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A NOTE ON ASYMPTOTICALLY ISOMETRIC COPIES OF l^1 AND c_0

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Abstract. Nonreflexive Banach spaces that are complemented in their bidual by an L-projection—like preduals of von Neumann algebras or the Hardy space H^1—contain, roughly speaking, many copies of l^1 which are very close to isometric copies. Such l^1-copies are known to fail the fixed point property. Similar dual results hold for c_0.

In [4] it is shown that an isomorphic l^1-copy does not necessarily contain asymptotically isometric l^1-copies although by James' classical distortion theorem it always contains almost isomorphic l^1-copies. (For definitions see below.) Within this context and the context of the fixed point property Dowling and Lennard [5] show that the presence of an asymptotic l^1-copy makes a Banach space fail the fixed point property. Then they prove that every nonreflexive subspace of $L^1[0,1]$ fails the fixed point property by observing that the proof of a theorem of Kadec and Pełczyński [10, Th. 6] yields an asymptotic l^1-copy inside such supspaces of $L^1[0,1]$. Alspach's example [1] may be considered as an early forerunner of these results.

In the present note we modify a construction of Godefroy in order to show that every nonreflexive subspace of any L-embedded Banach space contains an asymptotic l^1-copy and thus, in particular, fails the fixed point property. Analogous results hold for c_0 and M-embedded spaces.

Let (x_n) be a sequence of nonzero elements in a Banach space X.

We say that (x_n) *spans* l^1 *r-isomorphically* or just *isomorphically* if there exists $r > 0$ (trivially $r \leq 1$) such that $r\Big(\sum_{n=1}^\infty |\alpha_n|\Big) \leq \Big\|\sum_{n=1}^\infty \alpha_n x_n\Big\| \leq \sum_{n=1}^\infty |\alpha_n|$ for all scalars α_n. We say that (x_n) *spans* l^1 *almost isometrically* if it is such that there exists a sequence (δ_m) in $[0,1[$ tending to 0 such that $(1-\delta_m)\Big(\sum_{n=m}^\infty |\alpha_n|\Big) \leq \Big\|\sum_{n=m}^\infty \alpha_n x_n\Big\| \leq \Big(\sum_{n=m}^\infty |\alpha_n|\Big)$ for all $m \in \mathbb{N}$. Trivially the property of spanning l^1 almost isometrically passes to subsequences. Analogously, we say that (x_n) *spans* c_0 *almost isometrically* if there exists a sequence (δ_m) as above such that $(1-\delta_m)\sup_{m\leq n\leq m'} |\alpha_n| \leq \Big\|\sum_{n=m}^{m'} \alpha_n x_n\Big\| \leq (1+\delta_m)\sup_{m\leq n\leq m'} |\alpha_n|$ for all $m \leq m'$.

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Recall that James' distortion theorem [9] (or [3], [12]) for l^1 and c_0 says that every isomorphic copy of l^1 (of c_0) contains an almost isometric copy of l^1 (of c_0).

Following [4, 5, 6] we say that (x_n) *spans* l^1 *asymptotically isometrically* (or just that (x_n) *spans* l^1 *asymptotically*) if there is a sequence (δ_n) in $[0,1[$ tending to 0 such that $\sum_{n=1}^\infty (1-\delta_n)|\alpha_n| \leq \left\|\sum_{n=1}^\infty \alpha_n x_n\right\| \leq \sum_{n=1}^\infty |\alpha_n|$ for all scalars α_n. A sequence (x_n) is said to *span* c_0 *asymptotically isometrically* (or just *to span* c_0 *asymptotically*) if there is a sequence (δ_n) as above such that $\sup_{n\leq m}(1-\delta_n)|\alpha_n| \leq \left\|\sum_{n=1}^m \alpha_n x_n\right\| \leq \sup_{n\leq m}(1+\delta_n)|\alpha_n|$ for all $m \in \mathbb{N}$. (Note that in the definition of asymptotic c_0's and l^1's in [4] the sequence (δ_n) is supposed to be decreasing but that this difference is not essential.) Finally we say that a Banach space is isomorphic (respectively almost isometric, respectively asymptotically isometric) to l^1 (to c_0) if it has a basis with the corresponding property. Clearly a sequence spanning l^1 asymptotically spans l^1 almost isometrically. The main result of [4] states that the converse does not hold because there are almost isometric copies of l^1 which do not contain l^1 asymptotically. Analogously, it was proved in [4, Th. 3] that there exist almost isometric copies of c_0 which do not contain asymptotic c_0-copies.

Some notation: The results are stated for complex scalars but hold also for real Banach spaces. Operator means linear bounded map. As usual, we consider a Banach space as a subspace of its bidual omitting the canonical embedding. $[x_n]$ denotes the complete linear hull of (x_n), (e_n) is the standard basis of l^1. Basic properties and definitions of Banach space theory can be found in [3] or in [12, 13].

A Banach space X is said to have the *fixed point property* if any contractive (not necessarily linear) map $f : C \to C$ on any nonempty closed bounded convex subset $C \subset X$ has a fixed point where contractive means that $\|f(x) - f(y)\| < \|x - y\|$ for all $x, y \in X$.

Let Y be a subspace of a Banach space X and let P be a projection on X. P is called an *L-projection* provided $\|x\| = \|Px\| + \|(\mathrm{id}_X - P)x\|$ for all $x \in X$. A subspace $Y \subset X$ is called an *M-ideal in* X if its annihilator $Y^\perp$ in X' is the range of an L-projection on X'. Y is called an *L-summand in* X if it is the range of an L-projection on X. In the special case where $X = Y''$ and where Y is an M-ideal (respectively an L-summand) in Y'' we say that Y is *M-embedded* (respectively *L-embedded*). As examples we only mention that preduals of von Neumann algebras, in particular l^1 and L^1-spaces, furthermore the Hardy space H_0^1 and the dual of the disc algebra, are L-embedded. The sequence space c_0, the space of compact operators on a Hilbert space, and the quotient C/A of the continuous functions on the unit circle by the disc algebra A are examples among M-embedded spaces. The dual of an M-embedded space is L-embedded; the converse is false [8, III.1.3]. Throughout this note, if X is an L-embedded Banach space we will write X_s for the complement of (the canonical embedding of) X in X'', that is, $X'' = X \oplus_1 X_s$. In this case P (or P_X to avoid confusion) will denote the L-projection from X'' onto X. There is a useful criterion for L-embeddedness of subspaces of L-embedded spaces due to Li ([11] or [8, Th. IV.1.2]): *A closed subspace Y of an L-embedded Banach space X is L-embedded if and only if $PY^{\perp\perp} = Y$.* In this case if Y is L-embedded and if one identifies $Y'' = Y \oplus_1 Y_s$ and $Y^{\perp\perp} \subset X''$, then $Y_s = Y^{\perp\perp} \cap X_s$. Since biduals of L-embedded spaces are quite "big" and therefore difficult to handle we mention in passing that a theorem of Buhvalov-Lozanovskii ([2], [8, IV.3.4]) provides a characterisation of L-embeddedness of subspaces of $L^1[0,1]$ only in terms of the spaces themselves: Subspaces of $L^1[0,1]$ are L-embedded if and only if their unit balls are closed with respect to the measure topology [8, IV.3.5].

The standard reference for M- and L-embedded spaces is the monograph [8].

There are few stability results for L-embeddedness: Neither subspaces nor quotients inherit this property [8, IV.1] in general. Therefore the following lemma reveals a nice exception. In particular it underlines the idea that l^1-copies are the building blocks of L-embedded spaces.

Lemma 1. *Almost isometric copies of l^1 which are subspaces of L-embedded Banach spaces are L-embedded.*

Proof. Consider first the L-embedded subspaces $U_m = [e_n]_{n\geq m}$ of l^1, $m \in \mathbb{N}$. We have $U_m^{\perp\perp} = U_m \oplus_1 (c_0 \cap U_m^{\perp\perp})$. An element $\mu \in (l^1)_s = c_0^\perp$ belongs to each $U_m^{\perp\perp}$. [Denote by ρ_m the projection $(\alpha_n) \mapsto (0,\dots,0,\alpha_{m+1},\alpha_{m+2},\dots)$ on l^1. Then μ annihilates $\ker\rho_m'$ because $\ker\rho_m' \subset c_0$. Thus $\mu \in \operatorname{ran}(\rho'')$ and $\mu \in \overline{[e_n]}^{w^*}_{n\geq m} \cap c_0^\perp$.]

Let X be an L-embedded Banach space with L-decomposition $X'' = X \oplus_1 X_s$ and with L-projection P from X'' onto X. Let (x_n) be a sequence spanning an almost isometric copy Y of l^1 in X, put $Y_m = [x_n]_{n\geq m}$, $m \in \mathbb{N}$. Via the isomorphism between Y and l^1 induced by $x_n \mapsto e_n$ the situation described for l^1 carries over to Y. That is, there is $Z \subset Y^{\perp\perp} \subset X''$ such that $Y^{\perp\perp} = Y \oplus Z$, $Y_m^{\perp\perp} = Y_m \oplus (Z \cap Y_m^{\perp\perp})$ and $z \in Z \cap Y_m^{\perp\perp}$ for all $m \in \mathbb{N}$, $z \in Z$.

Since (x_n) is supposed to span Y almost isometrically, there are numbers $\eta_m \geq 0$ tending to 0 such that $\|y+z\| \geq (1-\eta_m)(\|y\|+\|z\|)$ for all $y \in Y_m$, $z \in Z \cap Y_m^{\perp\perp}$.

Let $z \in Z$. In order to show that Y is L-embedded it is enough to show that $Pz = 0$ because then Li's criterion $PY^{\perp\perp} = Y$ is fulfilled. But $z \in Z \cap Y_m^{\perp\perp}$ and by a quantitative version of Li's result (see [8, IV.1.4] or [14, Lem. 2]) applied to X and Y_m, we have $\|Pz\| \leq 3\eta_m^{1/2}\|z\|$ for all $m \in \mathbb{N}$. Hence $Pz = 0$.

In passing we note (for Corollary 3 below) that a w^*-accumulation point of $\{e_n|\ n \in \mathbb{N}\}$ belongs to $c_0^\perp \cap U_m^{\perp\perp}$ and has norm one. Accordingly, if z is a w^*-accumulation point of $\{x_n|\ n \in \mathbb{N}\}$, then $z \in X_s \cap Y_m^{\perp\perp}$ and $\|z\| = 1$, the latter because each $Y_m^{\perp\perp}$ is $(1-\delta_m)$-isomorphic to $(l^1)''$ with $\delta_m \to 0$. ☐

In addition to Lemma 1 we note that if Y is an almost isometric l^1-copy in an L-embedded space X, then X/Y is L-embedded, too, by [8, IV.1.3].

Here is a way of constructing asymptotically isometric l^1-copies in L-embedded Banach spaces. It is essentially due to Godefroy [8, IV.2.5] and was brought to my attention, some years ago, by D. Werner, under the name ace of $\diamond$ lemma.

Theorem 2. *Let X be an L-embedded Banach space with L-decomposition $X'' = X \oplus_1 X_s$, $(\Gamma, \preceq)$ a directed set and $(x_\gamma)_{\gamma\in\Gamma}$ a net in the unit ball of X. If $x_\gamma \overset{w^*}{\to} x_s \in X_s$ in the w^*-topology of X'' and $\|x_s\| = 1$, then there is a sequence $(x_{\gamma_n})_{n\in N}$ which spans l^1 asymptotically.*

Proof. Let (δ_n) be a sequence of strictly positive numbers converging to 0. Set $\eta_1 = \frac{1}{3}\delta_1$ and $\eta_{n+1} = \frac{1}{3}\min(\eta_n, \delta_{n+1})$ for $n \in \mathbb{N}$. By induction over $n \in \mathbb{N}$ we will construct $\gamma_n \in \Gamma$ such that

$$(1)\qquad \Big(\sum_{i=1}^n (1-\delta_i)|\alpha_i|\Big) + \eta_n \sum_{i=1}^n |\alpha_i| \leq \|\sum_{i=1}^n \alpha_i x_{\gamma_i}\| \leq \sum_{i=1}^n |\alpha_i| \ \ \forall n \in \mathbb{N},\ \alpha_i \in \mathbb{C}.$$

(The last inequality is trivial because $\|x_\gamma\| \le 1$.) We recall that the norm is w^*-lower semicontinuous. Thus $\liminf \|x_\gamma\| \ge \|x_s\| = 1$ and for the first induction step we choose x_{γ_1} such that $\|x_{\gamma_1}\| > 1 - \delta_1 + \eta_1$.

For the induction step $n \mapsto n+1$ we suppose $x_{\gamma_1}, \dots, x_{\gamma_n}$ to be constructed such that (1) holds. Fix an element $\alpha = (\alpha_i)_{i=1}^{n+1}$ in the unit sphere of l^1_{n+1} such that $\alpha_{n+1} \neq 0$. The w^*-convergence (along γ) of $(\sum_{i=1}^n \alpha_i x_{\gamma_i}) + \alpha_{n+1} x_\gamma$ to $(\sum_{i=1}^n \alpha_i x_{\gamma_i}) + \alpha_{n+1} x_s$ yields

$$\begin{aligned}
\liminf_\gamma \Big\| \Big(\sum_{i=1}^n \alpha_i x_{\gamma_i} \Big) + \alpha_{n+1} x_\gamma \Big\| &\ge \Big\| \Big(\sum_{i=1}^n \alpha_i x_{\gamma_i} \Big) + \alpha_{n+1} x_s \Big\| \\
&= \Big\| \sum_{i=1}^n \alpha_i x_{\gamma_i} \Big\| + \|\alpha_{n+1} x_s\| \\
&\overset{(1)}{\ge} \Big(\sum_{i=1}^n (1-\delta_i)|\alpha_i| \Big) + \eta_n \Big(\sum_{i=1}^n |\alpha_i| \Big) + |\alpha_{n+1}| \\
&= \Big(\sum_{i=1}^{n+1} (1-\delta_i)|\alpha_i| \Big) + \eta_n \Big(\sum_{i=1}^{n+1} |\alpha_i| \Big) \\
&\qquad - (\eta_n - \delta_{n+1})|\alpha_{n+1}| \\
&\ge \Big(\sum_{i=1}^{n+1} (1-\delta_i)|\alpha_i| \Big) + \min(\eta_n, \delta_{n+1})
\end{aligned}$$

because $\|\alpha\| = 1$ and $|\alpha_{n+1}| \le 1$. Thus there is an index $\gamma_0 \in \Gamma$ such that $\inf_{\gamma \succeq \gamma_0} \Big\| \Big(\sum_{i=1}^n \alpha_i x_{\gamma_i} \Big) + \alpha_{n+1} x_\gamma \Big\| > \Big(\sum_{i=1}^{n+1} (1-\delta_i)|\alpha_i| \Big) + 2\eta_{n+1}$; note that the subnet $(x_\gamma)_{\gamma \succeq \gamma_0}$ still w^*-converges to x_s.

Choose a finite η_{n+1}-net $(\alpha^l)_{l=1}^{L_{n+1}}$ in the unit sphere of l^1_{n+1} in the sense that for each α in the unit sphere of l^1_{n+1} there is $l \le L_{n+1}$ such that $\|\alpha - \alpha^l\| = \sum_{i=1}^{n+1} |\alpha_i - \alpha_i^l| < \eta_{n+1}$. Then we may repeat the reasoning above finitely many times for $l = 1, \dots, L_{n+1}$ in order to get $x_{\gamma_{n+1}}$ such that

$$\Big\| \sum_{i=1}^{n+1} \alpha_i^l x_{\gamma_i} \Big\| > \Big(\sum_{i=1}^{n+1} (1-\delta_i)|\alpha_i^l| \Big) + 2\eta_{n+1} \quad \forall l \le L_{n+1}.$$

For an arbitrary α in the unit sphere of l^1_{n+1} choose $l \le L_{n+1}$ such that $\|\alpha - \alpha^l\| < \eta_{n+1}$. Then

$$\begin{aligned}
\Big\| \sum_{i=1}^{n+1} \alpha_i x_{\gamma_i} \Big\| &\ge \Big\| \sum_{i=1}^{n+1} \alpha_i^l x_{\gamma_i} \Big\| - \Big\| \sum_{i=1}^{n+1} (\alpha_i - \alpha_i^l) x_{\gamma_i} \Big\| \\
&\ge \Big(\sum_{i=1}^{n+1} (1-\delta_i)|\alpha_i| \Big) + 2\eta_{n+1} - \|\alpha - \alpha^l\| \\
&\ge \Big(\sum_{i=1}^{n+1} (1-\delta_i)|\alpha_i| \Big) + \eta_{n+1} \\
&= \Big(\sum_{i=1}^{n+1} (1-\delta_i)|\alpha_i| \Big) + \eta_{n+1} \sum_{i=1}^{n+1} |\alpha_i|. \qquad (2)
\end{aligned}$$

This extends to all $\alpha \in l^1_{n+1}$ and thus ends the induction and the proof. □

Within L-embedded spaces James' distortion theorem is more efficient:

Corollary 3. *A sequence spanning l^1 almost isometrically in an L-embedded Banach space admits of a subsequence which spans l^1 asymptotically.*

Proof. We use the notation of the proof of Lemma 1 and the remark in the end of it. Hence there is a net (x_{n_γ}) which w^*-converges to a normalized w^*-accumulation point x_s of $\{x_n | \, n \in \mathbb{N}\}$. Theorem 2 applies. □

Corollary 4. *Every nonreflexive subspace of an L-embedded Banach space contains an asymptotic copy of l^1.*

In particular, every nonreflexive subspace of an L-embedded Banach space fails the fixed point property.

Proof. L-embedded spaces are w-sequentially complete ([7], or [8, IV.2.2]). Hence the first assertion follows from theorems of Eberlein-Šmuljan, Rosenthal, James and from Corollary 3. Combine this with [5, Th. 1.2] to get the second assertion. □

By [14] (or [8, IV.2.7]) the basis (x_{γ_n}) in Theorem 2 admits a subsequence whose closed linear span is complemented in X. In this way Corollary 4 recovers the theorem of Kadec and Pełczyński mentioned in the introduction.

From Lemma 1 we see that both almost isometric and asymptotically isometric l^1-copies are L-embedded provided they are contained in an L-embedded Banach space. But at the time of this writing it is not clear whether asymptotic l^1-copies are always L-embedded (or whether asymptotic c_0-copies are M-embedded). For almost isometric l^1-copies the situation is clearer:

Corollary 5. *There are almost isometric copies of l^1 which are not L-embedded.*

Proof. Combine Corollary 3 or 4 and the counterexample of [4]. □

Now we briefly turn to c_0. Some straightforward modifications of the proof of [14] (or [8, IV.2.7]) show that the dual of a nonreflexive L-embedded space contains asymptotic copies of c_0 although, of course, not every nonreflexive subspace in the dual of an L-embedded space contains c_0-copies. (Take l^∞ for example.) M-embedded spaces provide a more natural frame:

Proposition 6. *Every nonreflexive subspace of an M-embedded space contains an asymptotic c_0-copy.*

Proof. The result could be obtained by appropriate modifications of the proof of [8, III.3.4, 3.7a], but we suggest a more direct (and shorter) argument.

Since M-embeddedness passes to subspaces it is enough to prove that every nonreflexive M-embedded space Z contains an asymptotic c_0-copy. Let (δ_m) be a sequence in $]0,1[$ converging to 0. By induction over $n \in \mathbb{N}$ we will construct a sequence (z_n) in Z such that

$$(3) \quad \max_{i \le n}(1-(1-2^{-n})\delta_i)|\alpha_i| \le \| \sum_{i=1}^{n} \alpha_i z_i \| \le \max_{i \le n}(1+(1-2^{-n})\delta_i)|\alpha_i| \ \forall \alpha_i \in \mathbb{C}.$$

For the beginning of the induction we choose z_1 in the unit sphere of Z. Suppose that $z_1, \dots, z_n$ have been constructed and satisfy (3). Let P denote the L-projection on Z''' with range Z'. We have $P'|_Z = \mathrm{id}_Z$ and $Z'''' = Z^{\perp\perp} \oplus_\infty Z'^{\perp}$. There exists an element $z'^{\perp} \in Z'^{\perp}$ with $\|z'^{\perp}\| = 1$ because Z is not reflexive. Put $E = \operatorname{lin}(\{z_i | \ i \le n\} \cup \{z'^{\perp}\})$ and choose $\eta > 0$ such that

$$\begin{aligned}(1-\eta)(1-(1-2^{-n})\delta_i) &> (1-(1-2^{-(n+1)})\delta_i),\\ (1+\eta)(1+(1-2^{-n})\delta_i) &< (1+(1-2^{-(n+1)})\delta_i)\end{aligned}$$

for all $i \le n$. The principle of local reflexivity provides an operator $T_1 : E \to Z''$ and an operator $T_2 : T_1(E) \to Z$ such that $T = T_2 \circ T_1$ fulfills

$$T|_{E\cap Z} = \mathrm{id}_{E\cap Z} \quad \text{and} \quad (1-\eta)\|e\| \le \|Te\| \le (1+\eta)\|e\| \quad \forall e \in E. \tag{4}$$

Put $z_{n+1} = Tz'^{\perp}$. Then we get (3, $n+1$):

$$\begin{aligned}\max_{i\le n+1}(1-(1-2^{-(n+1)})\delta_i)|\alpha_i| &< (1-\eta)\max\Big(\max_{i\le n}(1-(1-2^{-n})\delta_i)|\alpha_i|, |\alpha_{n+1}|\Big)\\ &\overset{(3)}{\le} (1-\eta)\max\Big(\|\sum_{i=1}^{n}\alpha_i z_i\|, \|\alpha_{n+1}z'^{\perp}\|\Big)\\ &= (1-\eta)\|\Big(\sum_{i=1}^{n}\alpha_i z_i\Big) + \alpha_{n+1}z'^{\perp}\|\\ &\overset{(4)}{\le} \|\sum_{i=1}^{n+1}\alpha_i z_i\|\\ &\overset{(4)}{\le} (1+\eta)\|\Big(\sum_{i=1}^{n}\alpha_i z_i\Big) + \alpha_{n+1}z'^{\perp}\|\\ &= (1+\eta)\max\Big(\|\sum_{i=1}^{n}\alpha_i z_i\|, |\alpha_{n+1}|\Big)\\ &< \max_{i\le n+1}(1+(1-2^{-(n+1)})\delta_i)|\alpha_i|.\end{aligned}$$

This ends the induction and the proof. □

It is now immediate that every nonreflexive subspace of an M-embedded space fails the fixed point property (see [6, Prop. 7]).

The analogue of Lemma 1 holds trivially, simply because M-embeddedness passes to subspaces [8, III.1.6]; thus every c_0-copy inside an M-embedded space is M-embedded. And similarly as for Corollary 5, the counterexample of [4] combined with Proposition 6 provides an almost isometric c_0-copy which is not M-embedded.

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