{j_k}\lambda(V)^{i_k}\cdots\lambda(U)^{j_0}\lambda(V)^{i_0}$, where $j_k i_k \cdots j_0 i_0 \neq 0$. In this case, the previous arguments show that $|w(0)| \leq 1$.

Case 2. $w = \lambda(V)^{i_{k+1}}\lambda(U)^{j_k}\lambda(V)^{i_k}\cdots\lambda(U)^{j_0}\lambda(V)^{i_0}$, where $i_{k+1}j_k i_k\cdots j_0 i_0 \neq 0$. In this case again by the previous arguments, we have that $w(0) = \lambda(V)^{i_{k+1}}(z)$ for some $|z| \leq 1$. But $\lambda(V)^{i_{k+1}}(z) = z + 2i_{k+1} \neq \infty$.

Case 3. Otherwise, $w = \cdots\lambda(U)^{j_1}\lambda(V)^{i_1}\lambda(U)^{j_0}$, where $j_0 \neq 0$. Then since $\lambda(U)^{j_0}(0) = 0$, the problem reduces to Cases 1 and 2, so $w(0) \neq \infty$.

This finishes the proof for the first assertion. For the second assertion, just note that $\lambda(V)$ maps 0 to 2; hence if some $w \in \lambda(H)$ maps 2 to ∞, then $w\lambda(V)$ would map 0 to ∞, contradicting the first assertion. □

Lemma 3. *$T \notin H$ and $-T \notin H$.*

Proof. Note that $\lambda(T) = \lambda(-T)$; hence it suffices to prove $\lambda(T) \notin \lambda(H)$. For this just note further that $\lambda(T)(2) = \infty$ and then use Lemma 2. □

Next we show that $-I \notin H$. For this we need a different kind of argument, since $\lambda(-I) = \lambda(I) \in \lambda(H)$.

Lemma 4. *$-I \notin H$.*

Proof. We prove that $-I \neq U^{i_0}V^{j_0}\cdots U^{i_k}V^{j_k}$ for any $i_0, j_0, \cdots, i_k, j_k \in \mathbb{Z}$ (even if some of them are zero), by induction on k.

First note that

$$U^iV^j = \begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}\begin{pmatrix} 1 & 2j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2j \\ 2i & 1+4ij \end{pmatrix}.$$

Therefore when $k = 0$, $U^{i_0}V^{j_0} \neq -I$.

In general, it is easy by an induction to show that $U^{i_0}V^{j_0}\cdots U^{i_k}V^{j_k}$ is a matrix of the form

$$\begin{pmatrix} 1+4n & 2s \\ 2t & 1+4m \end{pmatrix}$$

since matrices of this form are closed under products. It is obvious that $-I$ cannot be written in this way. □

However, we have the following identity.

Lemma 5. *$T^2 + UV^{-1} = 0$.*

Proof. By a straightforward computation. □

Thus, $T^2 = -UV^{-1} \notin H$ (since $-I \notin H$), $T^3 = (-T)UV^{-1} \notin H$ (since $-T \notin H$), but $T^4 = UV^{-1}UV^{-1} \in H$.

3. Preliminaries about the action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$

For each element (m,n) of $\mathbb{Z}^2$ we denote the orbit of (m,n) under the $SL(2,\mathbb{Z})$ action by $[m,n]$.

Lemma 6. *$[1,1] = [-1,-1] \neq [n,n]$ for any $n \neq \pm 1$.*

Proof. Suppose $a, b, c, d, n \in \mathbb{Z}$ satisfy that

$$\begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \end{pmatrix}, \\ ad - bc = 1. \end{cases}$$

By comparing the sides we obtain $n(b + c) = n^2 - 1$, therefore $n|(n + 1)(n - 1)$, which is impossible unless $n = \pm 1$. □

Lemma 7. *If $n, m \in \mathbb{Z}$ and $\gcd(n, m) = 1$, then $[n, m] = [1, 1]$.*

Proof. Let $u, v \in \mathbb{Z}$ be such that $nu + mv = 1$. Then

$$\begin{pmatrix} n + v & -v \\ m - u & u \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ m \end{pmatrix}.$$

□

Lemma 8. *For any $n, m \in Z$, $[n, m] = [\gcd(n, m), \gcd(n, m)]$.*

Proof. Let $k = \gcd(n, m)$. Let n' and m' be such that $n = kn'$ and $m = km'$. Then there is some $A \in SL(2, \mathbb{Z})$ such that $A(1, 1) = (n', m')$. Therefore $A(k, k) = kA(1, 1) = (n, m)$. □

The above lemmas show that the orbits of the $SL(2, \mathbb{Z})$ action on $\mathbb{Z}^2$ are exactly $[n, n]$ for $n \in \mathbb{N}$.

For $(n, m) \in \mathbb{Z}^2$, we let $S(n, m)$ denote the stabilizer of (n, m). By the computation in the proof of Lemma 6, it is easy to see that $S(1, 1) = \langle T \rangle$, the subgroup of $SL(2, \mathbb{Z})$ generated by T. It then follows from the same argument as Lemma 8 that, for any $n > 0$, $S(n, n) = S(1, 1) = \langle T \rangle$. In general, the stabilizer of an arbitrary element is a conjugate of $S(1, 1)$. In particular, these are nontrivial. Thus we conclude that the action of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2$ is nowhere free. However, we shall find a free subgroup G of H so that $G \cap \langle T \rangle = \{I\}$. For $n > 0$, let $X_n = \{g(n, n) \,|\, g \in G\}$. Then the action of G is free on each X_n.

At the end of the last section we verified that $H \cap \langle T \rangle = \langle T^4 \rangle = \langle UV^{-1}UV^{-1} \rangle$. Let N be the normalizer of $UV^{-1}UV^{-1}$ in H. Then there is a canonical homomorphism φ from H onto H/N. Notice that the group H/N has a finite presentation

$$\langle a, b \,|\, ab^{-1}ab^{-1} = 1 \rangle,$$

which, if we let $c = ab^{-1}$, is equivalent to

$$\langle a, c \,|\, c^2 = 1 \rangle.$$

It is then easy to see that there is a nonabelian free subgroup of H/N, say the one generated by aca and cac. Let G be the subgroup of H generated by the elements $U(UV^{-1})U = U^2V^{-1}U$ and $(UV^{-1})U(UV^{-1}) = UV^{-1}U^2V^{-1}$. By considering the φ image of G we know that G is freely generated by these two elements and $G \cap \langle T \rangle = \{I\}$.

4. The main theorem

This section is devoted to a proof of the main theorem, stated below.

Theorem 1. *The orbit equivalence relation of the action of $SL(2, \mathbb{Z})$ on subsets of $\mathbb{Z}^2$ is a universal countable Borel equivalence relation.*

Let E be the equivalence relation in question. Let E_∞ be the orbit equivalence relation of the shift action of F_2 on its subsets, where F_2 is the free group with 2 generators. We are to define a Borel reduction θ to witness that $E_\infty \leq_B E$.

For the definition of θ, let S be a subset of F_2. Since the group G we defined at the end of the last section is isomorphic to F_2, we also regard S as a subset of G. Then let

$$\theta(S) = \{g(1,1) \mid g \in S\} \cup \{gU^2V^{-1}U(2,2) \mid g \in S\}.$$

For arbitrary $f \in F_2$, regarding f also as an element of G, we have

$$\theta(fS) = \{fg(1,1) \mid g \in S\} \cup \{fgU^2V^{-1}U(2,2) \mid g \in S\} = f(\theta(S)).$$

Therefore, $S_1 E_\infty S_2$ implies that $\theta(S_1) E \theta(S_2)$.

Now suppose S_1 and S_2 are subsets of F_2 with $\theta(S_1) E \theta(S_2)$. Without loss of generality assume that both S_1 and S_2 are nonempty. Then there is some element $A \in SL(2,\mathbb{Z})$ such that $\theta(S_2) = A(\theta(S_1))$. Let $Y_j^i = X_i \cap \theta(S_j)$ for $i,j = 1,2$, where $X_i = \{g(i,i) \mid g \in G\}$. Then by Lemma 6, $Y_2^i = A(Y_1^i)$, for $i = 1,2$.

Fix an arbitrary element s of S_1. Let $y_1 = s(1,1) \in Y_1^1$ and $y_2 = A(y_1) \in Y_2^1$. Then $y_1, y_2 \in X_1$. It follows that there is an element $g \in G$ such that $A \in gS(y_1)$. Furthermore, $S(y_1) = sS(1,1)s^{-1}$. Thus $A \in gs\langle T\rangle s^{-1}$. Therefore there is some $k_1 \in \mathbb{Z}$ such that $A = gsT^{k_1}s^{-1}$.

Denote $g_0 = U^2V^{-1}U \in G$. Let $z_1 = sg_0(2,2) \in Y_1^2$ and $z_2 = A(z_1) \in Y_2^2$. Then $z_1, z_2 \in X_2$. It then follows that there is an element $g' \in G$ such that $A \in g'S(z_1)$. But $S(z_1) = (sg_0)S(2,2)(sg_0)^{-1}$. Thus there is some $k_2 \in \mathbb{Z}$ such that $A = g'sg_0T^{k_2}g_0^{-1}s^{-1}$.

From these representations of A we get

$$g_0^{-1}s^{-1}g'^{-1}gsT^{k_1}g_0T^{-k_2} = I.$$

We would complete our proof if the following lemma is granted.

Lemma 9. *For any $g \in G$, $k,l \in \mathbb{Z}$, if $gT^kg_0T^l = I$, then $k = l = 0$.*

Granting the lemma and considering the equality we obtained above, we then have $k_1 = k_2 = 0$, which gives that $A \in G$. Therefore we can regard A as an element of F_2. Let $S_2' = A(S_1)$. Then $\theta(S_2') = A(\theta(S_1)) = \theta(S_2)$. Since θ is one-one, we have $S_2' = S_2$, or $S_2 = A(S_1)$. This finishes the proof that $\theta(S_1) E \theta(S_2)$ implies $S_1 E_\infty S_2$.

In the rest of this section we prove Lemma 9. First note a fact.

Lemma 10. *$UT^{-1} + TU^{-1} = 0$.*

Proof. By a straightforward computation. □

From Lemmas 5 and 10 we obtain

$$\begin{aligned} TUT^{-1} &= T(-TU^{-1}) \\ &= -T^2U^{-1} \\ &= UV^{-1}U^{-1} \end{aligned}$$

and

$$\begin{aligned} TVT^{-1} &= (TVU^{-1}T^{-1})(TUT^{-1}) \\ &= (TUV^{-1}T^{-1})^{-1}UV^{-1}U^{-1} \\ &= VU^{-1}UV^{-1}U^{-1} \\ &= U^{-1}. \end{aligned}$$

Now suppose $g \in G$, $k, l \in \mathbb{Z}$ and $gT^k g_0 T^l = I$. We consider several cases.

Case 1. $g = I$. In this case $T^k g_0 T^l = I$, or $g_0 = T^{-(k+l)}$. But $G \cap \langle T \rangle = \{I\}$. Contradiction.

Case 2. $g \neq I$ and $k = 0$. Then $gg_0 = T^{-l}$. Again by $G \cap \langle T \rangle = \{I\}$, $l = 0$.

Case 3. $g \neq I$ and $k \neq 0$. By the argument of Cases 1 and 2, we know that $l \neq 0$. We consider several subcases.

Subcase 3.1. k is odd and l is even. Let $k = 2k' + 1$ and $l = 2l'$. Then we have

$$\begin{aligned} T &= T^{-2k'} g^{-1} T^{-2l'} g_0^{-1} \\ &= (-UV^{-1})^{-k'} g^{-1} (-UV^{-1})^{-l'} g_0^{-1} \\ &= (-1)^{k'+l'} (VU^{-1})^{k'} g^{-1} (VU^{-1})^{l'} g_0^{-1}. \end{aligned}$$

This implies that either $T \in H$ or $-T \in H$, which contradicts Lemma 3.

Subcase 3.2. k is even and l is odd. Similar to Subcase 3.1.

Subcase 3.3. k is even and l is even. Let $k = 2k'$ and $l = 2l'$. Then $gT^{2k'} g_0 T^{2l'} = g(-UV^{-1})^{k'} g_0 (-UV^{-1})^{l'} = (-1)^{k'+l'} g(UV^{-1})^{k'} g_0 (UV^{-1})^{l'} = I$. It follows that $k' + l'$ is even, since otherwise the equality implies that $-I \in H$, contradicting Lemma 4. Furthermore, we yield

$$g(UV^{-1})^{k'} g_0 (UV^{-1})^{l'} = I, \text{ with } k', l' \neq 0.$$

Let $W = UV^{-1}$. Then H is also a free group generated by U and W. The above equality demonstrates that

$$g^{-1} = W^{k'} g_0 W^{l'} = W^{k'} UWUW^{l'}$$

for $k', l' \neq 0$ and $g \in \langle UWU, WUW \rangle$ with $g \neq I$. But this is easily seen to be impossible.

Subcase 3.4. k is odd and l is odd. Let $k = 2k' + 1$ and $l = 2l' - 1$. Then we have $gT^k g_0 T^l = g(-W)^{k'} T g_0 T^{-1} (-W)^{l'} = I$. Again we conclude that $k' + l'$ is even, since otherwise we have $-I \in HTHT^{-1}H \subseteq H$ by the computations following Lemma 10, contradicting Lemma 4. Furthermore we have

$$W^{k'} T g_0 T^{-1} W^{l'} = g^{-1} \in G.$$

Note that $Tg_0T^{-1} = TUWUT^{-1} = TUT^{-1}WTUT^{-1} = WU^{-1}WWU^{-1}$. It is then easy to see that $W^{k'} WU^{-1}WWU^{-1} W^{l'} \notin G$ for any k' and l'.

This finishes the proof of Lemma 9, hence the theorem.

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