

## CREATION AND PROPAGATION OF LOGARITHMIC SINGULARITIES BY INTERACTION OF TWO PIECEWISE SMOOTH PROGRESSING WAVES

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ABSTRACT. Our aim is to understand the non-conservation of the piecewise smooth regularity by a semi-linear interaction of two transverse progressing waves. Indeed, we know that this phenomenon occurs when the number of characteristic hypersurfaces passing through the locus of interaction, that is, a two-codimensional variety, is strictly inferior to the size of the considered first order strictly hyperbolic system. Thanks to the study of a significant example, we explain the obstruction to the piecewise smooth propagation by a loss of transmission property for the symbols describing the conormal singularities, which originates logarithmic singularities.

### 1. INTRODUCTION

We are interested in the local phenomena which are the creation and the propagation of singularities by the interaction of two transversal conormal waves for a semi-linear partial differential system. More precisely, the purpose is to analyze a phenomenon of a loss of piecewise smooth regularity shown by G. Métivier and J. Rauch in [6] and [7].

We consider a first order semi-linear system

$$(1.1) \quad Lu = f(x, u).$$

It is supposed strictly hyperbolic with respect to a timelike function  $t$ . The coefficients of  $L$  are  $M \times M$  smooth real matrix. The real function  $f$  is smooth. The variable  $x$  describes an open neighborhood  $\Omega$  of 0 into  $\mathbb{R}^n$ , for which the past is assumed to be a domain of determinacy.

Let  $\Sigma_1, \Sigma_2$  be two characteristic hypersurfaces intersecting transversely along a 2 codimension manifold  $\Gamma$ .

If  $\Sigma$  is a smooth hypersurface, we write  $pC_\Sigma^\infty$  as the space of the piecewise smooth functions with respect to  $\Sigma$ . Locally, their restriction to each open half-space defined by  $\Sigma$  extends in a  $C^\infty$  function defined on the whole space.

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We work under the following hypotheses:

1. The rays of the bicharacteristics issued from characteristic conormal vectors to  $\Gamma$  are transverse to  $\Gamma$ .
2. The  $\mathbb{R}^M$  valued function  $u$  is a solution of (1.1) in  $L_{loc}^\infty$ .
3.  $u|_{t < 0} \in pC_{\Sigma_1}^\infty + pC_{\Sigma_2}^\infty$ .

The hypothesis 1. implies there is a finite number of characteristic hypersurfaces  $\Sigma_1, \dots, \Sigma_m$  ( $2 \leq m \leq M$ ) passing through  $\Gamma$ . They are two-by-two transverse.

Thanks to J.M. Bony and G. Métivier’s works (see [1] (smooth case) and [5] (discontinuous case)), we know that the singularities of  $u$  are conormal with respect to  $\Sigma_i$  for  $i = 1, \dots, m$  and  $\Gamma$ , that is,  $WF(u) \subset \bigcup_{i=1, \dots, m} N(\Sigma_i) \cup N(\Gamma)$  if we write  $WF(u)$  as the wave front of  $u$  and  $N(\Sigma_i)$  (respectively  $N(\Gamma)$ ) as the conormal space of  $\Sigma_i$  (respectively  $\Gamma$ ) deprived its vanished section.

More precisely G. Métivier and J. Rauch [7] demonstrated notably that if  $\Gamma$  is included in a spacelike hypersurface, then  $u$  is piecewise smooth with respect to  $\bigcup_{i=1, \dots, m} \Sigma_i$ . Note that  $m = M$  in this case. Moreover they built an example showing that the hypothesis  $m = M$  is essential (see [6]).

In this work we propose a symbolic study of the singularities showing the microlocal mechanism of the creation and the propagation of non-piecewise smooth type singularities. We assume that the number  $m$  of characteristic hypersurfaces passing through the locus of interaction is strictly inferior to the size  $M$  of the system (1.1). We will see, by a significant example, that the elliptic part of the system originates the apparition of a non-piecewise smooth type singularity on the edge  $\Gamma$  which can propagate along the characteristic hypersurfaces  $\Sigma_1, \dots, \Sigma_m$  if the semi-linearity permits it.

The example we will present is inspired by [6]. Nevertheless our approach is microlocal: we explain the loss of piecewise smooth regularity by a loss of transmission property in the complete symbol of the solution. Note the symbolic forms which appear have generalizations (see [4]) which should allow us to treat more general cases.

We consider the following  $5 \times 5$  first order system on  $\mathbb{R}^3$ :

$$(1.2) \quad \left. \begin{aligned} \sqrt{2}(\partial_t + \partial_{x_1})u_1 + (\partial_t + \partial_{x_1} + \partial_{x_2})u_2 &= 0 \\ (\partial_t + \partial_{x_1} + \partial_{x_2})u_1 + \sqrt{2}(\partial_t + \partial_{x_2})u_2 &= 0 \end{aligned} \right\}$$

$$(1.3) \quad \partial_t v = u_1 w_1$$

$$(1.4) \quad \left. \begin{aligned} (\partial_t + 2\partial_{x_1})w_1 + 2\partial_{x_2}w_2 &= 0 \\ 2\partial_{x_2}w_1 + (\partial_t - 2\partial_{x_1})w_2 &= \psi(t)u_1 u_2 \end{aligned} \right\}$$

with  $\psi(t) = \begin{cases} 0 & \text{if } t < -\varepsilon \\ 1 & \text{if } t > -\frac{\varepsilon}{2} \end{cases}$  for a  $\varepsilon > 0$ .

This system is strictly hyperbolic with respect to the time variable  $t$ . Writing  $\mathbf{H}$  the Heaviside function, we impose the following conditions in the past  $\{t < -\varepsilon\}$ :

$$(1.5) \quad \begin{aligned} \text{for } t < -\varepsilon, \quad u_1 &= (t - x_1)^{k_1 - 1} \mathbf{H}(t - x_1) \text{ with } k_1 \geq 1, \\ u_2 &= (t - x_2)^{k_2 - 1} \mathbf{H}(t - x_2) \text{ with } k_2 \geq 1, \\ v = w_1 = w_2 &= 0. \end{aligned}$$

In fact, the system (1.2) yields explicitly  $u_j = (t - x_j)^{k_j - 1} \mathbf{H}(t - x_j)$  for  $j = 1, 2$ . If we consider successively (1.4) as a linear system with respect to  $w_1, w_2$ , and (1.3) as a linear equation with respect to  $v$ , we obtain a global solution  $u = (u_1, u_2, v, w_1, w_2)$  which is in  $pC_{\Sigma_1}^\infty + pC_{\Sigma_2}^\infty$  in the past with  $\Sigma_j = \{t - x_j\}$  for  $j = 1, 2$ . Both

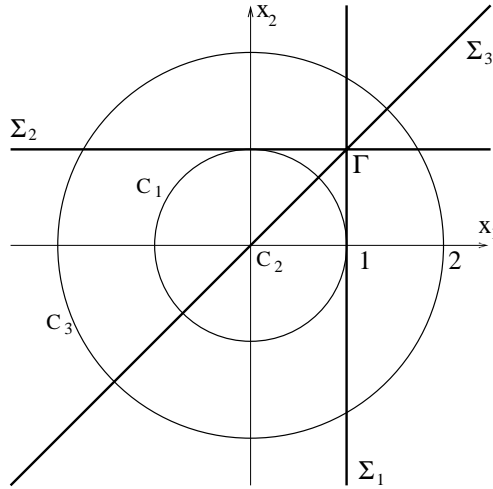


FIGURE 1.

characteristic hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  intersect transversely along the edge  $\Gamma = \{t = x_1 = x_2\}$ .

Figure 1 shows us that only three characteristic hypersurfaces contain  $\Gamma : \Sigma_1, \Sigma_2$  and  $\Sigma_3 = \{x_1 = x_2\}$ . We drew on the plan  $\{t = 1\}$  the traces of the light cones  $C_1, C_2, C_3$  issued from 0 respectively associated with (1.2), (1.3), (1.4).

There is not any characteristic hypersurface for (1.4) passing through the edge  $\Gamma$ . In other words the system (1.4) is microlocally elliptic on a neighborhood of  $N(\Gamma)$ . This partial ellipticity of the system (1.2), (1.3), (1.4) and the “sufficient nonlinearity” of the source term will create non-piecewise smooth type singularities after interaction.

We will prove

**Theorem 1.1.** *On a neighborhood of 0, near  $\Sigma_3^+ = \{x_1 = x_2\} \cap \{t > x_1\}$ , we have  $v(x) \equiv c(x_1 - x_2)^{2k_1+k_2-1} \ln|x_2 - x_1|$  modulo  $pC_{\Sigma_3}^\infty \cap C^{k_1+k_2-1}$  with  $c \neq 0$ .*

2. SYMBOLS

By the change of variables  $X_1 = t - x_1, X_2 = t - x_2, X_3 = x_1 + x_2$ , and writing again  $x_1, x_2, x_3$  the new variables, we obtain

$$(2.1) \quad \left. \begin{aligned} \sqrt{2}(\partial_{x_2} + \partial_{x_3})u_1 + 2\partial_{x_3}u_2 &= 0 \\ 2\partial_{x_3}u_1 + \sqrt{2}(\partial_{x_1} + \partial_{x_3})u_2 &= 0 \end{aligned} \right\}$$

$$(2.2) \quad (\partial_{x_1} + \partial_{x_2})v = u_1w_1$$

$$(2.3) \quad \left. \begin{aligned} (-\partial_{x_1} + \partial_{x_2} + 2\partial_{x_3})w_1 + 2(-\partial_{x_2} + \partial_{x_3})w_2 &= 0 \\ 2(-\partial_{x_2} + \partial_{x_3})w_1 + (3\partial_{x_1} + \partial_{x_2} - 2\partial_{x_3})w_2 &= \Psi(\frac{1}{2}[x_1 + x_2 + x_3])u_1u_2 \end{aligned} \right\}$$

with the new conditions in the past:

$$(2.4) \quad \text{for } x_1 + x_2 + x_3 < -2\varepsilon, \quad \begin{aligned} u_1 &= x_1^{k_1-1} \mathbf{H}(x_1) \text{ with } k_1 \geq 2, \\ u_2 &= x_2^{k_2-1} \mathbf{H}(x_2) \text{ with } k_2 \geq 2, \\ v &= w_1 = w_2 = 0. \end{aligned}$$

Now we have  $\Gamma = \{x_1 = x_2 = 0\}$ ,  $\Sigma_1 = \{x_1 = 0\}$ ,  $\Sigma_2 = \{x_2 = 0\}$ ,  $\Sigma_3 = \{x_1 = x_2\}$ , and for  $j = 1, 2$ ,

$$(2.5) \quad u_j = x_j^{k_j-1} \mathbf{H}(x_j) = U_j + g_j$$

with  $g_j \in C^\infty$ ,  $U_j = \int e^{ix_j \xi_j} c_j \frac{\chi(\xi_j)}{\xi_j^{k_j}} d\xi_j$ , and  $c_j = \frac{1}{2\pi} (-i)^{k_j} (k_j - 1)!$ .

Using a microlocal parametrix of (2.3) on a neighborhood of  $N(\Gamma)$ , and working near 0 so that  $\Psi(\frac{1}{2}[x_1 + x_2 + x_3]) = 1$ , we obtain

$$(2.6) \quad w_1 \equiv P(u_1 u_2) \text{ modulo } C^\infty$$

where  $P$  is a pseudo-differential operator of order  $-1$ . Near  $N(\Gamma)$  the symbol of  $P$  is equal to  $\frac{2(\xi_2 - \xi_3)}{i[-(\xi_1 + \xi_2)^2 + 4(\xi_1 - \xi_3)^2 + 4(\xi_2 - \xi_3)^2]}$ .

The microlocal study of the conormal singularities which appear after interaction necessitates the description of diverse class of symbols.

**Symbols with one variable of frequency.** One recalls that  $S^\mu(\mathbb{R}^n \times \mathbb{R})$  is the set of the symbols of order  $\mu$  that is the  $C^\infty$  function  $a(x, \xi)$  with  $x \in \mathbb{R}^n$  and  $\xi \in \mathbb{R}$ , such that for  $\alpha \in \mathbb{N}^n$ ,  $\beta \in \mathbb{N}$ , and  $\mathbf{K}$  a compact of  $\mathbb{R}^n$ ,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C(1 + |\xi|)^{\mu - \beta} \text{ for all } (x, \xi) \in \mathbf{K} \times \mathbb{R}.$$

For a hypersurface  $\Sigma$  and  $\mu \in \mathbb{R}$ , we write  $I^\mu(\Sigma)$  the space of the distributions  $f$  smooth out of  $\Sigma$  and which can be written

$$(2.7) \quad f(x) = \int e^{ix_1 \xi_1} a(x, \xi_1) d\xi_1, \quad a \in S^\mu(\mathbb{R}^n \times \mathbb{R})$$

near each point of  $\Sigma$  in coordinates so that  $\Sigma = \{x_1 = 0\}$ . Note the wave front of such distributions is included in  $N(\Sigma)$ .

With this notion of symbols we can characterize the piecewise smooth regularity by a transmission property of symbols. Indeed we have the well-known following result:

**Lemma 2.1.** Denoting  $x = (x_1, \dots, x_n) = (x_1, x')$  as the variable in  $\mathbb{R}^n$  and  $\Sigma = \{x_1 = 0\}$ ,  $pC_\Sigma^\infty$  is the set of the distributions  $f$  of  $I^{-1}(\Sigma)$  satisfying (2.7) with

$$(2.8) \quad a(x, \xi) \sim \sum_{j \geq 1} \frac{a_j(x')}{\xi_1^j} \text{ for } |\xi_1| > 1.$$

In this case  $a_j(x') = \frac{1}{2\pi i^j} [\partial_{x_1}^j f]_\Sigma$  where  $[\ ]_\Sigma$  designates the jump through  $\Sigma$ .

We write  $pC_k^\infty(\Sigma) = pC_\Sigma^\infty \cap I^{-k}(\Sigma)$  for  $k \in \mathbb{N}^*$ .

One recalls (see [3]) that the asymptotic expansion (2.8) means that for all  $N \in \mathbb{N}^*$ ,

$$a(x, \xi) - \sum_{j=1}^{N-1} a_j(x') \frac{\chi(\xi_1)}{\xi_1^j} \in I^{-N}(\Sigma)$$

where  $\chi$  is a  $C^\infty$  function vanishing near 0 so that  $\chi(\xi_1) = 1$  if  $\xi_1$  is tall enough.

**Symbols with two variables of frequency.** One calls  $\tilde{S}^{\mu_1, \mu_2}$  the set of the symbols with two variables of frequency defined as the  $C^\infty$  functions  $b(x, \xi_1, \xi_2)$  with  $x \in \mathbb{R}^n$  and  $\xi_1, \xi_2 \in \mathbb{R}$ , so that for  $\alpha \in \mathbb{N}^n$ ,  $\beta_1, \beta_2 \in \mathbb{N}$ , and  $\mathbf{K}$  a compact of  $\mathbb{R}^n$ ,

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} b(x, \xi_1, \xi_2)| \leq C(1 + |\xi_1|)^{\mu_1 - \beta_1} (1 + |\xi_2|)^{\mu_2 - \beta_2}$$

for all  $(x, \xi_1, \xi_2) \in \mathbf{K} \times \mathbb{R}^2$ .

For  $\mu_1, \mu_2 \in \mathbb{R}$ , let  $\tilde{I}^{\mu_1, \mu_2}$  be the space of the distributions  $f$  on  $\mathbb{R}^n$  smooth out of  $\Sigma_1 = \{x_1 = 0\}$  and  $\Sigma_2 = \{x_2 = 0\}$  such that  $f \in I^{\mu_i}(\Sigma_i)$  near each point of  $\Sigma_i \setminus \Gamma$  for  $i = 1, 2$  where  $\Gamma = \Sigma_1 \cap \Sigma_2$ , and

$$(2.9) \quad f(x) = \int e^{i(x_1 \xi_1 + x_2 \xi_2)} b(x, \xi_1, \xi_2) d\xi_1 d\xi_2, \quad b \in \tilde{S}^{\mu_1, \mu_2}$$

near each point of the edge  $\Gamma$ . The wave front of such distributions is contained by  $N(\Sigma_1) \cup N(\Sigma_2) \cup N(\Gamma)$ .

We write  $pC_{\Sigma_1 \cup \Sigma_2}^\infty$  as the space of the piecewise smooth functions with respect to  $\Sigma_1 \cup \Sigma_2$ . Locally, their restriction to each quarter space defined by  $\Sigma_1 \cup \Sigma_2$  extends in a  $C^\infty$  function defined on the whole space.

In this way we have a result analogous to Proposition 2.1:

**Lemma 2.2.** *We denote  $x = (x_1, \dots, x_n) = (x_1, x_2, x'')$  as the variable in  $\mathbb{R}^n$ ,  $\Sigma_i = \{x_i = 0\}$  for  $i = 1, 2$ .*

*Near each point of the edge  $\Gamma = \{x_1 = x_2 = 0\}$ , the functions of  $pC_{\Sigma_1 \cup \Sigma_2}^\infty$  are the distributions  $f \in \tilde{I}^{-1, -1}$  satisfying (2.9) with*

$$(2.10) \quad b(x, \xi_1, \xi_2) \sim \sum_{j_1, j_2 \geq 1} \frac{b_{j_1, j_2}(x'')}{\xi_1^{j_1} \xi_2^{j_2}} \text{ for } |\xi_1|, |\xi_2| > 1.$$

*In this case  $b_{j_1, j_2}(x'') = \frac{1}{4\pi^{2j_1 + j_2}} [\partial_{x_1}^{j_1} \partial_{x_2}^{j_2} f]_\Gamma$  where  $[ ]_\Gamma$  designates the jump through  $\Gamma$ , that is,*

$$[f]_\Gamma = f(0^+, 0^+, x'') - f(0^+, 0^-, x'') + f(0^-, 0^+, x'') - f(0^-, 0^-, x'').$$

Let's specify the meaning of the asymptotic expansion (2.10) : for all  $P, N \in \mathbb{N}$ ,

$$\begin{aligned} b(x, \xi_1, \xi_2) &- \sum_{\substack{1 \leq j_1 \leq P-1 \\ 1 \leq j_2 \leq N-1}} b_{j_1, j_2}(x'') \frac{\chi(\xi_1) \chi(\xi_2)}{\xi_1^{j_1} \xi_2^{j_2}} \\ &\equiv \sum_{1 \leq j_1 \leq P-1} b_{j_1}^N(x'', \xi_2) \frac{\chi(\xi_1)}{\xi_1^{j_1}} + \sum_{1 \leq j_2 \leq N-1} {}^P b_{j_2}(x'', \xi_1) \frac{\chi(\xi_2)}{\xi_1^{j_2}} \end{aligned}$$

modulo  $\tilde{S}^{-P, -N}$  with  $b_{j_1}^N \in S^{-N}(\mathbb{R}^{n-2} \times \mathbb{R})$  and  ${}^P b_{j_2} \in S^{-P}(\mathbb{R}^{n-2} \times \mathbb{R})$ .

For  $k, k' \geq 1$  we say that the symbol  $b$  verifying (2.10) is  $k, k'$ -classical if  $b_{j_1, j_2} = 0$  for  $j_1 \leq k$  or  $j_2 \leq k'$ , that is  $b \in \tilde{S}^{-k, -k'}$ .

Let's remark that the asymptotic expansion does not depend on  $x_1$  in (2.8) and  $x_1, x_2$  in (2.10). In fact we know (see [3]) that every distribution of  $I^\mu(\Sigma)$  can be written under the form (2.7) with a symbol  $a$  not depending on  $x_1$ . We have an analogous result of reduction of amplitude for the elements of  $\tilde{I}^{\mu_1, \mu_2}$  :

**Lemma 2.3.** *Working near a point of the edge  $\Gamma$ , a distribution  $f \in \tilde{S}^{\mu_1, \mu_2}$ ,  $\mu_1, \mu_2 \in \mathbb{R}$ , verifying (2.10) can be written in the form*

$$f(x) = \int e^{i(x_1 \xi_1 + x_2 \xi_2)} c(x'', \xi_1, \xi_2) d\xi_1 d\xi_2, \quad c \in \tilde{S}^{\mu_1, \mu_2},$$

with

$$(2.11) \quad c \sim \sum_{\alpha_1, \alpha_2 \in \mathbb{N}} \frac{1}{\alpha_1! \alpha_1!} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b|_{x_1=x_2=0}.$$

Here, the asymptotic expansion means that, for all  $N \in \mathbb{N}^*$ ,

$$c - \sum_{|(\alpha_1, \alpha_2)| \leq N-1} \frac{1}{\alpha_1! \alpha_1!} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b|_{x_1=x_2=0} \in \sum_{k+k'=N} \tilde{S}^{\mu_1-k, \mu_2-k'}.$$

**Action of the pseudo-differential operators.** We write  $\xi = (\xi_1, \dots, \xi_n)$  as the cotangent variables of  $\mathbb{R}^n$ . We need a new space of symbols :  $\tilde{S}^{\mu_1, \mu_2, \nu} = \tilde{S}^{\mu_1+\nu, \mu_2} \cap \tilde{S}^{\mu_1, \mu_2+\nu}$  for  $\mu_1, \mu_2, \nu \in \mathbb{R}$ , corresponding to a new space of conormal distributions:  $\tilde{I}^{\mu_1, \mu_2, \nu} = \tilde{I}^{\mu_1+\nu, \mu_2} \cap \tilde{I}^{\mu_1, \mu_2+\nu}$ .

Let's describe the action of a pseudo-differential operator on the conormal distributions (see [3] for an idea of the proof).

**Lemma 2.4.** *Let  $P$  be a proper pseudo-differential operator of degree  $d \leq 0$ . Let's write  $p(x, \xi)$  as its symbol.*

- 1) *If  $f \in I^\mu(\Sigma)$ ,  $\mu \in \mathbb{R}$ , is locally given by (2.7), then  $Pf \in I^{\mu+d}(\Sigma)$  can be locally written as*

$$Pf = \int e^{ix_1 \xi_1} c(x, \xi_1) d\xi_1$$

with

$$(2.12) \quad c(x, \xi_1) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x, \xi_1, 0, \dots, 0) D_x^{\alpha} a(x, \xi_1).$$

- 2) *If  $f \in \tilde{I}^{\mu_1, \mu_2}$ ,  $\mu_1, \mu_2 \in \mathbb{R}$ , is locally given by (2.9), then  $Pf \in \tilde{I}^{\mu_1, \mu_2, d}$  can be locally written as*

$$Pf = \int e^{i(x_1 \xi_1 + x_2 \xi_2)} c(x, \xi_1, \xi_2) d\xi_1 d\xi_2$$

with

$$(2.13) \quad \text{with } c(x, \xi_1, \xi_2) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x, \xi_1, \xi_2, 0, \dots, 0) D_x^{\alpha} b(x, \xi_1, \xi_2).$$

As defined in [3], the asymptotic expansion (2.12) means that for all  $N \in \mathbb{N}^*$ ,

$$c(x, \xi_1) - \sum_{|\alpha| \leq N-1} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x, \xi_1, 0, \dots, 0) D_x^{\alpha} a(x, \xi_1) \in I^{\mu+d-N}(\Sigma).$$

The meaning of the asymptotic expansion (2.13) is the following : for all  $N \in \mathbb{N}^*$ ,

$$c(x, \xi_1, \xi_2) - \sum_{|\alpha| \leq N-1} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x, \xi_1, \xi_2, 0, \dots, 0) D_x^{\alpha} b(x, \xi_1, \xi_2) \in \tilde{I}^{\mu_1, \mu_2, d-N}.$$

In that way, if  $P$  is a parametrix of a partial differential system elliptic on a neighborhood of  $N(\Gamma)$ , then its action on the functions of  $pC_{\Sigma_1 \cup \Sigma_1}^{\infty}$  yields the apparition of a new type of symbols. These symbols are an asymptotic sum of terms  $\sigma_{j_1, j_2, j} = b_{j_1, j_2, j}(x) \frac{\rho_j(x, \xi_1, \xi_2)}{\xi_1^{j_1} \xi_2^{j_2}}$  where  $\rho_j$  is a rational function for  $\xi_1, \xi_2$  of degree  $-j \leq 0$ , without any real pole, and smooth with respect to  $x$ , and  $b_{j_1, j_2, j}$  is a smooth function.

More precisely,

**Definition 2.5.** For  $k, k', l \in \mathbb{N}^*$ , we say that a symbol  $b \in \bar{S}^{-k, -k', -l}$  is  $k, k', l$ -prelogarithmic if

$$(2.14) \quad b \sim \sum_{\substack{k \leq j_1 \\ k' \leq j_2 \\ l \leq j}} b_{j_1, j_2, j}(x) \frac{\rho_j(x, \xi_1, \xi_2)}{\xi_1^{j_1} \xi_2^{j_2}} \text{ for } |\xi_1|, |\xi_2| > 1.$$

The asymptotic expansion (2.14) means : for all  $P, N \in \mathbb{N}^*$ ,

$$\begin{aligned} b(x, \xi_1, \xi_2) &- \sum_{\substack{k \leq j_1 \leq P-1 \\ k' \leq j_2 \leq N-1 \\ l \leq j \leq P+N-j_1-j_2-1}} \rho_{j_1, j_2, j}(x, \xi_1, \xi_2) \frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1^{j_1} \xi_2^{j_2}} \\ &\equiv \sum_{\substack{k \leq j_1 \leq P-1 \\ l \leq j \leq P-j_1-1}} b_{j_1}^N(x, \xi_2) \rho_j(x, \xi_1, \xi_2) \frac{\chi(\xi_1)}{\xi_1^{j_1}} \\ &\quad + \sum_{\substack{k' \leq j_2 \leq N-1 \\ l \leq j \leq N-j_2-1}} {}^P b_{j_2}(x, \xi_1) \rho_j(x, \xi_1, \xi_2) \frac{\chi(\xi_2)}{\xi_1^{j_2}} \end{aligned}$$

modulo  $\tilde{S}^{-P, -N}$  with  $b_{j_1}^N \in S^{-N}(\mathbb{R}^n \times \mathbb{R})$  and  ${}^P b_{j_2} \in S^{-P}(\mathbb{R}^n \times \mathbb{R})$ .

If  $b$  is such a symbol, the distribution  $f$  given by (2.9) is  $pC_{\Sigma_j}^\infty$  near each point of  $\Sigma_j \setminus \Gamma$  for  $j = 1, 2$ . Indeed, near a point of  $\Sigma_1 \setminus \Gamma$  for example, the variable  $x_2$  does not vanish so we can use the Taylor expansion of  $b$  at 0 with respect to  $\xi_2$  in order to make  $f$  under the form (2.7). This Taylor expansion yields a symbol like (2.8).

Nevertheless,  $f$  is not  $pC_{\Sigma_1 \cup \Sigma_1}^\infty$  near the edge  $\Gamma$  in general. For example, if  $b = \frac{\chi(\xi_1)\chi(\xi_2)}{\xi_1 \xi_2 (\xi_1^2 + \xi_2^2)}$ , then we have  $\Delta f \equiv 4\pi^2 \mathbf{H}(x_1) \mathbf{H}(x_2)$  modulo  $C^\infty$ , so  $\Delta \frac{\partial^2 f}{\partial x_1 \partial x_2} \equiv 4\pi^2 \delta$  and  $\frac{\partial^2 f}{\partial x_1 \partial x_2} \equiv \pi \ln(x_1^2 + x_2^2)$  modulo  $C^\infty$ .

**Integration transverse to  $\Sigma_1, \Sigma_2, \Gamma$ .** The following result originates from the properties of integration stated in [2]. Working on a neighborhood of a point on  $\Gamma$ , we have

**Lemma 2.6.** *Let  $f$  be under the form (2.9) with a symbol  $b(x'', \xi_1, \xi_2) \in \tilde{S}^{\mu_1, \mu_2}$  and  $\mu_1, \mu_2 < -1$ . If  $g$  satisfies*

$$(2.15) \quad (\partial_{x_1} + \partial_{x_2})g = f,$$

and if  $g$  is  $C^\infty$  on a neighborhood of each point of  $\Sigma_3^- = \{x_1 = x_2, x_1 < 0\}$ , then

$$(2.16) \quad g(x) \equiv \int e^{i(x_1 - x_2)\eta} b(x'', \eta, -\eta) d\eta \text{ modulo } C^\infty$$

on a neighborhood of each point of  $\Sigma_3^+ = \{x_1 = x_2, x_1 > 0\}$ . We note that  $b(x'', \eta, -\eta) \in S^{\mu_1 + \mu_2}(\mathbb{R}^{n-2} \times \mathbb{R})$ .

So if  $b$  is  $k, k'$ -classical or  $k, k', l$ -prelogarithmic, then  $g$  defined by (2.16) is respectively  $pC_{k+k'}^\infty(\Sigma_3)$  or  $pC_{k+k'+l}^\infty(\Sigma_3)$  near each point of  $\Sigma_3^\pm$ .

3. APPLICATION TO THE PROOF OF THEOREM 1.1

From the expression (2.5) of  $u_1$  and  $u_2$ , we have  $u_1 u_2 = U + \tilde{u}_1 + \tilde{u}_2$  with  $\tilde{u}_j \in pC_{k_j}^\infty(\Sigma_j)$  for  $j = 1, 2$  and

$$U(x) = \int e^{i(x_1 \xi_1 + x_2 \xi_2)} c_1 c_2 \frac{\chi(\xi_1) \chi(\xi_2)}{\xi_1^{k_1} \xi_2^{k_2}} d\xi_1 d\xi_2.$$

Using the expression (2.6) we obtain  $w_1 = W + P\tilde{u}_1 + P\tilde{u}_2$ , where  $W = PU$  and  $P\tilde{u}_j \in pC_{k_j+1}^\infty(\Sigma_j)$  for  $j = 1, 2$  thanks to Proposition 2.4 i).

So we can write  $u_1 w_1 = u_1 W + r_1 + r_2$  with  $r_1 = u_1 P\tilde{u}_1 \in pC_{k_1}^\infty(\Sigma_1)$  and  $r_2 = u_1 P\tilde{u}_2 \in pC_{\Sigma_1 \cup \Sigma_2}^\infty \cap \tilde{I}^{-k_1, -k_2-1}$ .

From equation (2.2) and the condition  $v = 0$  in the past, we obtain that  $v$  is smooth near  $\Sigma_3^-$ . So we can study the singularities of  $v$  along  $\Sigma_3^+$  using Proposition 2.6. The functions  $r_1, r_2$  originate  $pC_{k_1+k_2+1}^\infty(\Sigma_3)$  singularities for  $v$ .

So it remains to study the contribution of the product  $u_1 W$ . Proposition 2.4 shows that  $W = W_0 + r$  where  $r$  is  $C^\infty$  and

$$W_0(x) = \int e^{i(x_1 \xi_1 + x_2 \xi_2)} c_3 \frac{\chi(\xi_1) \chi(\xi_2)}{\xi_1^{k_1} \xi_2^{k_2-1} (\xi_1 - \alpha \xi_2) (\xi_1 - \bar{\alpha} \xi_2)} d\xi_1 d\xi_2,$$

with  $\alpha = \frac{1}{3} + i\frac{2\sqrt{2}}{2}$  and  $c_3 = \frac{2c_1 c_2}{3i}$ .

Hence from (2.5) we have  $u_1 W = (U_1 + g_1)(W_0 + r)$ . Because  $u_1 r$  is just piecewise smooth with respect to  $\Sigma_1$ , it does not contribute to any singularity of  $v$  along  $\Sigma_3^+$ . Let's treat the term  $g_1 W_0$ . The symbol describing its singularities is  $k_1, k_2 - 1, 2$ -prelogarithmic. Proposition 2.3 allows us to assume that this prelogarithmic symbol does not depend on the variable  $x_3$ . So we can apply Proposition 2.6 to conclude that the contribution of  $g_1 W_0$  to calculate  $v$  just yields  $pC_{k_1+k_2+1}^\infty(\Sigma_3)$  singularities near  $\Sigma_3^+$ .

Finally we just need to study the contribution of the term  $U_1 W_0$ . By convolution in the  $\xi_1$  variable, the symbol of  $U_1 W_0$  is:

$$(3.1) \quad \tilde{b}(\xi_1, \xi_2) = c_1 c_3 \frac{\chi(\xi_1)}{\xi_1^{k_1}} *_{\xi_1} \frac{\chi(\xi_1) \chi(\xi_2)}{\xi_1^{k_1} \xi_2^{k_2-1} (\xi_1 - \alpha \xi_2) (\xi_1 - \bar{\alpha} \xi_2)}.$$

We have

**Lemma 3.1.** *Let's write  $g(z) = [(\xi_1 - z)^{k_1} z^{k_1} \xi_2^{k_2-1} (z - \alpha \xi_2) (z - \bar{\alpha} \xi_2)]^{-1}$ .*

*Then the convolution defined by (3.1) is equivalent to*

$$\sigma(\xi_1, \xi_2) = 2i\pi c_1 c_3 \begin{cases} \text{Res}(g, \alpha \xi_2) & \text{if } \xi_2 > 0 \\ \text{Res}(g, \bar{\alpha} \xi_2) & \text{if } \xi_2 < 0 \end{cases}$$

*modulo a  $k_1, k_2 - 1, 2$ -prelogarithmic symbol plus a  $k_1, k_2 + 1$ -classical symbol.*

*The notation  $\text{Res}(g, z_0)$  designates the residue of  $g$  at the pole  $z_0$ .*

Let's assume this result provisionally. We obtain:

$$\sigma(\xi_1, \xi_2) = 2i\pi c_1 c_3 \begin{cases} \frac{\chi(\xi_2)}{(\xi_1 - \alpha \xi_2)^{k_1} \alpha^{k_1} (\alpha - \bar{\alpha}) \xi_2^{k_1+k_2}} & \text{if } \xi_2 > 0 \\ \frac{\chi(\xi_2)}{(\xi_1 - \bar{\alpha} \xi_2)^{k_1} \bar{\alpha}^{k_1} (\bar{\alpha} - \alpha) \xi_2^{k_1+k_2}} & \text{if } \xi_2 < 0, \end{cases}$$

and

$$\sigma(-\eta, \eta) = 2i\pi c_1 c_3 \begin{cases} \frac{\chi(\eta)}{2i[(-1 - \alpha)\alpha]^{k_1} \text{Im}(\alpha)\eta^{2k_1+k_2}} & \text{if } \eta > 0 \\ \frac{\chi(\eta)}{-2i[(-1 - \bar{\alpha})\bar{\alpha}]^{k_1} \text{Im}(\alpha)\eta^{2k_1+k_2}} & \text{if } \eta < 0 \end{cases}$$

is a symbol in  $S^{-2k_1-k_2}(\mathbb{R})$  which does not verify the transmission property characterizing the symbols of the piecewise smooth functions.

Finally Proposition 2.6 implies that near any point of  $\Sigma_3^+$ , we can write

$$v(x) \equiv \int e^{i(x_2-x_1)\eta} \sigma(-\eta, \eta) d\eta$$

modulo a function in  $pC_{k_1+k_2+1}^\infty(\Sigma_3)$ .

The calculus of Fourier’s transformation completes the proof of Theorem 1.1.

*Remark 3.2.* We need the non-linearity to be sufficient if we want the loss of transmission property to occur. For example, if we replace (1.3) by  $\partial_t v = w_1$  the solution is piecewise smooth along  $\Sigma_3^+$  and other types of singularities just appear on the edge  $\Gamma$ .

*Remark 3.3.* Proposition 3.1 creates a new type of symbol, called “logarithmic”. A systematic study of the classical, prelogarithmic and logarithmic symbols ough to allow us to generalize our approach.

*Proof of Lemma 3.1.* Our aim is to calculate

$$I_{\mathbb{R}} = \int_{\mathbb{R}} \chi(\xi_1 - z)\chi(z)g(z)dz.$$

Let  $\mathbf{K}$  be a compact neighborhood of 0 in  $\mathbb{R}$  so that  $\chi = 1$ . At first we study  $I_{\mathbf{K}} = \int_{\mathbf{K}} \chi(\xi_1 - z)\chi(z)g(z)dz$ . The compactness of the domain of integration allows us to expand in power series the functions  $z \mapsto [(z - \alpha\xi_2)(z - \bar{\alpha}\xi_2)]^{-1}$  and  $z \mapsto \frac{\chi(\xi_1-z)}{(\xi_1-z)^{k_1}}$ . We obtain that  $I_{\mathbf{K}}$  is a  $k_1, k_2 + 1$ -classical symbol.

Let’s treat  $I_{\mathbf{K}(\xi_1)} = \int_{\xi_1-z \in \mathbf{K}} \chi(\xi_1 - z)\chi(z)g(z)dz = \int_{\mathbf{K}} \chi(z)\chi(\xi_1 - z)g(\xi_1 - z)dz$ . We carry out the study of  $I_{\mathbf{K}}$  by replacing the first expansion by the expansion of  $z \mapsto [(\xi_1 - z - \alpha\xi_2)(\xi_1 - z - \bar{\alpha}\xi_2)]^{-1}$ . So  $I_{\mathbf{K}(\xi_1)}$  is a  $k_1, k_2 - 1, 2$ -prelogarithmic symbol.

Finally we prove that  $I_{\mathbb{R}} \equiv \int_{\mathbb{R} \setminus (\mathbf{K} \cup \mathbf{K}(\xi_1))} g(z)dz$  modulo the classical and prelogarithmic symbols. The function  $g$  is analytic on  $\mathbb{C} \setminus \{0, \xi_1, \alpha\xi_2, \bar{\alpha}\xi_2\}$  and can be integrated in a complex path.

As for  $I_{\mathbf{K}}$  and  $I_{\mathbf{K}(\xi_1)}$ ,  $I_{\gamma} = \int_{\gamma} g(z)dz$  is still a  $k_1, k_2 + 1$ -classical symbol and  $I_{\gamma(\xi_1)} = \int_{\xi_1-z \in \gamma} g(z)dz$  a  $k_1, k_2 - 1, 2$ -prelogarithmic one, when  $\gamma$  is the half circle defined by Figure 2.

Accordingly, for a tall enough  $R > 0$ ,  $I_{\mathbb{R}}$  is equivalent to  $I_{\gamma_R} = \int_{\gamma_R} g(z)dz$  modulo classical and prelogarithmic symbols, where  $\gamma_R$  is the closed path defined on the figure.

The Residue Theorem completes the proof. □

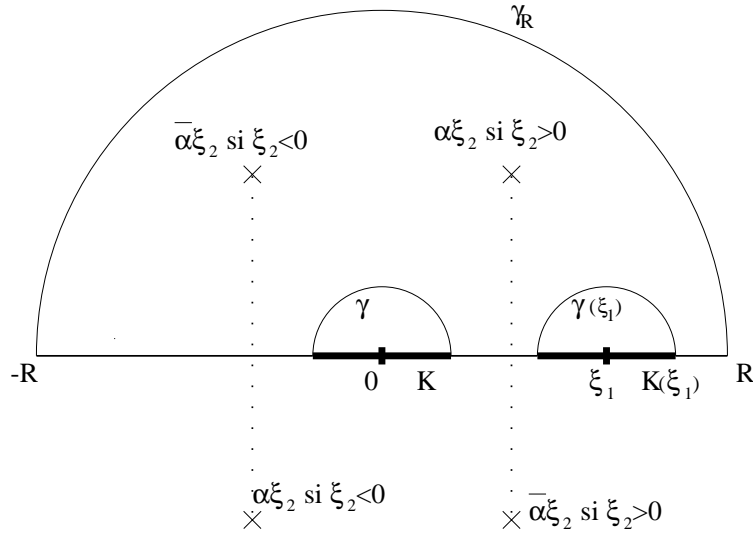


FIGURE 2.

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