

## HOLOMORPHIC PERTURBATION OF FOURIER COEFFICIENTS

THOMAS VILS PEDERSEN

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ABSTRACT. Let  $\mathbb{T}$  be the unit circle, let  $\mathcal{B}$  be a Banach space continuously embedded in  $L^1(\mathbb{T})$  and suppose that  $\mathcal{B}$  is a Banach  $L^1(\mathbb{T})$ -module under convolution. We show that if  $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$  and  $F$  is holomorphic in a neighbourhood  $U$  of 0 with  $F(0) = 0$  and  $a_n \in U$  ( $n \in \mathbb{Z}$ ), then  $\sum_{n=-\infty}^{\infty} F(a_n) z^n \in \mathcal{B}$ .

Recently, Render ([3]) proved the result stated above for the case where  $\mathcal{B}$  is the space  $H^\infty(\mathbb{D})$  of bounded analytic functions on the open unit disc  $\mathbb{D}$ . In this note, we show that Render's result follows easily from the theory of so-called abstract Segal algebras, and that this approach allows us to generalize his result to a large class of spaces of functions. We also give a direct proof of this result.

For  $f, g \in L^1(\mathbb{T})$ , we define the convolution product  $f * g$  by

$$(f * g)(e^{it}) = \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{is}) g(e^{i(t-s)}) ds \quad (e^{it} \in \mathbb{T}).$$

Then  $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$ , where  $\widehat{f}(n)$  ( $n \in \mathbb{Z}$ ) are the Fourier coefficients of  $f$ .

**Proposition 1.** *Let  $\mathcal{B}$  be a Banach space continuously embedded in  $L^1(\mathbb{T})$  and suppose that  $\mathcal{B}$  is a Banach  $L^1(\mathbb{T})$ -module under convolution. Let  $E = \{n \in \mathbb{Z} : \text{there exists } f \in \mathcal{B} \text{ with } \widehat{f}(n) \neq 0\}$ . Considered as a Banach algebra under convolution, every character on  $\mathcal{B}$  is of the form*

$$f \mapsto \widehat{f}(n) \quad (f \in \mathcal{B})$$

for some  $n \in E$ .

*Proof.* Let  $L_E^1(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for } n \in \mathbb{Z} \setminus E\}$ . For  $n \in E$ , choose  $f \in \mathcal{B}$  with  $\widehat{f}(n) \neq 0$ . Then  $e^{int} = (e^{int} * f) / \widehat{f}(n) \in \mathcal{B}$ , so  $\mathcal{B}$  contains every trigonometric polynomial in  $L_E^1(\mathbb{T})$ , and in particular  $\mathcal{B}$  is dense in  $L_E^1(\mathbb{T})$ . Hence  $\mathcal{B}$  is an abstract Segal algebra with respect to  $L_E^1(\mathbb{T})$  in the sense of [1, Definition 1.1], so the result follows from [1, Theorem 2.1] (see [2, Theorem 6.2.4] for the case  $E = \mathbb{Z}$ ).  $\square$

*Remark 2.* Let  $\mathcal{B}$  be a Banach space continuously embedded in  $L^1(\mathbb{T})$  and for  $f \in \mathcal{B}$  and  $e^{is} \in \mathbb{T}$ , let  $f_s(e^{it}) = f(e^{i(t-s)})$  ( $e^{it} \in \mathbb{T}$ ). If  $f_s \in \mathcal{B}$  with  $\|f_s\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$  and

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$f_s \rightarrow f$  in  $\mathcal{B}$  as  $e^{is} \rightarrow 1$ , or if  $\mathcal{B}$  is a dual space (like  $H^\infty(\mathbb{D})$ ) and  $f_s \rightarrow f$  weak star in  $\mathcal{B}$  as  $e^{is} \rightarrow 1$ , then

$$g * f = \frac{1}{2\pi} \int_{\mathbb{T}} g(e^{is}) f_s ds \in \mathcal{B}$$

and  $\|g * f\|_{\mathcal{B}} \leq \|g\|_{L^1(\mathbb{T})} \|f\|_{\mathcal{B}}$  for  $g \in L^1(\mathbb{T})$ , so  $\mathcal{B}$  is a Banach  $L^1(\mathbb{T})$ -module under convolution. This provides a large class of spaces of functions to which our results apply.

With  $\mathcal{B}$  as above, let  $\mathcal{A} = \mathcal{B} \oplus \mathbb{C}\delta_1$  be the Banach algebra obtained by adjoining the identity  $\delta_1$  (the Dirac measure at  $z = 1$ ) to  $\mathcal{B}$ .

**Corollary 3.** For  $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$ , we have  $\sigma_{\mathcal{A}}(f) = \{a_n : n \in E\} \cup \{0\}$ .

Our main result now follows from the holomorphic functional calculus for Banach algebras (see, for instance, [4, Theorem 10.2.7]).

**Theorem 4.** Let  $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$  and let  $F$  be holomorphic in a neighbourhood  $U$  of 0 with  $F(0) = 0$  and  $a_n \in U$  ( $n \in \mathbb{Z}$ ). Then  $\sum_{n=-\infty}^{\infty} F(a_n) z^n \in \mathcal{B}$ .

*Remark 5.* Writing  $F(z) = zG(z)$  ( $z \in U$ ), the conclusion of the theorem takes the form  $\sum_{n=-\infty}^{\infty} G(a_n) a_n z^n \in \mathcal{B}$ ; that is, the sequence  $(G(a_n))$  acts as a “local coefficient multiplier” on  $f$ .

We wish to point out that it is possible to give a proof of Corollary 3 (and thus of Theorem 4) which does not depend on the theory of abstract Segal algebras: Let  $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$  and suppose that  $\lambda \notin \{a_n : n \in E\} \cup \{0\}$ . It is well known that every character on  $L^1(\mathbb{T})$  is of the form  $g \mapsto \widehat{g}(n)$  ( $g \in L^1(\mathbb{T})$ ) for some  $n \in \mathbb{Z}$ , so it follows that  $\lambda \notin \sigma_{L^1(\mathbb{T}) \oplus \mathbb{C}\delta_1}(f)$ . Since the Fourier series of  $\delta_1$  is  $\sum_{n=-\infty}^{\infty} z^n$ , we thus have

$$(\lambda\delta_1 - f)^{-1} = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda - a_n} z^n = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \frac{a_n}{\lambda - a_n} z^n + \frac{1}{\lambda} \delta_1 \in L^1(\mathbb{T}) \oplus \mathbb{C}\delta_1.$$

Moreover, since  $\mathcal{B}$  is a Banach  $L^1(\mathbb{T})$ -module under convolution, we have

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{\lambda - a_n} z^n = (\lambda\delta_1 - f)^{-1} * f \in \mathcal{B},$$

so  $(\lambda\delta_1 - f)^{-1} \in \mathcal{A}$ , and we deduce that  $\lambda \notin \sigma_{\mathcal{A}}(f)$ .

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*E-mail address:* vils@math.u-bordeaux.fr